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Positivity and stabilization of driver support systems[1]

by

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Abstract: In this paper, we first show that the transfer function from the front steering angle to yaw rate is strictly positive real, irrespective of the uncertain mass and uncertain velocity. We then show how to determine the positivity margin for this transfer function. Some stabilization results are obtained. Finally, we show how to check the positivity of a controller family.

Keywords: driver support systems, PID controllers, positivity, uncertain parameters, stabilization

1. Introduction

Robust Vehicle Control has attracted much attention in control community over the past few years. Achieving safe and comfortable travel by robust control is one of the most challenging problems in control and systems research. Recently, great progress has been made in this field, e.g., practical driving tests have shown essential safety advantages for a robust steering control law which is based on feedback of yaw rate into active front wheel steering. By the control law, robust unilateral decoupling of the lateral and yaw motions of the car is achieved. Interested readers can find more comprehensive development in the recent Bode Lecture presented at IEEE Conference on Decision and Control (Kobe, Japan, December, 1996).

The aim of this paper is to establish some positivity results for driver support systems, and use these results to solve some stabilization problems. The

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motivation of this research stems mainly from the numerous simulation results performed at German Aerospace Research Establishment (DLR). The difficulty in this theoretic study is that very complicated nonlinear uncertainty structure is involved, i.e., the uncertain parameters enter into system equation in a nonlinear fashion. Thus, most mathematically appealing robust stability criteria, e.g., Kharitonov's Theorem, Edge Theorem, Convex Direction Criterion, Tsypkin-Polyak Criterion, Garloff-Wagner Theorem etc., see Ackermann (1993), Kharitonov (1978), Bartlett, Hollot and Huang (1988), Tsypkin and Polyak (1991), Garloff and Wagner (1996), simply do not apply. Gridding or overbounding is not desirable, and can lead to erroneous or very conservative conclusions, Ackermann (1993).

In this paper, we first show that the transfer function from the front steering angle to yaw rate is strictly positive real, irrespective of the uncertain mass and uncertain velocity. We then show how to determine the positivity margin for this transfer function. Some stabilization results are obtained. Finally, we show how to check the positivity of a controller family.

2. Strict positive realness

Consider the driver support system, Ackermann (1993): the transfer function from the front steering angle δ_f to yaw rate r is

$$G(s) = \frac{\frac{c_f}{\tilde{J}}\left(l_f s + \frac{c_r l}{\tilde{m} v}\right)}{s^2 + \left(\frac{c_r + c_f}{\tilde{m} v} + \frac{c_r l_r^2 + c_f l_f^2}{\tilde{J} v}\right) s + \left(\frac{c_f c_r l^2}{\tilde{m} \tilde{J} v^2} + \frac{c_r l_r - c_f l_f}{\tilde{J}}\right)} \tag{1}$$

where the physical parameters are
c_r: cornering stiffness for rear wheels
c_f: cornering stiffness for front wheels
l_r: the distance from the center of gravity to the rear axle
l_f: the distance from the center of gravity to the front axle
$\tilde{m}$: virtual mass, unknown but fixed, $\tilde{m} \in [\tilde{m}^- \tilde{m}^+]$
v: velocity, unknown but fixed, $v \in [v^- v^+]$
l: $= l_r + l_f$, wheelbase
$\tilde{J}$: $= i^2 \tilde{m}$, moment of inertia
i: inertial radius

DEFINITION 2.1 *A proper transfer function* $\frac{p(s)}{q(s)}$ *is said to be strictly positive real (SPR), if*

1) $q(s)$ *is Hurwitz stable*

(2)

2) $\Re \frac{p(j\omega)}{q(j\omega)} > 0 \quad \forall \omega \in R$

For the city bus O 305, it was shown by Ackermann (1993) that the transfer function from the front steering angle δ_f to yaw rate r

$$\frac{\frac{c_f}{\tilde{J}}\left(l_f s + \frac{c_r l}{\tilde{m}v}\right)}{s^2 + \left(\frac{c_r + c_f}{\tilde{m}v} + \frac{c_r l_r^2 + c_f l_f^2}{\tilde{J}v}\right)s + \left(\frac{c_f c_r l^2}{\tilde{m}\tilde{J}v^2} + \frac{c_r l_r - c_f l_f}{\tilde{J}}\right)} \tag{3}$$

is strictly positive real, irrespective of the uncertain virtual mass $\tilde{m}$ and uncertain velocity v. In what follows, we will show that this property is shared by all vehicles.

LEMMA 2.1 *The transfer function*

$$\frac{a_1 s + a_0}{s^2 + b_1 s + b_2} \tag{4}$$

with positive coefficients is strictly positive real, if and only if

$$a_1 b_1 \geq a_0 \tag{5}$$

Proof: Obviously, $s^2 + b_1 s + b_2$ is Hurwitz stable. Moreover

$$\Re\frac{a_1(j\omega) + a_0}{(j\omega)^2 + b_1(j\omega) + b_2} = \frac{a_0 b_2 + (a_1 b_1 - a_0)\omega^2}{(b_2 - \omega^2)^2 + (b_1\omega)^2} \tag{6}$$

Since

$$(b_2 - \omega^2)^2 + (b_1\omega)^2 > 0, \quad \forall \omega \in R \tag{7}$$

and all coefficients a_0, a_1, b_1, b_2 are positive, the result obviously follows. ∎

THEOREM 2.1 *The transfer function from the front steering angle δ_f to yaw rate r*

$$\frac{\frac{c_f}{\tilde{J}}\left(l_f s + \frac{c_r l}{\tilde{m}v}\right)}{s^2 + \left(\frac{c_r + c_f}{\tilde{m}v} + \frac{c_r l_r^2 + c_f l_f^2}{\tilde{J}v}\right)s + \left(\frac{c_f c_r l^2}{\tilde{m}\tilde{J}v^2} + \frac{c_r l_r - c_f l_f}{\tilde{J}}\right)} \tag{8}$$

is robustly strictly positive real.

Proof: By Lemma 2.1, it suffices to show that

$$\frac{c_f l_f}{\tilde{J}}\left(\frac{c_r + c_f}{\tilde{m}v} + \frac{c_r l_r^2 + c_f l_f^2}{\tilde{J}v}\right) \geq \frac{c_f c_r l}{\tilde{J}\tilde{m}v} \tag{9}$$

Since $\tilde{J} = i^2\tilde{m} = l_r l_f \tilde{m}$, we only need to show that

$$\frac{c_r l_f}{\tilde{m}v} + \frac{c_f l_f}{\tilde{m}v} + \frac{c_r l_r}{\tilde{m}v} + \frac{c_f l_f^2}{\tilde{m}v l_r} \geq \frac{c_r l_r}{\tilde{m}v} + \frac{c_r l_f}{\tilde{m}v} \tag{10}$$

3. Robust exponential stabilization

LEMMA 3.1 *(Ackermann, 1993) Under the unity negative feedback, the closed-loop system with a strictly positive real controller and a strictly positive real plant is Hurwitz stable.*

DEFINITION 3.1 *Given $\alpha \geq 0$, a polynomial $p(s)$ is said to be α-stable, if $p(s-\alpha)$ is Hurwitz stable. A proper transfer function $G(s)$ is said to be α-SPR, if $G(s-\alpha)$ is SPR (strictly positive real).*

THEOREM 3.1 *Under the unity negative feedback, the closed-loop system with an α-SPR controller and an α-SPR plant is α-stable.*

Proof: Since the controller $C(s)$ and the plant $P(s)$ are α-SPR, by definition, $C(s-\alpha)$ and $P(s-\alpha)$ are strictly positive real. Hence, by Lemma 3.1, under the unity negative feedback, the closed-loop system with controller $C(s-\alpha)$ and plant $P(s-\alpha)$ is Hurwitz stable. Henceforce, the closed-loop system with controller $C(s)$ and plant $P(s)$ is α-stable.

Next, we will discuss the following problem: Given a SPR plant, how can we determine the largest α such that the plant is α-SPR?

Consider the SPR transfer function

$$G(s) = \frac{\frac{c_f}{\tilde{J}}\left(l_f s + \frac{c_r l}{\tilde{m} v}\right)}{s^2 + \left(\frac{c_r + c_f}{\tilde{m} v} + \frac{c_r l_r^2 + c_f l_f^2}{\tilde{J} v}\right) s + \left(\frac{c_f c_r l^2}{\tilde{m}\tilde{J} v^2} + \frac{c_r l_r - c_f l_f}{\tilde{J}}\right)} \tag{11}$$

For city bus O 305, we have the following data

- $l_f = 3.67$
- $l_r = 1.93$
- $c_f = 198000$
- $c_r = 470000$
- $i^2 = 10.85$
- $v = 20$
- $\tilde{m} = 32000$

Now, consider the shifted transfer function

$$G(s-\alpha) = \frac{\frac{c_f}{\tilde{J}}\left(l_f(s-\alpha) + \frac{c_r l}{\tilde{m} v}\right)}{(s-\alpha)^2 + \left(\frac{c_r + c_f}{\tilde{m} v} + \frac{c_r l_r^2 + c_f l_f^2}{\tilde{J} v}\right)(s-\alpha) + \left(\frac{c_f c_r l^2}{\tilde{m}\tilde{J} v^2} + \frac{c_r l_r - c_f l_f}{\tilde{J}}\right)} \tag{12}$$

$$= \left(\frac{c_f}{\tilde{J}}\left(l_f s + \frac{c_r l}{\tilde{m} v} - \alpha l_f\right)\right) / \left(s^2 + \left(\frac{c_r + c_f}{\tilde{m} v} + \frac{c_r l_r^2 + c_f l_f^2}{\tilde{J} v} - 2\alpha\right) s + \right.$$

$$\left.\left[\alpha^2 - \left(\frac{c_r + c_f}{\tilde{m} v} + \frac{c_r l_r^2 + c_f l_f^2}{\tilde{J} v}\right)\alpha + \left(\frac{c_f c_r l^2}{\tilde{m}\tilde{J} v^2} + \frac{c_r l_r - c_f l_f}{\tilde{J}}\right)\right]\right) \tag{13}$$

For $G(s-\alpha)$ to be SPR, all coefficients of $G(s-\alpha)$ must be positive. This leads to the following inequalities

$$\frac{c_r l}{\tilde{m}v} - \alpha l_f > 0 \tag{14}$$

$$\frac{c_r + c_f}{\tilde{m}v} + \frac{c_r l_r^2 + c_f l_f^2}{\tilde{J}v} - 2\alpha > 0 \tag{15}$$

$$\alpha^2 - \left(\frac{c_r + c_f}{\tilde{m}v} + \frac{c_r l_r^2 + c_f l_f^2}{\tilde{J}v}\right)\alpha + \left(\frac{c_f c_r l^2}{\tilde{m}.\tilde{J}v^2} + \frac{c_r l_r - c_f l_f}{\tilde{J}}\right) > 0 \tag{16}$$

With the given data for the city bus O 305, the above inequalities lead to

$$\alpha < 1.1206 \tag{17}$$

$$\alpha < 0.8400 \tag{18}$$

$$\alpha^2 - 1.68\alpha + 1.1764 > 0 \Longrightarrow \alpha < +\infty \tag{19}$$

Moreover, by Lemma 2.1, we must also have

$$l_f\left(\frac{c_r + c_f}{\tilde{m}v} + \frac{c_r l_r^2 + c_f l_f^2}{\tilde{J}v} - 2\alpha\right) \geq \frac{c_r l}{\tilde{m}v} - \alpha l_f \tag{20}$$

This leads to

$$\alpha \leq 0.5594 \tag{21}$$

Therefore, whenever $\alpha \leq 0.5594$, the transfer function $G(s-\alpha)$ is always SPR. Namely, $G(s)$ is α-SPR.

Note that the largest α was determined under the assumption of maximal velocity and maximal virtual mass. In other cases, significant improvement can be achieved. For example, if

- $v = 10$
- $\tilde{m} = 16000$

then the largest α is 2.2373; if

- $v = 1$
- $\tilde{m} = 32000$

then the largest α is 11.1868.

Consider the lead-lag controller

$$C(s) = K\frac{s + k_n}{s + k_d} \tag{22}$$

with $K > 0, k_n > 0, k_d > 0$. It is easy to verify that $C(s)$ is strictly positive real.

COROLLARY 3.1 *For any lead-lag controller, the closed-loop system with the plant transfer function $G(s)$ is robustly stable.*

From the above discussion, we see that the transfer function $G(s)$ from the front steering angle δ_f to yaw rate r is not only SPR, but also α-SPR for some $\alpha \in [0, \alpha_{max}]$. Suppose $k_n > \alpha, k_d > \alpha$. Then, it is easy to see that $C(s)$ is also α-SPR. Hence, by Theorem 3.1, we have

COROLLARY 3.2 *For any lead-lag controller*

$$C(s) = K\frac{s + k_n}{s + k_d} \tag{23}$$

with $K > 0, k_n > \alpha, k_d > \alpha$, the closed-loop system with the plant transfer function $G(s)$ is robustly α-stable.

In what follows, we will discuss the following problem, see Wang (1994): Given a family of controllers, how can we check whether every controller is α-SPR?

Consider the controller of the following form

$$C(s) = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots\cdots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots\cdots + b_1 s + b_0} \tag{24}$$

with $0 < a_i^- \le a_i \le a_i^+, i = 0, 1, \cdots\cdots, m$ and $0 < b_j^- \le b_j \le b_j^+, j = 0, 1, \cdots\cdots, n$.

For notational simplicity, denote

$$\Gamma = \{C(s) \mid a_i \in [a_i^-, a_i^+], b_j \in [b_j^-, b_j^+]\} \tag{25}$$

$$\Gamma^* = \{C(s) \mid a_i \in \{a_i^-, a_i^+\}, b_j \in \{b_j^-, b_j^+\}\} \tag{26}$$

THEOREM 3.2 *Every controller in Γ is α-SPR if and only if every controller in Γ^* is α-SPR.*

Proof: Necessity: Obvious.

Sufficiency: Since for every controller $C(s)$ in Γ^*, we have

$$\Re C(j\omega - \alpha) > 0, \forall \omega \in R \tag{27}$$

For any $a_i \in [a_i^-, a_i^+], b_j \in \{b_j^-, b_j^+\}$, $C(j\omega - \alpha)$ can be expressed as the convex combination of $\{C(j\omega - \alpha) \mid C(s) \in \Gamma^*\}$. Thus, we have

$$\Re C(j\omega - \alpha) > 0, \forall \omega \in R \tag{28}$$

Moreover, for any $a_i \in [a_i^-, a_i^+], b_j \in [b_j^-, b_j^+]$, since similar arguments lead to the same conclusion. Hence, for every controller $C(s)$ in Γ, we have

$$\Re C(j\omega - \alpha) > 0, \forall \omega \in R \tag{30}$$

This also shows that the set of denominators of Γ evaluated along $j\omega - \alpha$ always excludes zero. By zero exclusion principle, we know that every denominator of Γ is α-stable. Hence, every controller $C(s)$ in Γ is α-SPR. This completes the proof.

From the above results, we have the following conclusion: Given an α-SPR plant and a family Γ of controllers, suppose all the corner controllers in Γ are α-SPR. Then, every controller in Γ α-stabilizes the plant. Namely, the resulting closed-loop system is α-stable. In other words, the closed-loop system not only is stable, but also meets some performance specifications, see Wang and Ackermann (1997).

4. Conclusions

We have shown that the transfer function from the front steering angle to yaw rate is strictly positive real, irrespective of the uncertain mass and uncertain velocity. We have also shown how to determine the positivity margin for this transfer function. Some stabilization results have been obtained. Finally, we have shown how to check the positivity of a controller family.

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