Portfolio optimization – two rules approach

by

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Abstract: The new approach to the portfolio optimization, based on the concept of two-factor utility function, is proposed. The first factor describes the expected average profit, while the second – the worse case profit. Then, two rules enabling one to compose an optimum portfolio are formulated. The first rule determines the level of acceptance for all assets with given risk/return ratio. The second rule enables one to allocate the investment fund among all the accepted assets.

The methodology proposed does not require to specify the individual utility function in an explicit form. It can be used to optimize portfolios composed of equities as well as bond and other securities, using a passive or – active management strategy.

Keywords: portfolio optimization, utility function, investment allocation, risk aversion, expected return, worse case return, optimum investment startegies, portfolio variance

1. Introduction

The existing portfolio theory can be viewed as an impressive collection of different approaches and methodologies, such as e.g.:

- the classical mean-variance theory, originated by H. Markowitz,
- the capital assets pricing models,
- the discounted cash flow and fundamental analysis,
- the arbitrage pricing theory,
- the technical analysis, etc. (see Elton, Gruber, 1994; Levy, Sarnat, 1994).

Each methodology can be applied to a specific class of securities and it requires a specific, sometimes complicated, optimization technique.

On the other hand, many practically oriented investors find the existing theory too abstract and complicated. They would like to have a unified approach, which appeals to individual motivations and risk attitudes, and which is based on simple and understandable rules of portfolio management.

In the present paper an attempt has been done to present a unified approach to the portfolio optimization and management. It is based on the concept of two-factor utility function, described in Kulikowski (1998, 1994). The first factor represents an average profit, which motivates the investor to accept an asset. The second factor represents the impact of risk in the form of "worse case" profit. That factor restrains the investor to accept the asset. Such an approach has an appealing psychological interpretation, which stems from the well known motivation theory of Atkinson (1964). From the formal point of view the two-factor utility (unlike the classical single-factor utility) incorporates the risk factor into the utility structure.

Following such an approach the portfolio optimization can be performed using two simple rules. The first rule, called "the rule of acceptance", says that a risk averse investor accepts in his, or her, portfolio the assets with large return-to-risk ratio only. The level of acceptance depends on the individual risk attitude, such as, for instance, the accepted worse-case frequency of occurrence.

The second rule, called the capital allocation rule, starts with a set of weakly correlated assets which have been accepted (by the first rule). It says how much, out of the total investment fund, should be invested in a particular security, characterized by the given coefficient of assurance. The coefficients of assurance depend on the parameters, characterizing the investors risk-attitude and the security's risk-to-return ratios.

The strategy of capital allocation, based on the second rule, does not depend on the explicit form of individual utility function and even for a large-size portfolio it is computationally simple.

The second part of the paper deals with derivation of coefficients of assurance for different securities. In Section 4 the portfolio composed of equities is analysed. Two main sources of information (historical and anticipated (ex ante)) are discussed. The two management styles: passive, with long planning horizon, and active, with short planning horizon are also analysed. The active strategy is based on the binomial forecasting model. In Section 5 the portfolio composed of bonds is analysed using an active management policy. The result enables one to evaluate the expected excess return by using bonds with given duration and the probabilities of two-states return.

A numerical example, illustrating the optimization of a portfolio, which is composed of risk-free security, stocks and bonds, is also given.

2. Rule of acceptance

One can assume that before the asset is purchased by an investor it is evaluated and compared with other assets.

In the case of equities (stocks) the investor can, in particular, analyse the monetary return of the i-th asset ($i = 1, \ldots, n$):

where

$P_i(t)$ is the price of equity at the t-th period,

$D_i(t)$ is the dividend paid to the investor at t-th period.

It is usually assumed that the non-monetary returns

$$R_i(t) = z_i(t) : P_i(t), \quad \forall t$$

are random, normally distributed, with given expected values $E(R_i)$, denoted by $\bar{R}_i$, and variances, denoted by σ_i^2.

The numerical values of $\bar{R}_i$, σ_i are usually derived using the historical (ex post) data or – anticipated (ex ante) values.

In order to evaluate the possible worse case outcome value, one can introduce the notion of expected profit in the worse period (or day):

$$Y_i = P_i\left[\bar{R}_i - \kappa\sigma_i\right], \tag{1}$$

where κ is the investor's "degree of risk aversion". The notion introduced has a simple economic interpretation. The κP_i can be interpreted as the investor's marginal cost of bearing the risk σ_i. The marginal cost κP_i is a product of the objective factor P_i and the subjective factor κ, representing the value of a worry or "not sleeping well", due to the possible worse case outcome.

If the expected (by the investor) income is $P_i(1 + \bar{R}_i)$ and the expected "worse case" cost is $P_i(1 + \kappa_i\sigma_i)$, the value of (1) is the upper bound for the expected worse case profit.

In the assessment of κ the notion of probability p or – the frequency of worst case occurrence, is also helpful.

Using the relation

$$p\left\{\left|R - \bar{R}\right| > \kappa\sigma\right\} = \frac{2}{\sqrt{2\pi}} \int_\kappa^\infty e^{-\frac{u^2}{2}} du = 1 - erf\left(\frac{\kappa}{\sqrt{2}}\right),$$

one can construct the graph of $\kappa(p)$, which is shown in Fig. 1.

Using that graph one can find for a given frequency p the corresponding value of κ. For example, when the investor accepts the frequency $p = 1/6$ he gets $\kappa \cong 1$. That means that once out of six cases the investor will get the worse case (w.c.) profit amounting to not more than

$$Y = P(\bar{R} - \sigma),$$

where (P) is the price of the risky asset.

At the same time he will also get once in six cases the profit equal (at least) $\bar{R}_i$ and four times out of six – the profit will be contained within the bounds $[P(\bar{R} - \sigma), P(\bar{R} + \sigma)]$.

By increasing (decreasing) κ above (below) the $\kappa = 1$ level one gets lower (higher) frequency of losses (p) but at some time he gets also smaller (larger) value of w.c. profit (Y). When $\kappa \to 0$ ($p \to 1/2$), Y approaches the average
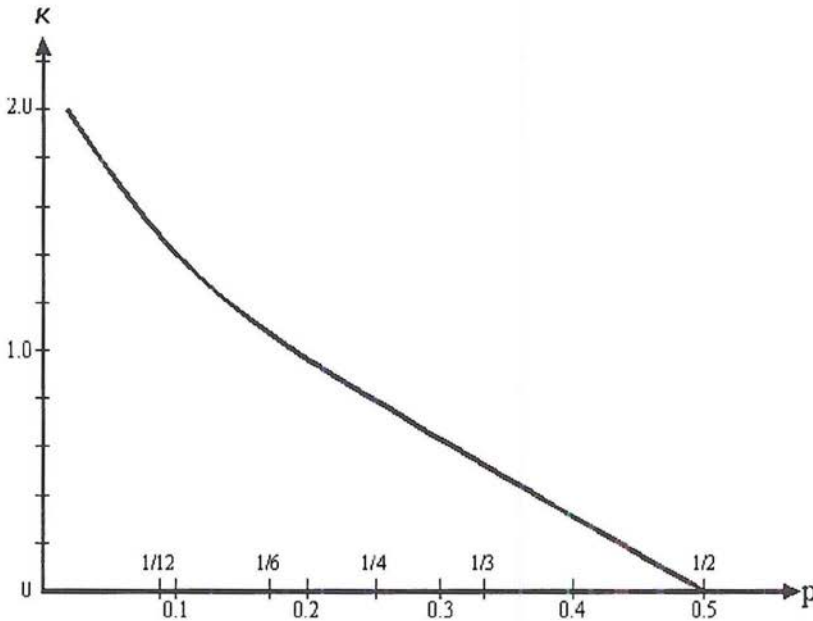
Figure 1. The graph of the relation $\kappa(p)$

Both extreme values of κ seem unacceptable for a risk-averse investor and a positive but finite κ seems more reasonable. The concrete, numerical value of κ may depend, however, on the individual's characteristics, such as the age and wealth level. For example, one can assume that when the value of asset price (P) to the wealth (W) ratio, i.e. P/W, increases the investor becomes more risk averse and κ should increase. Such an assumption requires that κ be an increasing function of P/W. If the asset prices P_i do not differ much one can make an assumption that $P/W = \text{const}$. It should be also noted that the stock splitting processes help to keep $\kappa(P_i/W)$ constant.

In the model of investor behaviour one assumes that the decision to accept the asset depends on two factors:

- the expected profit $P\bar{R}$ (when $P\bar{R}$ is large enough, i.e. $P\bar{R} > PR_F$, where PR_F is the risk-free profit; it stimulates the investor to accept the asset),
- the worse case profit $Y = P(\bar{R} - \kappa\sigma)$ (a small value of $Y/P\bar{R}$ restrains the investor from accepting the asset).

The ratio

$$A = \frac{Y}{P\bar{R}} = 1 - \kappa\frac{\sigma}{\bar{R}},\tag{2}$$

called the coefficient of assurance, characterizes the degree of investor's confi-

also that A increases (up to $A = 1$ level) along with an increase of the return/risk ratio R/σ.

The value of $\kappa\sigma_i$ can be regarded as a risk premium for the risky asset, acquired by an investor. Denoting the risk free return by R_F one can define the minimum required return on the risky asset:

$$\bar{R}_i = R_F + \kappa\sigma_i. \tag{3}$$

The relation (3) can be called "the individual assets pricing model", or – IAPM.

The same asset can be also evaluated by the capital assets pricing model (CAPM) using the relation from Elton, Gruber (1994):

$$\bar{R}_i = R_F + \beta_i (R_m - R_F),$$

where $\beta_i = \sigma_{im}/\sigma_m^2$; R_m, σ_m are the market returns and the standard deviation, respectively.

When one deals with the well diversified portfolio so that $\sigma_i = \beta_i\sigma_m$ (see e.g. Elton, Gruber, 1994) one can compare the risk premium offered by IAPM, i.e. $\kappa\beta_i\sigma_m$, and CAPM, i.e. $\beta_i (R_m - R_F)$.

One can see that κ in IAPM corresponds to the market price of risk $[(R_m - R_F)/\sigma_m]$ in CAPM model. For that reason κ can be also called "the individual price of risk".

It is obvious that when the excess return $\bar{R}_i - R_F$ is bigger then $\kappa\sigma_i$, the asset should be accepted by the risk averse investor. Since $\bar{R}_i - R_F \geq \kappa\sigma_i$ is equivalent to $\bar{R}_i/R_F \geq A^{-1}$ one can formulate the following rule of acceptance:

A risk averse investor evaluating the assets, characterized by $\bar{R}_i/\sigma_i$, accepts to his portfolio all the assets satisfying the rule $\bar{R}_i \geq R_F/A_i , \forall i$.

Since κ depends on the individual characteristics (age, wealth level etc.) the acceptance, as well, depends on these and possibly also additional factors, characterizing the individual utility function (introduced in the next section).

EXAMPLE 2.1 *Consider a risk averse investor, characterized by $\kappa = 1/2$, who is confronted with an acceptance problem of two stocks described by:*
1. $\bar{R}_1 = 15\%$, $\sigma_1 = 17\%$
2. $\bar{R}_2 = 10\%$, $\sigma_2 = 14\%$
and $R_F = 5\%$.

Since $A_1 = 1 - 0.5\frac{17}{15} = 0.43$, $A_2 = 1 - 0.5\frac{14}{10} = 0.30$ one gets $R_F/A_1 = 5/0.43 = 11.6\%$ $R_F/A_2 = 5/0.3 = 16.6\%$ so that the first stock should be accepted and the second – dropped from the portfolio.

It should be observed that by changing κ one can change the acceptance decision. For example, if $\kappa = 1/3$ one gets $A_1 = 0.62$, $A_2 = 0.53$ and $R_F/A_1 =$

3. Capital allocation rule

The rule of acceptance can help one to determine the subset of equities which can be included in the portfolio. It does not, however, allow one to find out how much of the total investment capital should be spend on each concrete asset.

In order to solve the present problem using the proposed methodology based on two factors approach Kulikowski (1998, 1994), assume that for each i-th asset the investor is concerned with

- worse case monetary profit
$$Y_i = P_i \left[\bar{R}_i - \kappa \sigma_i \right] \tag{4}$$
- average monetary profits
$$Z_i = P_i \bar{R}_i x_i, \tag{5}$$

where $x_i=$ the number of assets of i-th category, as well as the coefficients of assurances $A_i = Y_i / P_i \bar{R}_i$, $\forall i$, are known explicitly.

It is also assumed that the investor's utility function $U(Y, Z)$, depending on the sets $Y_i \equiv Y$, $Z_i \equiv Z$ exists and can be written in an additive form:

$$U(Y, Z) = \sum_{i=1}^n U(Y_i, Z_i) \tag{6}$$

Since Y_i, Z_i are expressed in monetary terms it is natural to assume that $U(Y_i, Z_i)$ is a constant return to scale (CRS) function, i.e. it is homogenous degree one. Such a function can be written in an equivalent form

$$U(Y_i, Z_i) = Y_i F \left(\frac{Z_i}{Y_i} \right) = Y_i F \left(\frac{x_i}{A_i} \right) \tag{7}$$

where $F(\cdot)$ is a single-factor utility function.

It is usually assumed that $F(\cdot)$ is increasing and strictly concave, or – that $F(\cdot) > 0$, $F'(\cdot) > 0$, $F''(\cdot) < 0$.

It should be noted that the basic idea underlying the present approach rests on the decision motivation. The decision motivation (Atkinson, 1964) stems from the classical Lewinian-Tolmanian tradition in psychology. According to that approach behaviour is determined by the relative strength of motives: to avoid failure and achieve success. In the case of investment in a risky assets the motive to avoid failure is a large anticipated worse case profit (expressed by $P(\bar{R} - \kappa \sigma)$) while the motive to achieve success – the large anticipated profit $(P\bar{R})$.

The analytic form of the utility function U should possess two properties. First of all it should not generate an additional utility by a change in the monetary units of Z_i and Y_i, e.g. by changing US \$ into cents. For that reason the function $U(Y_i, Z_i)$ is assumed to be homogenous, degree one.

The second property requires that the function $F(\cdot)$ be increasing and strictly concave with respect to the investor's growing wealth. The growth of wealth is

In other words the second property requires that the increase of the utility, resulting from the purchase of an additional asset, have a negative acceleration, i.e. that it decrease along with the wealth level represented by x_i.

Now one can formulate the main portfolio optimization problem. Assume there are n given (i.e. accepted) assets with known Y_i, A_i, and initial prices $P_i, \forall_i$.

The investor requires the total return of the portfolio (Z) to be given. It is necessary to find the optimum numbers $x_i \triangleq \hat{x}_i, \forall_i$, of assets, such that:

$$U(\hat{x}) = \max_{x \in \Omega} \sum_{i=1}^{n} Y_i F\left(\frac{x_i}{A_i}\right), \tag{8}$$

where

$$\Omega = \left\{ x_i : \sum_{i=1}^{n} x_i P_i \bar{R}_i = Z, \quad x_i \geq 0, \quad \forall_i \right\}$$

When $F(\cdot)$ is a strictly concave function and $Y_i > 0$, $\forall_i$ one can show, Kulikowski (1998, 1994), that a unique strategy:

$$\hat{x}_i = \frac{A_i}{Y} Z, \quad \forall_i \tag{9}$$

where

$$A_i = 1 - \kappa \frac{\sigma_i}{\bar{R}_i},$$

exists, and

$$U(\hat{x}) = Y F\left(\frac{Z}{Y}\right), \quad Y = \sum_{i=1}^{n} Y_i \tag{10}$$

Multiplying both sides of (9) by P_i and summing up one gets the total capital invested X:

$$X = \sum_{i=1}^{n} P_i \hat{x}_i = \frac{Z}{Y} \sum_{i=1}^{n} P_i A_i = \frac{Z}{Y} P, \quad P \triangleq \sum_{i=1}^{n} P_i A_i$$

Then $Z/Y = X/P$, and the optimum share of capital X invested in i-th asset becomes

$$u_i = \frac{P_i A_i}{X} \cdot \frac{Z}{Y} = \frac{P_i A_i}{P}.$$

Obviously, $u_i > 0$, $\forall_i$, and $\sum_{i}^{n} u_i = 1$.

The following Capital Allocation Rule, based on formula (9), can be formu-

The optimum share of capital invested in an asset $(P_i\hat{x}_i/X)$ should be proportional to the product of asset price P_i and its coefficient of assurance A_i.

It is interesting to observe that the optimum strategy (9) does not depend on the individual, investor's utility function $F(\cdot)$. Since generally the exact analytic form of $F(\cdot)$ is unknown, that property is an obvious advantage of the present approach.

Being universal, the optimum strategy can, however, take into account the individual risk-averse attitude (represented by the parameters κ), as well as the values of coefficients of assurance:

$$A_i = 1 - \kappa\,\sigma_i/\bar{R}_i, \qquad \forall i.$$

Using the utility function (7) one can analise also the impact of investor's utility on the assets acceptance rule. According to that rule, based on IAPM model, an asset with the return

$$R \geq R_F/A, \tag{11}$$

should be accepted.

Since the return of a single $(x = 1)$ risky asset R having price P produces the utility

$$PYF(1/A) = PRAF(1/A)$$

and the risk-free asset produces $PR_F F(1)$, it is possible to find such a value of R which makes the utility $RAF(1/A)$ certainty equivalent to the risk-free asset utility, i.e.

$$RAF(1/A) = R_F F(1).$$

Consequently when

$$R \geq \frac{R_F}{A} \cdot \frac{F(1)}{F(1/A)} \tag{12}$$

the utility of a risky asset is bigger than that of a risk-free asset and it should be accepted by the risk averse investor.

Since F is generally unknown, (12) is inconvenient in applications. The function $F(\cdot)$ can be, however, approximated by

$$F(\cdot) = \text{const}(\cdot)^{\beta},$$

where β is a given number $(0 \leq \beta \leq 1)$.

Then, (12) can be written

$$R \geq R_F A^{\beta-1}.$$

One can observe that in the present case the elasticity

$$\frac{dU}{\quad} \cdot \frac{dA}{\quad} = 1 - \beta$$

attains maximum for $\beta = 0$ (i.e. the most sensitive or risk averse investor) and (12) becomes then equivalent to (11).

Using the acceptance rule it is also possible to take into account the individual anticipations regarding "the state of the world", i.e. stock indices, rates of return risk indices (e.g. β) etc. In the next sections one assumes that anticipations can be expressed by the probabilities attached to the possible world states or scenarios.

Such an approach is used, in particular, by the fundamental and technical analysts who reject the efficient market hypothesis and look for arbitrage opportunities.

It should be observed that so far one did not consider the portfolio composed of correlated assets explicitly.

If one wants to apply the present methodology to the correlated assets it is necessary to express the risks (σ_i) of the assets by parameters which characterize correlations.

As shown in Kulikowski (1998), the CAPM coefficients

$$\beta_i = \frac{cov\,(x_i, x_m)}{\sigma_m^2}, \quad \forall i$$

σ_m^2 – variance of market portfolio; can be used for that purpose.

Using β_i the risk (σ_i) can be decomposed (see Levy, Sarnat, 1994) into the nonsystematic (diversifiable) component σ_i^{NS} and the nondiversifiable (systematic) component $\beta_i\sigma_m$, i.e.

$$\sigma_i = \sigma_i^{NS} + \beta_i\sigma_m, \quad \forall i.$$

Consequently the coefficients of assurance become

$$A_i = 1 - \kappa\,\frac{\sigma_i^{NS} + \beta_i\sigma_m}{\bar{R}_i}, \quad \forall i.$$

Suppose the investor owns the market (diversified) portfolio with $A_m = 1 - \kappa\,\frac{\sigma_m}{\bar{R}_m}$. He, or she, considers the acceptance of correlated assets having equal parameters, except β_i. The best acceptable asset in such a case is that with the smallest β_i (large A_i), e.g. the asset which is negatively correlated with the market. Such an asset will also get preferences when one uses the Capital Allocation Rule.

4. Portfolio composed of equities

The derivation of coefficients of assurance (A_i) in the case of equities requires the knowledge of $\bar{R}_i$, σ_i for each individual asset.

There are two main sources of data:

1. historical (ex post) based on time series analysis, which yields $\bar{R}_{hi}$, σ_{hi},

2. anticipated (ex ante) based on the projected scenarios, which yield $\bar{R}_{ai}$, σ_{ai}, $\forall_i$.

The derivation of $\bar{R}_{ai}$, σ_{ai} is usually more difficult. It requires the application of the so called fundamental analysis, which is based on the discounting of future cash flows.

In a simple model of that category one analyses the single scenario only. It is assumed:

1. that dividends the stock holders expect to receive will grow at the same rate g up to infinity,
2. that the firm will earn a stable return (r) on new investments,
3. that the firm's current "stock price (P) to earnings (E) ratio" is known.

Then, deriving the present value of dividends one arrives at the well known formula, Copeland, Koller, Murrin, (1990):

$$P/E = \frac{1 - g/r}{k - g}, \tag{13}$$

where k is the "internal rate of return". The internal rate of return can be regarded as the anticipated rate of return, i.e. $k = R_a$. Then one gets

$$R_a = \frac{E}{P}\,(1 - g/r) + g. \tag{14}$$

The formula (13) employes a simple firm model. A more accurate model can be based on the so called "firm valuation model" (see e.g. Copeland, Koller, Murrin, 1990):

$$V = V_1^T + V_{T+1}^\infty,$$

where

$$V_1^T = \sum_{\tau=1}^{T} \frac{s(\tau)}{(1 + k)^\tau},$$

is the discounted cash flow $(s(t))$ value within $[0, T]$, while

$$V_{T+1}^\infty = \frac{N_{T+1}}{(1 + k)^T} \frac{1 - g/r}{k - g},$$

$N_{T+1} = $ net operational profit less adjusted taxes at $t = T + 1$.

The anticipated return R_a becomes

$$R_a = \frac{V(q) - V(0)}{V(0)} : n, \tag{15}$$

where
$V(q) = V_q^T$, $q = $ given planning period $= 1$ year,
$n = $ number of shares outstanding

Using the single scenario model one is unable to estimate the anticipated standard deviation (σ_a).

It is, however, possible to employ here a multi-scenario model. Assume for example that there are given J anticipated g_j, r_j parameters, each with probability of occurrence p_j, with $\sum_j p_j = 1$.

Using the relation (14) one can derive the anticipated values of k_j:

$$k_j = \frac{E}{P}\left(1 - g_j/r_j\right) + g_j, \quad j = 1, \ldots, J \tag{16}$$

The expected (ex ante) rate of return becomes

$$\bar{R}_a = \sum_{j=1}^{J} p_j k_j$$

The ex ante standard deviation (σ_a) can be also derived

$$\sigma_a = \left\{ \sum_{j=1}^{J} p_j \left[\bar{R}_a - k_j\right]^2 \right\}^{\frac{1}{2}}. \tag{17}$$

It can be observed that using the historical and ex ante analysis one arrives at two sets A_{hi}, A_{ai}, of assurance coefficients:

$$A_{hi} = 1 - \kappa\,\frac{\sigma_{hi}}{\bar{R}_{hi}}, \quad \forall i, \tag{18}$$

$$A_{ai} = 1 - \kappa\,\frac{\sigma_{ai}}{\bar{R}_{ai}}, \quad \forall i. \tag{19}$$

In order to derive the resulting A_i coefficients, using the historical A_{hi} and ex ante A_{ai} information one can introduce a weight parameter (w) and write

$$A_i = wA_{hi} + (1-w)A_{ai}, \quad 0 \le w \le 1, \tag{20}$$

In the case of unreliable historical information (when there is e.g. a short observation time) one can assume $w \approx 0$. If the information on the discounted cash flow is unreliable one should assume $w \approx 1$.

Using the discounted cash flow models one assumes a long planning horizon, i.e. one is interested in a long term capital returns. In such a case one is not inclined to change the composition of portfolio much in time and his portfolio management style is called passive.

An active portfolio manager, on the contrary, will try to use all the available information, within a short planning horizon q, to find assets which offer excess returns and he will make profit by a process of arbitrage.

To exercise the active management a class of short-term returns forecasting models is needed

The simple, binomial, single-period, forecasting model can be characterized by two scenario-states ($u =$ up, $d =$ down):

$$R_u = \bar{R}_h + \sigma_h \text{ with probability } p_1,$$

$$R_d = \bar{R}_h - \sigma_h \text{ with probability } p_2 = 1 - p_1.$$

In the present model one believes that in the given period (duration q) the given historical return R_h will change up or down, with anticipated probabilities (p_1 and $p_2 = 1 - p_2$), by the historical deviation σ_h.

The ex ante anticipated return becomes

$$
\begin{aligned}
\bar{R}_a &= [P(1 + R_u p_1 + R_d p_2) - P] \frac{1}{P} \\
&= p_1 R_u + p_2 R_d = \bar{R}_h + \sigma_h (p_1 - p_2),
\end{aligned}
\tag{21}
$$

where P is the initial price of the asset.

The ex ante variance can be also derived

$$\sigma_a^2 = p_1 \left[\bar{R}_a - \bar{R}_h - \sigma_h\right]^2 + p_2 \left[\bar{R}_a - \bar{R}_h + \sigma_h\right]^2 = 4 p_1 p_2 \sigma_h^2 \tag{22}$$

One can observe that for $p_1 = 1$ ($p_2 = 1$) one gets $\bar{R}_a = \bar{R}_h + \sigma_h$, ($\bar{R}_a = \bar{R}_h - \sigma_h$) and $\sigma_a = 0$. When $p_1 = p_2 = 1/2$ one gets also $\bar{R}_a = \bar{R}_h$, $\sigma_a = \sigma_h$.

The notion of an "excess return", denoted by R_e, can be attached to the returns $\bar{R}_a$ and $\bar{R}_h$:

$$R_e = \bar{R}_a - \bar{R}_h = (p_1 - p_2) \sigma_h. \tag{23}$$

One can say that the excess return is the additional gain the investor can achieve if he has an "excess information", which enables him to believe that $p_1 > p_2$.

The excess information can be used effectively when the investor gets the information prior to the rest of investors on the capital market. Since $p_1 - p_2$ is subjective, the value of R_e is as well, subjective. It can be, however, supported by an analysis of fundamental factors, economic trends etc.

It is interesting to analyse the effect of anticipation on the acceptance policy.

According to the rule of acceptance the asset with given $\bar{R}_a$, σ_a should be accepted when

$$\bar{R}_a \geq R_F + \kappa \sigma_a.$$

Using (21), (22) one can formulate the rule in terms of R_h and σ_h:

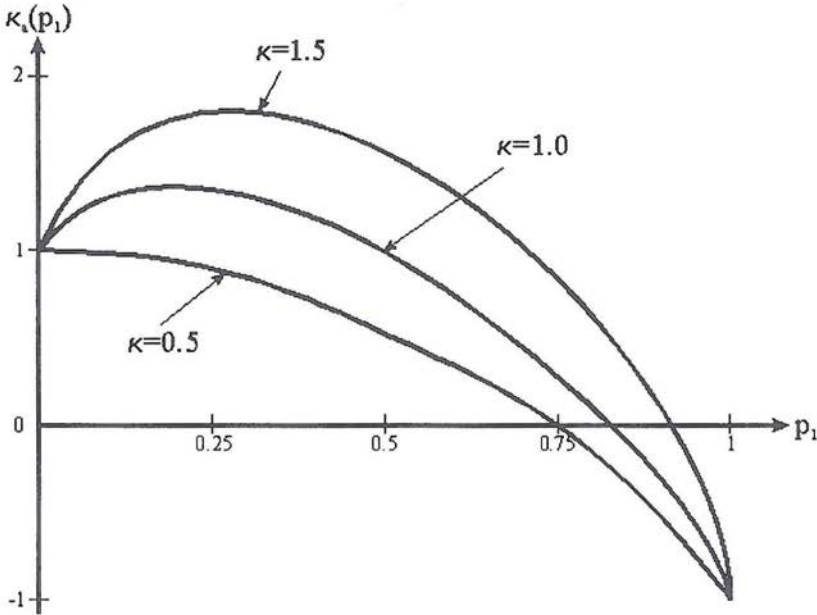$$\bar{R}_h \geq R_F + \kappa_a(p_1)\sigma_h, \tag{24}$$

where

Figure 2. The graph of $\kappa_a(p_1)$

can be called "the anticipated risk price".

In Fig. 2 the relation $\kappa_a(p_1)$, for different κ, is exhibited. One can observe that for the increasing assurance (i.e. $p_1 \to 1$, and so $\bar{R}_a \to \bar{R}_h + \sigma_h$) the anticipated risk-price $\kappa_a(p_1)$ goes down and in the limit it attains $\kappa_a(1) = -1$.

At the same time the anticipated coefficient of assurance

$$A_a = 1 - \kappa_a(1)\frac{\sigma_h}{\bar{R}_h} = 1 + \sigma_h/\bar{R}_h$$

becomes bigger than the risk-free assurance $(A_F = 1)$.

One should observe that when the active management, based on the binomial model, is used, it is necessary to readjust portfolio composition according to the anticipations expressed in terms of p_1/p_2 ratio and assurances (20). When the anticipated A_i coefficients are derived for each i-th asset one can use the rule of acceptance to determine all the assets which should be kept in the portfolio and – afterwards – he can use the rule of capital allocation, to find the best composition of assets kept in the portfolio.

Since portfolio readjustment involves the transaction costs, the readjustment should be carried out only in the case when one has definite and strong

5. Portfolio composed of bonds

Bonds, unlike equities, have a fixed maturity time T. When one keeps high quality bond up to maturity it is a risk free security, with the rate of return equal R_F. This is, however, a passive portfolio managing strategy.

An active bond management strategy requires that one sets the planning horizon (q) and chooses the bond duration (D) according to the anticipated rate of return (R_a) or – to the so called, realized rate of return (R_r).

When formulating the bond rate of return it is necessary to start with the relation describing the present value of a bond (V) as the sum of discounted cash flow s_t, received annually:

$$V = \sum_{t=1}^{T} s_t(1 + R)^{-t}.$$

With V and s_t fixed the resulting R is the internal rate of return called "yield to maturity" (YTM).

The duration D of the bond is the elasticity of price (value), with respect to $(1 + R)^{-1}$, Bierwag (1987):

$$D = - \frac{dV}{dR} \cdot \frac{1 + R}{V}. \tag{25}$$

When one buys the bond he, or she, is promised to receive the return $R = R_p$. Assume that instantly after the investment $V(R_p)$ is made, the annual yield to maturity changes to R_a and the value of the investment changes to $V(R_a)$. If no further changes occur in the planning interval q the value $V(R_a)$ accumulates, after q periods, to $(1 + R_a)^q V(R_a)$. Then, the initial investment $V(R_p)$ must accumulate, during q periods, to $(1+R_r)^q V(R_p)$, which is equal $(1+R_a)^q V(R_a)$:

$$(1 + R_r)^q V(R_p) = (1 + R_a)^q V(R_a). \tag{26}$$

The value of R_r, defined by (26), i.e.:

$$R_r = (1 + R_a) \left[\frac{V(R_a)}{V(R_p)} \right]^{1/q} - 1, \tag{27}$$

is called the realized rate of return. It is the rate at which the initial investment R_p must grow in order to be equal to the realized value of the investment found after q periods.

As shown in Bierwag (1987) the linear approximation of the relation $R_r(R_a)$, described by (27), can be used:

$$R_r = R_p + (1 - D/q)(R_a - R_p). \tag{28}$$

After computing expected values of both sides of (28) one gets

It can be observed that the excess return

$$\bar{R}_e = \bar{R}_r - \bar{R}_p = (1 - D/q)(\bar{R}_a - \bar{R}_p) \tag{30}$$

is positive when sign $(1 - D/q) = \text{sign}\,(\bar{R}_a - \bar{R}_p)$, i.e. when the anticipated return $\bar{R}_a > \bar{R}_p$ ($\bar{R}_a < \bar{R}_p$) the duration D should be less (more) the planning interval q. In other words the investor should exchange long for short term bonds ("go short") when he believes the return R_a will increase above $\bar{R}_p$. He, or she, should "go long", when $\bar{R}_a < \bar{R}_p$, by exchanging short for long term bonds. In order to use these rules effectively it is helpful to employ the binomial, single-period, forecasting model, described already in Section 4.

Assume that the anticipated return attains two states:

$$R_u = \bar{R}_p + \sigma_p \text{ with probability } p_1,$$

$$R_d = \bar{R}_p - \sigma_p \text{ with probability } p_2 = 1 - p_1.$$

The expected value $\bar{R}_a$, based on the binomial model, becomes

$$\bar{R}_a = p_1 \left[\bar{R}_p + \sigma_p\right] + p_2 \left[\bar{R}_p - \sigma_p\right] = \bar{R}_p + (p_1 - p_2)\sigma_p \tag{31}$$

The variance σ_a^2 of the anticipated return can be also derived

$$\sigma_a^2 = p_1 \left[\bar{R}_a - R_u\right]^2 + p_2 \left[\bar{R}_a - R_d\right]^2 = 4\,p_1 p_2 \sigma_p^2 \tag{32}$$

Then, using (31), (32) one gets by (29):

$$\bar{R}_r = \bar{R}_p + (1 - D/q)\,(p_1 - p_2)\,\sigma_p, \tag{33}$$

and

$$\sigma_r = |1 - D/q|\,\sigma_a = 2\,\sqrt{p_1 p_2}\,|1 - D/q|\,\sigma_p. \tag{34}$$

The excess return $\bar{R}_e$ can be written in the following form

$$\bar{R}_e = \begin{cases} (1 - D/q)(p_1 - p_2)\sigma_p, & \text{for } D < q,\ p_1 > p_2, \\ 0 & \text{for } D = q, \\ (1 - D/q)(p_1 - p_2)\sigma_p, & \text{for } D > q,\ p_1 < p_2. \end{cases} \tag{35}$$

According to (35) an excess return can be earned when one is "almost sure" $\bar{R}_a$ will grow ($p_1/p_2 > 1$), and it pays to go short, or – in the opposite case ($p_1/p_2 < 1$) – to go long. Observing the current market rates of return the investor can change the bond portfolio duration when the market moves not according to his expectation. Assume that, e.g., the investor believes ($p_1 > p_2$) that R_a will increase, so he goes short. He observes, however, that the market, contrary to his expectation, yields a decreasing $\bar{R}_a$. In such a case, in order to prevent losses, the investor may reverse his strategy. He can, e.g., set $D = q$ and get an immunized portfolio, or – when he comes to a conclusion that $\bar{R}_a$

term bonds. Such a portfolio management style is known as the "contingent immunization" Bierwag (1987).

According to the proposed methodology, in order to conduct an active management of portfolio composed of bonds, one has to watch the values of coefficients of assurance:

$$A_i = 1 - \kappa \, \frac{\sigma_{r_i}}{\bar{R}_{r_i}}, \quad \forall i, \tag{36}$$

where

$$\bar{R}_{r_i} = \bar{R}_{p_i} + (1 - D_i/q)(p_1 - p_2)\,\sigma_{p_i}$$

$$\sigma_{r_i} = 2\sqrt{p_1 p_2}\,|1 - D_i/q|\,\sigma_{p_i}$$

When the bonds are default free, so that one can assume $\frac{\bar{R}_{p_i}}{\sigma_{p_i}} = c = \text{const}$, one gets

$$\frac{\sigma_{ri}}{\bar{R}_{ri}} = \frac{2\sqrt{p_1 p_2}\left|1 - \frac{D_i}{q}\right|}{C + (p_1 - p_2)\left(1 - \frac{D_i}{q}\right)}, \quad \forall i. \tag{37}$$

The assurances depend in such a case on the bond durations D_i only. The acceptance rule for bonds becomes

$$\bar{R}_p \ge R_F + \kappa_a(p_1)\sigma_p, \tag{38}$$

where

$$\kappa_a(p_1) = 2\,|1 - D/q|\,\sqrt{p_1(1 - p_1)}\,\kappa - (1 - D/q)(2p_1 - 1).$$

Observe that for $D = q$ the bond-portfolio is immunized, and so the anticipated risk price $\kappa_a(p_1) = 0$, for each value of p_1.

The allocation of the initial capital X among bonds with different A_i can be derived by the capital allocation rule (9). That rule can be also used in the case when portfolio is composed of bonds, as well as the equities.

EXAMPLE 5.1 *Consider an actively minded investor having initial capital X who is analysing a purchase of the portfolio consisting of three noncorrelated categories of securities (which he wants to sell in one year $(q = 1)$):*

1. *the risk-free security (e.g. the government bonds maturing in $q = 1$) having the price P_F and yielding the return $R_F = 10\%$,*
2. *the diversified portfolio[1] of stocks with $\bar{R}_h = 25\%$, $\sigma_h = 31\%$, and price $P_s = 0.25\,P_F$; the anticipated excess return is characterized by $p_1 = 0.6$, $p_2 = 0.4$,*

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[1] Using a diversified portfolio of stocks enables one to reduce the unsystematic risk compo-

3. the high quality bonds with $R_p = 17\%$, $\sigma_p = 5\%$, and price $P_b = 1.4\,P_F$; the anticipated excess return for $D/q = 2$ is characterized by $p_1 = 0.25$, $p_2 = 0.75$.

Using the rule of acceptance the investor checks whether the securities are acceptable. He believes that $\kappa = 0.5$.

Using, for the stocks, the formula (24) he gets $\kappa_a = 2\sqrt{p_1(1 - p_1)}\kappa - 2p_1 + 1 = 0.29$ and

$$\bar{R}_h \geq R_F + \kappa_a \sigma_h = 18.9\%.$$

Since $\bar{R}_h = 25\%$, the stock can be accepted.

In a similar way for the bonds one obtains by (38):

$$\kappa x a = 2\,|1 - D/q|\,\sqrt{p_1(1 - p_1)}\kappa - (1 - D/q)(2p_1 - 1) = -0.067$$

$$\bar{R}_p > R_F + \kappa_a \sigma_p = 9.7\%,$$

So the bonds can be also accepted.

Now one can derive the coefficients of assurance:
1. for risk-free security $A_F = 1$,
2. for stocks $A_s = 1 - \kappa \sigma_a / \bar{R}_a = 0.513$,
3. for bonds $A_b = 1 - \kappa \sigma_r / \bar{R}_r = 0.89$.

Using the capital allocation rule (9) one gets the following allocation of capital:
1. for risk-free security
$$P_F x_F / X = \frac{P_F A_f}{P} = \frac{1}{1 + 0.513\,P_s/P_F + 0.89\,P_b/P_F}$$
$$= \frac{1}{1 + 0.513 \cdot 0.25 + 0.89 \cdot 1.4} = 0.421.$$

2. for stocks
$$P_s x_s / X = \frac{A_s P_s}{P} = 0.421 \cdot \frac{A_s P_s}{P_F} = 0.421 \cdot 0.513 \cdot 0.25 = 0.046.$$

3. for bonds
$$P_b x_b / X = \frac{A_b P_b}{P} = 0.421 \cdot \frac{A_b P_b}{P_F} = 0.421 \cdot 0.89 \cdot 1.4 = 0.526.$$

When the numerical values of X, P_F, P_s. P_b are specified, one can easily find the numbers of securities of each category (x_F, x_s, x_b).

It should be noted that when (due to the passage to time) the investor gets a new information which changes the p_1/p_2 ratio he (or she) can readjust the portfolio.

The readjustment is, of course, possible in the case when the transaction costs, connected with selling and buying of new securities, are not too large.

It should be also noted that for the negative A_s the values of $P_s x_s / X$ become negative. In such a case the investor, instead of buying the stock, should rather engage in the so called "short selling" process.

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