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Optimal location of the actuator for the pointwise stabilization at high frequencies of a Bernoulli–Euler beam

by

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Abstract: We consider the optimization of the actuator problem for a Bernoulli-Euler beam. By using Riesz basis theory, we show, at high frequencies, that the optimal location of the actuator is the middle of the beam.

Keywords: Bernouli–Euler beam, stabilization, high frequencies.

1. Introduction

In this paper we study the optimal location of the actuator for the pointwise stabilization of Bernoulli–Euler beam modelling the vibrations of a beam with pointwise damping. More precisely we consider the following initial and boundary value problem:

$$\frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial^4 u}{\partial x^4}(x,t) + \frac{\partial u}{\partial t}(\xi,t)\delta_\xi = 0,\ 0 < x < \pi,\ t > 0, \tag{1.1}$$

$$u(0,t) = \frac{\partial^2 u}{\partial x^2}(0,t) = 0,\ t > 0, \tag{1.2}$$

$$\frac{\partial u}{\partial x}(\pi,t) = \frac{\partial^3 u}{\partial x^3}(\pi,t) = 0,\ t > 0, \tag{1.3}$$

$$u(x,0) = u^0(x),\ \frac{\partial u}{\partial t}(x,0) = u^1(x),\ 0 < x < \pi, \tag{1.4}$$

where δ_ξ is the Dirac mass concentrated at the point $\xi \in (0,\pi)$. Simple calculations show that (1.1) is equivalent to the equations modelling the vibrations of two beams with a dissipative joint.

The problem of finding the optimal decay rate for systems with distributed interior damping is difficult and in general has not found a complete answer. We refer to S. Cox and E. Zuazua (1994, 1995), P. Freitas (1999) and to references therein. The novelty brought in by this paper is that we give, at high frequencies, the optimal location of the actuator. More precisely, we show that the fastest exponential decay rate is obtained if the actuator is located the middle of the beam. A similar problem for a string was studied in K. Ammari, A. Henrot and M. Tucsnak (2000). One of the main ingredients of the proof is a result showing that the generalized eigenfunctions of the associated dissipative operator form a Riesz basis in the energy space.

The paper is organized as follows. In the next section we give precise statements of the main results. Section 3 contains some technical results needed in the following sections. In Section 4 we give the proof of the main result. The last section is devoted to a remark for optimizing the location of the actuator in the case of low frequencies.

2. Statement of the main result

If u is a solution of (1.1)–(1.4) we define the energy of u at instant t by

$$E(u(t)) = \frac{1}{2}\int_0^\pi \left(\left|\frac{\partial u}{\partial t}(x,t)\right|^2 + \left|\frac{\partial^2 u}{\partial x^2}(x,t)\right|^2 \right) dx. \tag{2.1}$$

Simple formal calculations show that a sufficiently smooth solution of (1.1)–(1.4) satisfies the energy estimate

$$E(u(0)) - E(u(t)) = \int_0^t \left|\frac{\partial u}{\partial t}(\xi,s)\right|^2 ds, \quad \forall\, t \geq 0. \tag{2.2}$$

We check that equations (1.1)–(1.4) are well posed in the space $V \times L^2(0,\pi)$ where

$$V = \left\{ \phi \in H^2(0,\pi) \middle| \frac{d\phi}{dx}(\pi) = \phi(0) = 0 \right\}.$$

Let n_0 be fixed and sufficiently large. Introduce the notation

$$\mathcal{X}_{n_0} = \left\{ (u,v) = \sum_{n\in\mathbb{N}} a_n F_{\lambda_n} \in V \times L^2(0,\pi),\ a_n \in l^2,\ a_n = 0,\ \forall\, n \leq n_0 \right\},$$

where F_{λ_n} is defined by (3.6).

The uniform stability result is given in the proposition below. This result was proved in K. Ammari and M. Tucsnak (2000) and R. Rebarber (1995).

PROPOSITION 2.1 *The system described by* (1.1)–(1.4) *is exponentially stable in* $V \times L^2(0,\pi)$ *if and only if* $\frac{\xi}{\pi}$ *is a rational number with coprime factorization*

$$\frac{\xi}{\pi} = \frac{p}{q},\ \textit{where } p \textit{ is odd.} \tag{2.3}$$

In order to state the result on the optimal location of the actuator, we define the decay rate, as function of ξ, as

$$\begin{aligned} &\omega_1(\xi) = \inf\{\omega|\text{ there exists } C = C(\omega) > 0 \text{ such that} \\ &E(u(t)) \leq C(\omega)e^{2\omega t}E(u(0)), \\ &\text{for every solution of (1.1)–(1.4) with initial data in } V \times L^2(0,\pi)\}, \quad (2.4) \\ &\omega_2(\xi) = \inf\{\omega|\text{ there exists } C = C(\omega) > 0 \text{ such that} \\ &E(u(t)) \leq C(\omega)e^{2\omega t}E(u(0)),] \\ &\text{for every solution of (1.1)–(1.4) with initial data in } \mathcal{X}_{n_0}\}, \quad (2.5) \end{aligned}$$

where $E(u(t))$ is defined by (2.1). Our main result on the optimal location of the actuator is

THEOREM 2.2 *1. If n_0 is sufficiently large, then the inequality $\omega_2(\xi) \geq \omega_2(\frac{\pi}{2}) = -\frac{1}{2}$ holds true for any $\xi \in (0, \pi)$. In other words, for n_0 sufficiently large, the fastest decay rate of the solutions of (1.1)–(1.4) with initial data in $\mathcal{X}_{n_0}$ is obtained if the actuator is located in the middle of the beam.*

2. The inequality $\omega_1(\xi) \geq -\frac{1}{2}$ holds true for any $\xi \in (0, \pi)$.

3. Some technical results

Take the following rotations

$$Y = [H^3(0,\pi) \cap H^4(0,\xi) \cap H^4(\xi,\pi)] \times H^2(0,\pi), \quad (3.1)$$

$$\mathcal{D}(A) = \left\{(u,v) \in Y,\ u(0) = v(0) = 0,\ \frac{du}{dx}(\pi) = \frac{dv}{dx}(\pi) = 0,\right.$$
$$\left.\frac{d^2u}{dx^2}(0) = \frac{d^3u}{dx^3}(\pi) = 0,\ \frac{d^3u}{dx^3}(\xi+) - \frac{d^3u}{dx^3}(\xi-) = -v(\xi)\right\}. \quad (3.2)$$

Consider the unbounded linear operators:

$$A : \mathcal{D}(A) \to V \times L^2(0,\pi),\ A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\dfrac{d^4u}{dx^4} - v(\xi)\delta_\xi \end{pmatrix} \quad (3.3)$$

where $\mathcal{D}(A)$ is defined in (3.2),
and

$$A_c : \mathcal{D}(A_c) \to V \times L^2(0,\pi),\ A_c\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\dfrac{d^4u}{dx^4} \end{pmatrix} \quad (3.4)$$

where

$$\mathcal{D}(A_c) = \left\{(u,v) \in H^4(0,\pi) \times H^2(0,\pi),\ u(0) = v(0) = 0,\right.$$
$$\left.\frac{du}{dx}(\pi) = \frac{dv}{dx}(\pi) = 0,\ \frac{d^2u}{dx^2}(0) = \frac{d^3u}{dx^3}(\pi) = 0\right\}.$$

First we give the following characterization of the eigenvalues and eigenfunctions of A. This result is not stated nor proved in K. Ammari and M. Tucsnak (2000).

Lemma 3.1 *A complex number $\lambda = i\tau^2$, $\tau = re^{i\theta}$, with $r > 0$, $\theta \in [-\frac{\pi}{2}, 0]$, is an eigenvalue of A if and only if*

$$\begin{aligned} &-2\tau ch(\tau\pi)\cos(\tau\pi) - i[-ch[\tau(\xi-\pi)]sh(\tau\xi)\cos(\tau\pi) \\ &+ ch(\tau\pi)\cos[\tau(\pi-\xi)]\sin(\tau\xi)] = 0. \end{aligned} \tag{3.5}$$

Moreover, the corresponding eigenfunction F_λ is given by

$$F_\lambda(x) = \begin{pmatrix} \frac{1}{\lambda}\phi_\lambda(x) \\ \phi_\lambda(x) \end{pmatrix}, \tag{3.6}$$

where

$$\phi_\lambda(x) = \begin{cases} -ch[\tau(\pi-\xi)]\cos(\tau\pi)sh(\tau x) \\ +ch(\tau\pi)\cos[\tau(\pi-\xi)]\sin(\tau x), & 0 \le x \le \xi, \\ -sh(\tau\xi)\cos(\tau\pi)ch[\tau(\pi-x)] + \\ ch(\tau\pi)\sin(\tau\xi)\cos[\tau(\pi-x)], & \xi \le x \le \pi. \end{cases} \tag{3.7}$$

The following two propositions concern the spectrum of the operator A defined in (3.3).

Proposition 3.2 *There is a family of eigenvalues $\lambda_n = i\ \tau_n^2$ of A for all sufficiently large positive integer n, satisfying the following asymptotic expression*

$$Re\lambda_n = -\sin^2\left(\frac{2n-1}{2}\xi\right) + O\left(\frac{1}{n}\right).$$

Before stating the second proposition we recall that a Riesz basis in a Hilbert space is, by definition, isomorphic to an orthonormal basis*.

Proposition 3.3 *For an arbitrary $\xi \in (0,\pi)$, the generalized eigenfunctions of A form a Riesz basis in $V \times L^2(0,\pi)$.*

Proof of Proposition 3.2. It is easily verified that A_c is a skew-adjoint operator in $V \times L^2(0,\pi)$ with compact resolvent. We see that

$$\sigma(A_c) = \left\{ i\left(\frac{2n-1}{2}\right)^2, -i\left(\frac{2n-1}{2}\right)^2 \right\}.$$

We write a characteristic equation (3.5), in a small neighborhood of $\frac{2n-1}{2}$

$$\cos(\tau\pi) = \frac{i}{2\tau}\left[\frac{1}{2}\cos(\tau\pi) - \cos[\tau(\pi-\xi)]\sin(\tau\xi)\right] + O(e^{-\pi Re\tau}). \tag{3.8}$$

* Let $(u_n)_{n\ge 0}$ be an orthonormal basis in the Hilbert space H, and let $(v_n)_{n\ge 0} \subset H$. If there exists an isomorphism T from H onto H such that $T(u_n) = v_n$, $\forall n \ge 0$, then $(v_n)_{n\ge 0}$ is a Riesz basis in H.

Applying Rouche's Theorem, we obtain that

$$\tau_n = \frac{2n-1}{2} + O\left(\frac{1}{n}\right) \tag{3.9}$$

is a solution of (3.8).
Substituting (3.9) into (3.8), we find

$$O\left(\frac{1}{n}\right) = \frac{i}{2n-1}\sin^2\left(\frac{2n-1}{2}\,\xi\right) + O\left(\frac{1}{n^2}\right)$$

and so

$$\lambda_n = i\tau_n^2 = i\left(\frac{2n-1}{2}\right)^2 - \sin^2\left(\frac{2n-1}{2}\,\xi\right) + O\left(\frac{1}{n}\right). \tag{3.10}$$

This is achieves the proof. ■

Before giving the proof of Proposition 3.3, we need a technical lemma on Riesz basis generation for discrete operators in general Hilbert spaces. This lemma was proved in (B. Z. Guo and K. Y. Chan, 2001, Theorem 2] (see e.g., B. Rao, 1997).

LEMMA 3.4 *Let $\{\widetilde{\Phi}_{n,c}\}_0^\infty$ be a Riesz basis in $V \times L^2(0,\pi)$. If there are an $N \geq 0$ and an w-linearly independent*[†] *subset of generalized eigenfunctions $\{\widetilde{\Psi}_n\}_{N+1}^\infty$ of A corresponding to $\{\lambda_n\}_{N+1}^\infty$, where λ_n is as in (3.10), such that*

$$\sum_{N+1}^{\infty} \|\widetilde{\Phi}_{n,c} - \widetilde{\Psi}_n\|^2_{V\times L^2(0,\pi)} < \infty,$$

then there are generalized eigenfunctions $\{\widetilde{\Psi}_n\}_0^N$ of A corresponding to $\{\lambda_n\}_0^N$ such that $\{\widetilde{\Psi}_n\}_0^\infty$ forms a Riesz basis in $V \times L^2(0,\pi)$. Hence, $\sigma(A) = \{\lambda_n, \bar{\lambda}_n\}_0^\infty$ counting algebraic multiplicity. Therefore λ_n are algebraic simple for sufficiently large n as that they are distinct for sufficiently large n.

REMARK 3.5 *According to the preceding lemma, we remark in particular that except for at most a finite set, all $\{\lambda_n, \bar{\lambda}_n\}$ determined by (3.10) consist of all eigenvalues of A.*

Proof of Proposition 3.3. A similar result, for a slightly different situation, was obtained in (B. Z. Guo and K. Y. Chan, 2001, Theorem 4). But for the sake of completness we give a proof.

[†] Let $(g_j)_{j\geq 0}$ be a sequence of vectors in the Hilbert space H. Then, $(g_j)_{j\geq 0}$ is said to be w-linearly independent, if the equality $\sum_{j\geq 0} c_j g_j = 0$ is impossible for $0 < \sum_{j\geq 0} |c_j|^2 \|g_j\|_H^2 < \infty$.

Let $\lambda_n = i\tau_n^2$, $\tau_n = \frac{2n-1}{2} + O(\frac{1}{n})$, for sufficiently large positive integer n, be an eigenvalue of A. Then the corresponding eigenfunction $(\frac{1}{\lambda_n}\phi_n, \phi_n)$, where ϕ_n is given by (3.7) with $\tau = \tau_n$, has the following asymptotic expression

$$\left\{ \begin{array}{l} \dfrac{2e^{-\tau_n \pi}}{\lambda_n} \dfrac{d^2\phi_n}{dx^2}(x) \\ 2e^{-\tau_n\pi}\phi_n(x) \end{array} \right. = \left\{ \begin{array}{l} i(-1)^{n+1}\sin(\frac{2n-1}{2}\xi)\sin(\frac{2n-1}{2}x) + O(\frac{1}{n}) \\ (-1)^{n+1}\sin(\frac{2n-1}{2}\xi)\sin(\frac{2n-1}{2}x) + O(\frac{1}{n}) \end{array} \right. \tag{3.11}$$

$\forall x \in (0,\xi) \cup (\xi,\pi)$.
We see that (3.11) is also valid for $\tau_n = \frac{2n-1}{2}$. Let

$$\Phi_{n,c} = \begin{pmatrix} \frac{1}{\mu_n}\phi_{n,c} \\ \phi_{n,c} \end{pmatrix}, \bar{\Phi}_{n,c} = \begin{pmatrix} \frac{-1}{\mu_n}\bar{\phi}_{n,c} \\ \bar{\phi}_{n,c} \end{pmatrix}$$

be the eigenfunction of A_c corresponding to $\mu_n = i(\frac{2n-1}{2})^2$ and $\bar{\mu}_n = -i(\frac{2n-1}{2})^2$, where $\phi_{n,c}$ is given by (3.7) with $\tau = \frac{2n-1}{2}$, $\bar{\phi}_{n,c}$ is the conjugate of $\phi_{n,c}$. Then $\phi_{n,c}$ satisfies (3.11). The set of eigenfunctions of A_c is $\{\Phi_{n,c}, \bar{\Phi}_{n,c}\}_0^\infty$.
Since A_c is skew-adjoint in $V \times L^2(0,\pi)$ and its resolvent is compact, each eigenvalue of A_c is geometrically simple and hence algebraically simple. We know that $\{\Phi_{n,c}, \bar{\Phi}_{n,c}\}_0^\infty$ forms an orthogonal basis in $V \times L^2(0,\pi)$. From (3.11) and Proposition 3.2 there are an $N > 0$ and a family of eigenfunctions $\Psi_n = \begin{pmatrix} \frac{1}{\lambda_n}\phi_n \\ \phi_n \end{pmatrix}$ of A corresponding to $\lambda_n = i\tau_n^2$, τ_n being determined by (3.9), satisfying

$$\sum_{n>N} \{\|e^{-\tau_n\pi}\Psi_n - e^{-\frac{2n-1}{2}\pi}\Phi_{n,c}\|^2_{V\times L^2(0,\pi)} + \|e^{-\bar{\tau}_n\pi}\bar{\Psi}_n - e^{-\frac{2n-1}{2}\pi}\bar{\Phi}_{n,c}\|^2_{V\times L^2(0,\pi)}\} < \infty.$$

Thus, since $\{\widetilde{\Phi}_{n,c} = e^{-\frac{2n-1}{2}\pi}\Phi_{n,c}, \overline{\widetilde{\Phi}}_{n,c} = e^{-\frac{2n-1}{2}\pi}\bar{\Phi}_{n,c}\}_0^\infty$ is a Riesz basis in $V \times L^2(0,\pi)$, then according to Lemma 3.4, there are generalized eigenfunctions $\{\widetilde{\Psi}_n = e^{-\tau_n\pi}\Psi_n\}_0^N$ of A such that $\{\widetilde{\Psi}_n = e^{-\tau_n\pi}\Psi_n\}_0^\infty$ forms a Riesz basis in $V \times L^2(0,\pi)$. ∎

4. Proof of Theorem 2.2

To prove Theorem 2.2 we need the following lemma:

LEMMA 4.1 *For any $\xi \in (0,\pi)$ satisfying (2.3), we have*

$$\left|\sin\left(\frac{2n-1}{2}\xi\right)\right|^2 \leq \frac{1}{2},\ \forall n \in \mathbb{Z}. \tag{4.1}$$

Proof. Since ξ satisfies (2.3), $\frac{\xi}{\pi} = \frac{p}{q}$, where p is odd. Then, if j is integer

$$\sin^2\left(\frac{2n-1}{2}\,\xi - j\pi\right) = \sin^2\left(\frac{p(2n-1)}{2q}\,\pi\right).$$

Note that

$$\left|\frac{p(2n-1)}{2q}\,\pi - j\pi\right| = \frac{\pi}{2q}\,|p(2n-1) - 2qj| \geq \frac{\pi}{2q}. \tag{4.2}$$

The last inequality follows from the fact that $p(2n-1)$ is odd and $2qj$ is even. Therefore, the inequality (4.2) is optimal, i.e.,

$$\inf_{j\in\mathbb{Z},\, n\in\mathbb{Z},\, p\in\mathbb{N}^*}\left\{\left|\frac{p(2n-1)}{2q}\,\pi - j\pi\right|\right\} = \frac{\pi}{2q}. \tag{4.3}$$

Suppose now that (4.1) is not true. Then, there exists $n \in \mathbb{Z}$ such that

$$\left|\sin\left(\frac{2n-1}{2}\,\xi\right)\right|^2 > \frac{1}{2}, \tag{4.4}$$

so there exists $j \in \mathbb{Z}$ such that

$$\frac{\pi}{4} < \left|\frac{2n-1}{2}\,\frac{p}{q}\,\pi - j\pi\right| < \frac{3\pi}{4}.$$

According to (4.3) we deduce that $\frac{\pi}{4} < \frac{\pi}{2q}$, this implies $q < 2$ which shows that (4.4) cannot be true and end the proof of the lemma. ■

By Proposition 3.2 we have

$$\begin{aligned}&\forall\varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \text{such that}\\ &\forall|n| > n_0,\ -\varepsilon - \sin^2\left(\frac{2n-1}{2}\,\xi\right) \leq Re\lambda_n \leq \varepsilon - \sin^2\left(\frac{2n-1}{2}\,\xi\right).\end{aligned} \tag{4.5}$$

Thus, for an arbitrary $\xi \in (0, \pi)$ such that $\frac{\xi}{\pi}$ is a rational number with coprime factorization and $\frac{\xi}{\pi} = \frac{p}{q}$, where p is odd, by Lemma 4.1 there exists an eigenvalue λ_n of A such that $\mathcal{R}e\,\lambda_n \geq -\frac{1}{2} - \varepsilon$. If we consider the solution of (1.1)-(1.4) with initial data

$$\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} = F_{\lambda_n},$$

where F_{λ_n} is an eigenfunction corresponding to the eigenvalue λ_n, we obviously get

$$\begin{pmatrix} u(t) \\ u'(t) \end{pmatrix} = e^{\lambda_n t}\begin{pmatrix} u^0 \\ u^1 \end{pmatrix}, \quad \forall\, t \geq 0,$$

so

$$\omega_i(\xi) \geq -\frac{1}{2} - \varepsilon, \quad \forall i = 1, 2, \ \forall \varepsilon > 0, \quad \forall \xi \in (0, \pi), \text{ satisfying (2.3).}$$

Then,

$$\omega_i(\xi) \geq -\frac{1}{2}, \quad \forall i = 1, 2, \ \forall \xi \in (0, \pi), \text{ satisfying (2.3).} \tag{4.6}$$

On the other hand, for $\xi = \frac{\pi}{2}$, by (4.5) the eigenvalues λ_n of A are such that $\mathcal{R}e\lambda_n \leq -\frac{1}{2} + \varepsilon$ for sufficiently large integer n. We know from Proposition 3.3 that the generalized eigenfunctions $\widetilde{\Psi}_n$ of A form a Riesz basis in $V \times L^2(0, \pi)$. Using this fact and Lemma 3.4 we obtain that, for initial data

$$\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} = \sum_{n \in \mathbb{N}} a_n \widetilde{\Psi}_n = \sum_{n > n_0} a_n \widetilde{\Psi}_n, \ (a_n) \subset l^2(\mathbb{C}) \text{ and } a_n = 0, \ \forall n \leq n_0,$$

where n_0 is a positive integer sufficiently large, the solution

$$\begin{pmatrix} u(t) \\ u'(t) \end{pmatrix} = \sum_{n > n_0} a_n e^{\lambda_n t} \widetilde{\Psi}_n,$$

of (1.1)–(1.4) satisfies the estimate

$$E(u(t)) \leq Ce^{(-1+2\varepsilon)t} E(u(0)), \quad \forall t \geq 0,$$

where C is a positive constant. It follows that

$$\omega_2\left(\frac{\pi}{2}\right) \leq -\frac{1}{2} + \varepsilon, \quad \forall \varepsilon > 0.$$

Then,

$$\omega_2\left(\frac{\pi}{2}\right) \leq -\frac{1}{2}. \tag{4.7}$$

Inequalities (4.6) and (4.7) give the conclusion of the theorem. ∎

5. Remark on optimal location of the actuator at low frequencies

In order to see what happens in the case of pointwise stabilization at low frequencies of a Bernoulli–Euler beam one can make some numerical experiments. As an example we calculate the energy of numerical solution of problem (1.1)–(1.4) given by finite difference discretisation. Figure 1 shows the energy of this numerical solution with pointwise damping concentrated in three differents points:
$\xi_1 = \frac{\pi}{10}$ (line 1.), $\xi_2 = \frac{\pi}{2}$ (line 2.), and $\xi_3 = \frac{2\pi}{3}$ (line 3.). We see that the fastest

decay rate of the energy is obtained with pointwise damping concentrated in the middle of the beam ($\xi_2 = \frac{\pi}{2}$). This results allows us to expect the same results as in previous sections and the optimal location of the actuator at low frequencies is also the middle of the beam.

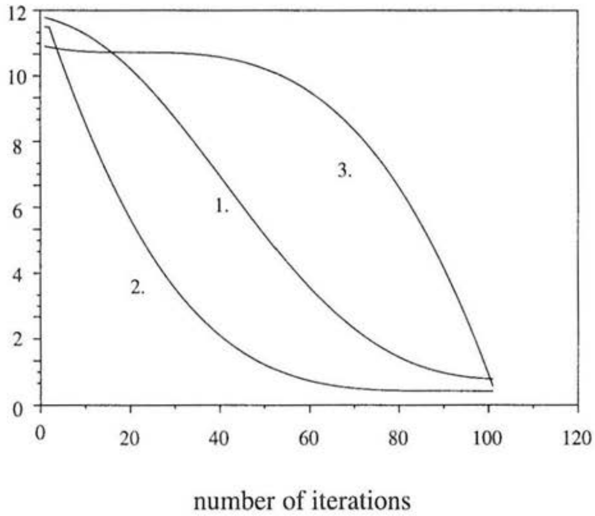

Figure 1. Energy of numerical solution

A related question

A question related to the problem studied in this paper is the optimal location of the actuator for the pointwise stabilization at high frequencies of a Bernoulli–Euler beam with moment feedback (see R. Rebarber 1995, for an appropriate model).

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