Control and Cybernetics

vol. **29** (2000) No. 2

Optimal control of semilinear elliptic equation with state constraint: maximum principle for minimizing sequence, regularity, normality, sensitivity

by

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Abstract: This article deals with state constrained optimal control problem for semilinear elliptic equation in a domain Ω. The state constraint is lumped on the compactum $X \subset \overline{\Omega}$ and contains a functional parameter $q \in C(X)$. It is shown that any minimizing approximate solution (m.a.s.) in the sense of J. Warga satisfies the pointwise maximum principle (the maximum principle for m.a.s.) if the problem is meaningful, i.e., the value of the problem is finite. It is also shown that a condition of Slater's type is sufficient for the normality in the so-called "linear-convex" problem, and the normality of the problem for some fixed value of the parameter $q \in C(X)$ implies the Lipschitz continuity of its value function in a neighborhood of q. The paper contains illustrative examples.

Keywords: parametric optimal control, minimizing sequence, elliptic equation, pointwise state constraint, maximum principle, regularity, normality, sensitivity.

1. Introduction

Optimal control problems for systems with distributed parameters governed by semilinear elliptic and parabolic equations with pointwise state constraints have been the subject of many publications during the last ten years (see, e.g., Alibert and Raymond, 1994, Bergounioux, 1992, Bonans, 1991, Bonans and Casas, 1991, 1992, 1995, Casas, 1993, 1997, 1998, Li and Yong, 1995, Raymond and Zidani, 1998, 1999). The primary attention in the references listed was devoted

This work was supported by the Russian Foundation for Basic Research, project no. 98-01-00793

to deriving Pontryagin's maximum principle and to the question how stability of an optimal control problem (in some natural sense) under one-dimensional perturbation of the state constraint is connected with realizability of both the maximum principle and the regular maximum principle.

We can formulate the following main differences with respect to papers referred to. First, we take as a "basic element of the theory" a minimizing sequence (m.s.) of ordinary controls (but not optimal control) or, in other words, the so-called minimizing approximate solution (m.a.s.) in the sense of Warga (1971) (see Warga, 1971, Ch.III, for the advantages of m.a.s. from the viewpoint of applications). Such approach allows us to consider optimal control problems (a family of problems) in the broadest generality without certain suppositions ensuring the existence of optimal elements (ordinary or relaxed) and does not use the relaxation of the optimal control problem in the sense of Warga (1971). We call the obtained necessary conditions for m.a.s. the maximum principle for m.a.s. All results derived in this way may be "closed" and rewritten in terms of optimal relaxed controls if the relaxation of the optimal control problem is possible, i.e. the maximum principle for m.a.s. turns into the ordinary maximum principle "in the limit" when there exists a relaxed (or usual) optimal control. In particular, this may by done with the results of the present paper. At the same time, we do not rewrite our results in terms of relaxed controls in this paper for two reasons. One of them consists in space limitation. Another reason is the larger practical (engineering) significance of the results expressed in terms of usual controls and not containing the relaxed controls (abstract measures). In other words, all information about optimal controls is contained in the maximum principle for m.a.s. and, in this sense, the maximum principle for m.a.s. "rather" than the ordinary maximum principle for optimal controls. The more so as m.a.s. gives for the optimal control problem, generally speaking, a deeper "minimum" than the usual optimal control (see Examples 8.1, 8.2, here, and also Warga, 1971, Ch.III, for details).

Second, we consider the so-called parametric optimal control problem. More precisely, we study the problem containing an infinite-dimensional parameter q in the state constraint. This parameter is put into the most natural space of perturbations $C(X)$. Here $X \subset \Omega$ is a compactum where the state constraint must be fulfilled and Ω is a domain where an elliptic boundary value problem is defined.

Third, we discuss together with the regularity and normality conditions of the maximum principle for m.a.s. (the corresponding concepts, Sumin, 1995, 1996, 1997a, 1997b, are defined in the article) the sensitivity problem as well. We show that a condition of Slater's type is sufficient for the normality in the so-called "linear-convex" problem, and the normality of the optimal control problem for some fixed value of the parameter q implies Lipschitz continuity of its value function in a neighborhood of q. We show also that regularity is a typical property of the similar optimal control problems in the sense that it the set of all values of the parameter for which the problem "has a meaning" (the value function is finite).

Finally, we offer an alternative method, Sumin (1986), for investigation of the problem with the state constraint differing from the methods of the papers quoted. According to this method we first approximate the primal problem (a family of problems) with the state constraint (with continuum of functional constraints) by a sequence of problems (of families of problems) with a finite number of functional constraints. Then, we derive the maximum principle for m.a.s. in the approximating problems and, at last, pass to the limit in the derived results as the number of functional constraints converges to infinity. In particular, because of the results in Sumin (1995, 1996, 1997a, 1997b) such approximation allows us simultaneously to write normality conditions for approximating problems with a finite number of the functional constraints and to use them for investigation of the sensitivity properties in the primal problem with the state constraint.

2. Problem statement

Let Ω be a bounded domain in R^n. Given a compactum $U \subset R^m$ and a set $\mathcal{D} \equiv \{u \in L_\infty(\Omega) : u(x) \in U \text{ a.e. on } \Omega\}$, consider the family of optimization problems depending on functional parameter q

$$I_0(u) \to \inf,\ I_1(u) \in \mathcal{M} + q,\ u \in \mathcal{D},\ q \in C(X), \tag{P_q}$$

where

$$I_0(u) \equiv \int_\Omega F(x, z[u](x), u(x))\, dx,\ I_1(u) \equiv G(\cdot, z[u](\cdot)),$$

$\mathcal{M} \subset C(X)$ is a convex closed set of all continuous nonpositive functions on $X, X \subset \overline{\Omega}$ is a compactum, $z[u] \in \mathring{W}_2^1(\Omega)$ is a weak solution, in the sense of Ladyzhenskaya and Ural'tseva (1973), of the Dirichlet problem for semilinear elliptic equation with a divergent principal part

$$\frac{\partial}{\partial x_i} a_{i,j}(x) z_{x_j} + a(x, z, u(x)) = 0,\ z(x) = 0,\ x \in S, \tag{2.1}$$

corresponding to the control $u \in \mathcal{D}$

Assume that the following conditions hold for the initial data of Problem (P_q):

(i) functions $G, \partial G/\partial z : \overline{\Omega} \times R^1 \to R^1$ are continuous in (x, z), functions $F, \partial F/\partial z : \Omega \times R^1 \times R^m \to R^1$, $a, \partial a/\partial z : \Omega \times R^1 \times R^m \to R^1$, are Lebesgue measurable in (x, z, u) and continuous in (z, u) for a.e. x, functions $a_{i,j}$:

(ii) estimates

$$\nu|\xi|^2 \le a_{i,j}(x)\xi_i\xi_j \le \mu|\xi|^2,\ \nu,\mu > 0,\ a_{i,j}(x) = a_{j,i}(x),$$
$$|a(x,0,u)| + |\partial a(x,z,u)/\partial z| \le a_z(x) + N(M)$$
$$\forall x \in \Omega,\ z \in S^1_M,\ u \in U,$$
$$\partial a(x,z,u)/\partial z \le 0\ \forall (x,z,u) \in \Omega \times R^1 \times U,$$

hold, where $a_z \in L_{q/2}(\Omega)$, $q > n$ (see p. 181 in Ladyzhenskaya and Ural'tseva, 1973), $S^n_M \equiv \{x \in R^n : |x| < M\}$;

(iii) the following estimates hold

$$|F(x,0,u)| + |\partial F(x,z,u)/\partial z| \le f_z(x) + N(M)$$
$$\forall x \in \Omega,\ z \in S^1_M,\ u \in U,$$
$$|G(x,z)|, |\partial G(x,z)/\partial z| \le N(M)\ \forall (x,z) \in \Omega \times S^1_M,$$

where $f_z \in L_{q/2}(\Omega)$, $q > n$, $N(M) > 0$ is a positive nondecreasing function of $M > 0$;

(iv) the boundary $S \equiv \partial\Omega$ is Lipschitz.

We equip the set $\mathcal{D}$ with Ekeland's metric $d(u^1,u^2) \equiv meas\{x \in \Omega : u^1(x) \ne u^2(x)\}$, to then convert this set into a complete metric space. According to Warga (1971), the sequence $u^i \in \mathcal{D}$, $i = 1,2,\ldots$, is called a minimizing approximate solution (m.a.s.) in Problem (P_q) if

$$I_0(u^i) \le \beta(q) + \gamma^i,\ u^i \in \mathcal{D}^{\epsilon^i}_q,\ \gamma^i, \epsilon^i \ge 0,\ \gamma^i, \epsilon^i \to 0,\ i \to \infty, \tag{2.2}$$

where $\mathcal{D}^\epsilon_q \equiv \{u \in \mathcal{D} : \rho(I_1(u), \mathcal{M} + q) \le \epsilon\}$, $\rho(I, \mathcal{M} + q) \equiv \inf_{m \in \mathcal{M}} |m + q - I|^{(0)}_X$, $\beta(q) \equiv \beta_{+0}(q) \equiv \lim_{\epsilon \to +0} \beta_\epsilon(q) \le \beta_0(q)$, $\beta_\epsilon(q) \equiv \inf_{u \in \mathcal{D}^\epsilon_q} I_0(u)$, $\beta_\epsilon(q) \equiv +\infty$, if $\mathcal{D}^\epsilon_q = \emptyset$, $|q|^{(0)}_X \equiv \|q\|_{C(X)}$.

3. Auxiliary results

In this section we give auxiliary lemmas necessary for proving the main results. These lemmas contain information about properties of solutions of the boundary value problem (2.1) and of the adjoint boundary value problem.

LEMMA 3.1 *There exists a unique solution $z[u] \in \mathring{W}^1_2(\Omega) \cap C^{(\alpha)}(\overline{\Omega})$ of the Dirichlet problem (2.1) for every control $u \in \mathcal{D}$ such that*

$$\|z[u]\|^{(1)}_{2,\Omega} + |z[u]|^{(\alpha)}_{\overline{\Omega}} \le C_1,$$

where $\alpha \in (0,1)$, $C_1 > 0$ are constants independent of $u \in \mathcal{D}$. Moreover, the solution $z[u]$ depends on $u \in \mathcal{D}$ continuously in metric $W^1_2(\Omega) \cap C^{(\alpha)}(\overline{\Omega})$.

For the proof of the lemma see Bonans and Casas (1991), Bonans and Casas

LEMMA 3.2 *The functional $I_0 : \mathcal{D} \to R^1$ and the operator $I_1 : \mathcal{D} \to C(X)$ are bounded and continuous. The value function $\beta : C(X) \to R^1 \cup \{+\infty\}$ is lower semicontinuous.*

Proof. The first assertion of the lemma is a trivial consequence of Lemma 3.1 and conditions (i), (iii) for functions F, G. The proof of the second one is the same as a proof of the similar assertion in Sumin (1996, 1997a).

LEMMA 3.3 *The adjoint problem*

$$\frac{\partial}{\partial x_j} a_{i,j}(x)\eta_{x_i} + \nabla_z a(x, z[u](x), u(x))\eta = \psi(x), \;\; \eta(x) = 0, \;\; x \in S, \qquad (3.1)$$

has a unique solution $\eta[u,\psi]$ in the class $\mathring{W}_2^1(\Omega) \cap C^{(\alpha)}(\overline{\Omega})$ for any control $u \in \mathcal{D}$ and any function $\psi \in L_{q/2}(\Omega)$, $q > n$. The solution $\eta[u,\psi]$ satisfies the estimates

$$|\eta[u,\psi]|_{\overline{\Omega}}^{(\alpha)} \le C_2 \|\psi\|_{q/2,\Omega}, \;\; \|\eta[u,\psi]\|_{2,\Omega}^{(1)} + |\eta[u,\psi]|_{\infty,\Omega} \le C_3 \|\psi\|_{q/2,\Omega},$$

with constants $\alpha \in (0,1)$, $C_2, C_3 > 0$ independent of $u \in \mathcal{D}$, $\psi \in L_{q/2}(\Omega)$.

The proof of the lemma may be found in Bonans and Casas (1991) (see also Bonans and Casas, 1995, Lemma 2.2, Gilbarg and Trudinger, 1977, Theorems 8.6, 8.16, 8.29).

In order to obtain the necessary information about the Dirichlet problem for a linear elliptic equation with a Radon measure in the right-hand part we apply the results of Bonans and Casas (1995), Lemma 2.4. We will denote by $M(\Omega)$ the space of real regular Borel measures in Ω, which is identified with the dual space of $C_0(\Omega)$, the space formed by all the real continuous functions defined in Ω and vanishing on S.

LEMMA 3.4 *For every nonpositive function $b \in L_q(\Omega)$ and every Radon measure $\mu \in M(\Omega)$ there exists a unique solution $\eta[\mu] \in \mathring{W}_\sigma^1(\Omega)$, for all $\sigma < n/(n-1)$, of the problem*

$$\frac{\partial}{\partial x_j} a_{i,j}(x)\eta_{x_i} + b(x)\eta = \mu, \;\; \eta(x) = 0, \;\; x \in S. \qquad (3.2)$$

Moreover, there exists a constant $C > 0$ independent of b such that

$$\|\eta[\mu]\|_{\sigma,\Omega}^{(1)} \le C|\mu|.$$

In particular, there exists a unique solution $\eta[u,\mu] \in \mathring{W}_\sigma^1(\Omega)$, $\sigma < n/(n-1)$, of the problem

$$\frac{\partial}{\partial x_j} a_{i,j}(x)\eta_{x_i} + \nabla_z a(x, z[u](x), u(x))\eta = \mu, \;\; \eta(x) = 0, \;\; x \in S \qquad (3.3)$$

for every control $u \in \mathcal{D}$ and every Radon measure $\mu \in M(\Omega)$. This solution satisfies the estimate

$$\|\eta[u,\mu]\|^{(1)} \le C|\mu| \;\forall u \in \mathcal{D}, \; \mu \in M(\Omega)$$

4. Maximum principle for m.a.s. in the problem with state constraint

We prove in this section a theorem about necessary conditions for elements of arbitrary m.a.s. in Problem (P_q). These necessary conditions are also called maximum principle for m.a.s. To formulate the maximum principle we introduce the following notations:

$$H_0(x,z,u,\eta) \equiv \eta a(x,z,u) - F(x,z,u),\ H(x,z,u,\eta) \equiv \eta a(x,z,u),$$
$$\mathcal{H}(x,z,u,\eta,\mu_0) \equiv \eta a(x,z,u) - \mu_0 F(x,z,u).$$

THEOREM 4.1 *Let $\beta(q) < \infty$ and u^s, $s = 1,2,\dots$ be an m.a.s. in the sense of (2.2) of Problem (P_q). Then, there exists a sequence of numbers $\gamma^s \geq 0$, $s = 1,2,\dots$, $\gamma^s \to 0$, $s \to \infty$ and a sequence of pairs (μ_0^s, λ^s), $\mu_0^s \geq 0$, $\lambda^s \in M(\Omega)$, $\mu_0^s + |\lambda^s| = 1$, with a positive Radon measure λ^s having a support in $\{x \in X : |G(x, z[u^s](x)) - q(x)| \leq \gamma^s\}$ such that*

$$\int_\Omega \max_{v\in U}\{\mathcal{H}(x, z[u^s](x), v, \psi^s[u^s](x), \mu_0^s)$$
$$- \mathcal{H}(x, z[u^s](x), u^s(x), \psi^s[u^s](x), \mu_0^s)\}\, dx \leq \gamma^s, \tag{4.1}$$

where $\psi^s[u^s] \equiv \mu_0^s\eta_0[u^s] + \eta^s[u^s]$, $\eta^s[u^s] \equiv \eta[u^s, -\nabla_z G(\cdot, z[u^s](\cdot))\lambda^s]$, i.e., it is the solution of the adjoint problem (3.3) with $u = u^s$, $\mu = -\nabla_z G(\cdot, z[u^s](\cdot))\lambda^s$.

REMARK 4.1 *It follows from Theorem 4.1 that if a control $u^0 \in \mathcal{D}_q^0$ satisfies the equality $I_0(u^0) = \beta(q)$ then it satisfies the usual maximum principle ($\mu_0^s = \mu_0$, $\lambda^s = \lambda$, $\gamma^s = \gamma = 0$). Some modification of the proof stated below allows us to prove also that the same maximum principle is correct for any such control $u^0 \in \mathcal{D}_q^0$ for which $I_0(u^0) = \beta_0(q)$.*

Proof. We use the method of Sumin (1986) in the proof. Since u^s, $s = 1,2,\dots$ is the m.a.s. in Problem (P_q), it follows that it is the m.a.s. also in the problem

$$J(u) \equiv \max\{I_0(u) - \beta(q),\ G(x, z[u](x)) - q(x),\ x \in X\} \to \inf,\ u \in \mathcal{D}$$

with zero lower bound, i.e.,

$$J(u^s) \leq \inf_{u\in\mathcal{D}} J(u) + \epsilon_s,\ \epsilon_s \geq 0,\ \epsilon_s \to 0,\ s \to \infty. \tag{4.2}$$

By (4.2), we apply the Ekeland's variational principle, Ekeland (1974), to the functional J. As a result, we find a control $w^s \in \mathcal{D}$ providing the minimum in the problem

$$J^s(u) \equiv J(u) + \sqrt{\epsilon_s}d(u, w^s) \to \inf,\ u \in \mathcal{D}, \tag{4.3}$$

and satisfying the inequalities

Further, let $\widehat{X}$ be a denumerable dense net of the compactum X. Let also $\widehat{X}_k \equiv \{x^{k,1}, \dots, x^{k,l_k}\} \subset \widehat{X}$ be a finite $1/k$ net of the compactum X, $\widehat{X}_k \subset \widehat{X}_{k+1}$, $k = 1, 2, \dots$. Approximate the problem (4.3) by a sequence of problems

$$J^{s,k}(u) \equiv \max\{I_0(u) - \beta(q),\ I_j^k(u) - q(x^{k,j}),\ j = 1, \dots, l_k\}$$
$$+ \sqrt{\epsilon_s} d(u, w^s) \to \inf,\ u \in \mathcal{D},\ k = 1, 2, \dots, \tag{4.5}$$

where the functional I_j^k is defined by the equality $I_j^k(u) \equiv G(x^{k,j}, z[u](x^{k,j}))$. We may assert that

$$\inf_{u \in \mathcal{D}} J^{s,k}(u) \to \inf_{u \in \mathcal{D}} J^s(u) = J^s(w^s) = J(w^s),\ k \to \infty. \tag{4.6}$$

Indeed, on the one hand, it is easy to see that

$$\limsup_{k \to \infty} \inf_{u \in \mathcal{D}} J^{s,k}(u) \le J(w^s). \tag{4.7}$$

On the other hand, suppose that for some $\delta > 0$

$$\liminf_{k \to \infty} \inf_{u \in \mathcal{D}} J^{s,k}(u) \le -\delta + J(w^s). \tag{4.8}$$

Then, there exists a sequence of controls $v^k \in \mathcal{D}$, $k = 1, 2, \dots$, such that

$$J^{s,k}(v^k) \le -\frac{\delta}{2} + J(w^s),\ k = 1, 2, \dots. \tag{4.9}$$

From (4.9), by uniform boundedness and equicontinuity of the family of solutions $\{z[u] : u \in \mathcal{D}\} \subset C(\overline{\Omega})$ (see the estimate of Lemma 3.1) and by conditions on functions F, G, in turn, follows the existence of a number $k_0 > 0$ such that

$$J(v^{k_0}) + \sqrt{\epsilon_s} d(v^{k_0}, w^s) \le -\frac{\delta}{4} + J(w^s).$$

The last inequality contradicts the optimality of the control w^s. Consequently, inequality (4.8) is not true. Then, by (4.7), correctness of (4.6) follows.

Obviously, by (4.6), we have

$$J^{s,k}(w^s) \le \inf_{u \in \mathcal{D}} J^{s,k}(u) + \delta_k,\ \delta_k \ge 0,\ \delta_k \to 0,\ k \to \infty.$$

Due to this inequality, we may apply the Ekeland's variational principle, Ekeland (1974), once more, but now to the functional $J^{s,k}(\cdot)$. Find a control $w^{s,k} \in \mathcal{D}$, providing the minimum in the problem

$$J^{s,k}(u) + \sqrt{\delta_k} d(u, w^{s,k}) \to \inf,\ u \in \mathcal{D} \tag{4.10}$$

and satisfying the inequalities

Now, approximate each of the problems (4.10) by the family of "smoothing" problems

$$J^{s,k,h}(u) \equiv \max\{I_0(u) - \beta(q),\ I_j^{k,h}(u) - q(x^{k,j}),\ j = 1, \ldots, l_k\} + \sqrt{\epsilon_s} d(u, w^s) + \sqrt{\delta_k} d(u, w^{s,k}) \to \inf,\ u \in \mathcal{D}, \quad h \in [0, h_0],\ h_0 > 0, \tag{4.12}$$

where

$$I_j^{k,h}(u) \equiv 1/meas\,(S_h(x^{k,j}) \cap \Omega) \int_{S_h(x^{k,j}) \cap \Omega} G(x, z[u](x))\, dx,$$

$S_h(x^{k,j})$ is a ball in R^n of radius h and center at $x^{k,j}$. Thanks to uniform boundedness and equicontinuity of the family of solutions $\{z[u] : u \in \mathcal{D}\} \subset C(\overline{\Omega})$ (see the estimate of Lemma 3.1), we may assert again that

$$\inf_{u \in \mathcal{D}} J^{s,k,h}(u) \to \inf_{u \in \mathcal{D}} J^{s,k}(u) = J^{s,k}(w^{s,k}). \tag{4.13}$$

Therefore we may write once more

$$J^{s,k,h}(w^{s,k}) \leq \inf_{u \in \mathcal{D}} J^{s,k,h}(u) + \gamma_h,\ \gamma_h \geq 0,\ \gamma_h \to 0,\ h \to 0, \tag{4.14}$$

i.e., the control $w^{s,k}$ is γ_h optimal in the problem (4.12). Here we note that the functional $J^{s,k,h}$ has the same form as the functional $J_{\xi,\rho}$ in Sumin (1989). For this reason, we may apply the results of Sumin (1989) for deriving necessary conditions for suboptimality of the control $w^{s,k}$ in the problem (4.12). We turn our attention briefly to the scheme of deriving these suboptimality conditions.

Due to inequality (4.14), we may apply the Ekeland's variational principle to the functional $J^{s,k,h}$. We find a control $w^{s,k,h} \in \mathcal{D}$ providing the minimum in the problem

$$J^{s,k,h}(u) + \sqrt{\gamma_h} d(u, w^{s,k,h}) \to \inf,\ u \in \mathcal{D}, \tag{4.15}$$

and satisfying the inequalities

$$d(w^{s,k}, w^{s,k,h}) \leq \sqrt{\gamma_h},\ J(w^{s,k,h}) \leq J(w^{s,k}). \tag{4.16}$$

The necessary conditions for optimality of the control $w^{s,k,h}$ in the problem (4.15) can also be treated as conditions for suboptimality of the control $w^{s,k}$ in the problem (4.12). Let U^* be a denumerable dense subset of U ($U^* = U$ if U is a denumerable or a finite set). Define the variation $w^{s,k,h,\epsilon}$, $0 \leq \epsilon \leq \epsilon_0$, of the control $w^{s,k,h}$

$$w^{s,k,h,\epsilon}(x) \equiv \begin{cases} w^{s,k,h}(x),\ x \in \Omega \setminus \bigcup\limits_{\substack{p=1,\ldots,p_1 \\ r=1,\ldots,r_p}} \Omega^{\epsilon}_{p,r}, \end{cases}$$

where $\Omega^{\epsilon}_{p,r} \equiv \{x \equiv (x_1, \ldots, x_n) \in R^n : x^p_1 - \epsilon \sum\limits_{m=1}^{r} \gamma^{p,m} < x_1 \leq x^p_1 - \epsilon \sum\limits_{m=1}^{r-1} \gamma^{p,m}$, $x^p_i - \epsilon r < x_i \leq x^p_i - \epsilon(r-1)$, $i = 2, \ldots, n\}$, $x^p \in \Omega$, $p = 1, \ldots, p_1$, is a finite collection of Lebesgue points of the functions $a(x, z[w^{s,k,h}](x), v) - a(x, z[w^{s,k,h}](x), w^{s,k,h}(x))$, $F(x, z[w^{s,k,h}](x), v) - F(x, z[w^{s,k,h}](x), w^{s,k,h}(x))$, $x \in \Omega$, simultaneously, for all $v \in U^*$; $\gamma^{p,r}$, $p = 1, \ldots, p_1$, $r = 1, \ldots, r_p$, is a finite collection of nonnegative numbers such that $\sum\limits_{p=1}^{p_1} \sum\limits_{r=1}^{r_p} \gamma^{p,r} \leq 1$; $u^{p,r} \in U^*$, $p = 1, \ldots, p_1$, $r = 1, \ldots, r_p$, is a finite collection of vectors; $\epsilon_0 > 0$ is a sufficiently small number, depending on the collections $\gamma^{p,r}$, x^p, such that the sets $\Omega^{\epsilon_0}_p \equiv \{[x^p_1 - \epsilon_0 \sum\limits_{m=1}^{r_p} \gamma^{p,m}, x^p_1]\} \times \{\prod\limits_{i=2}^{n} [x^p_i - \epsilon_0 r_p, x^p_i]\}$, $p = 1, \ldots, p_1$, are not mutually disjoint.

Denote by N the set of all finite collections $\mathsf{n} \equiv \{x^p, \gamma^{p,r}, u^{p,r}, p = 1, \ldots, p_1, r = 1, \ldots, r_p\}$, defining the variation $w^{s,k,h,\epsilon}$ and satisfying all previously mentioned conditions for the points x^p, for the numbers $\gamma^{p,r}$, and for the vectors $u^{p,r}$. We may assert that the following lemma, which is similar to Lemma 7 in Sumin (1989), is valid.

LEMMA 4.1 *The following equalities hold for the first variations*

$$\delta I_0(w^{s,k,h}, \mathsf{n}) \equiv \lim_{\epsilon \to 0} (I_0(w^{s,k,h,\epsilon}) - I_0(w^{s,k,h}))/\epsilon^n,$$
$$\delta I^{k,h}_j(w^{s,k,h}, \mathsf{n}) \equiv \lim_{\epsilon \to 0} (I^{k,h}_j(w^{s,k,h,\epsilon}) - I^{k,h}_j(w^{s,k,h}))/\epsilon^n$$

of the functionals I_0, $I^{k,h}_j$, $j = 1, \ldots, l_k$, for arbitrary fixed collection $\mathsf{n} \in \mathsf{N}$

$$\delta I_0(w^{s,k,h}, \mathsf{n})$$
$$= -\sum_{p=1}^{p_1} \sum_{r=1}^{r_p} \gamma^{p,r} (H_0(x^p, z[w^{s,k,h}](x^p), u^{p,r}, \eta_0[w^{s,k,h}](x^p))$$
$$- H_0(x^p, z[w^{s,k,h}](x^p), w^{s,k,h}(x^p), \eta_0[w^{s,k,h}](x^p))),$$
$$\delta I^{k,h}_j(w^{s,k,h}, \mathsf{n})$$
$$= -\sum_{p=1}^{p_1} \sum_{r=1}^{r_p} \gamma^{p,r} (H(x^p, z[w^{s,k,h}](x^p), u^{p,r}, \eta^{k,h}_j[w^{s,k,h}](x^p))$$
$$- H(x^p, z[w^{s,k,h}](x^p), w^{s,k,h}(x^p), \eta^{k,h}_j[w^{s,k,h}](x^p))),$$

where $\eta_0[w^{s,k,h}], \eta^{k,h}_j[w^{s,k,h}] \in \mathring{W}^1_2(\Omega)$ are the solutions of the adjoint problem (3.1) for $u = w^{s,k,h}$ and for $\psi(x) = -\nabla_z F(x, z[w^{s,k,h}](x), w^{s,k,h}(x))$, $\psi(x) = -1/meas(S_h(x^{k,j}) \cap \Omega) \chi^{k,h}_j(x) \nabla_z G(x, z[w^{s,k,h}](x))$, respectively, Let $\{i_1, \dots, i_\kappa\} \subset \{0, 1, \dots, l_k\}$ be the set of all active subscripts, i.e., the set of all subscripts j for which

$$J^{s,k,h}(w^{s,k,h}) - \phi_j(w^{s,k,h}) - \sqrt{\epsilon_s} d(w^{s,k,h}, w^s) - \sqrt{\delta_k} d(w^{s,k,h}, w^{s,k}) = 0$$

where we use notation $\phi_0(u) \equiv I_0(u) - \beta(q)$, $\phi_j(u) \equiv I_j^{k,h}(u) - q(x^{k,j})$, $j = 1, \dots, l_k$.

Thanks to affinity with respect to parameters $\gamma^{p,r}$ of the expressions for $\delta I_0(w^{s,k,h}, \mathsf{n})$, $\delta I_j^{k,h}(w^{s,k,h}, \mathsf{n})$, the set of all vectors of the first variations $\mathcal{K} \equiv \{(\delta I_0(w^{s,k,h}, \mathsf{n}), \delta I_1^{k,h}(w^{s,k,h}, \mathsf{n}), \dots, \delta I_{l_k}^{k,h}(w^{s,k,h}, \mathsf{n})) \in R^{l_k+1} : \mathsf{n} \in \mathsf{N}\} \subset R^{l_k+1}$ is convex (this fact may by proved by argument usual for optimal control). Denote by $\mathcal{K}_\kappa$ a projection of $\mathcal{K}$ on the subspace R^κ of vectors $(y_{i_1}, \dots, y_{i_\kappa})$ of the space R^{l_k+1} (of vectors $(y_0, y_1, \dots, y_{l_k})$). A well-known argument allows one to prove that the convex set $\mathcal{K}_\kappa$ does not intersect with the convex set $\mathcal{K}_\kappa^- \equiv \{(y_{i_1}, \dots, y_{i_\kappa}) \in R^\kappa : y_{i_j} < -2(\sqrt{\epsilon_s} + \sqrt{\delta_k} + \sqrt{\gamma_h}),\ j = 1, \dots, \kappa\}$ (the intersection of the sets $\mathcal{K}_\kappa$, $\mathcal{K}_\kappa^-$ contradicts the optimality of the control $w^{s,k,h}$). Thus, these two sets are separated by a vector $\mu^{s,k,h} \in R^\kappa$, $\mu^{s,k,h} \equiv (\mu_{i_1}^{s,k,h}, \dots, \mu_{i_\kappa}^{s,k,h})$, $\mu_{i_j}^{s,k,h} \geq 0$, $j = 1, \dots, \kappa$, $|\mu^{s,k,h}| = 1$. Completing this vector with zero components (corresponding to the passive components) to vector $\mu^{s,k,h} \in R^{l_k+1}$ (we preserve for this vector the previous notation) $\mu^{s,k,h} \equiv (0, \dots, 0, \mu_{i_1}^{s,k,h}, 0, \dots, 0, \mu_{i_\kappa}^{s,k,h}, 0, \dots, 0)$, we derive that

$$\begin{aligned}
&\mu_0^{s,k,h} \delta I_0(w^{s,k,h}, \mathsf{n}) + \sum_{j=1}^{l_k} \mu_j^{s,k,h} \delta I_j^{k,h}(w^{s,k,h}, \mathsf{n}) \\
&\geq -2 \sum_{j=0}^{l_k} \mu_j^{s,k,h} (\sqrt{\epsilon_s} + \sqrt{\delta_k} + \sqrt{\gamma_h}) \ \forall \mathsf{n} \in \mathsf{N}, \\
&\mu_0^{s,k,h} (J^{s,k,h}(w^{s,k,h}) - I_0(w^{s,k,h}) + \beta(q) \\
&- \sqrt{\epsilon_s} d(w^{s,k,h}, w^s) - \sqrt{\delta_k} d(w^{s,k,h}, w^{s,k})) = 0, \\
&\mu_j^{s,k,h} (J^{s,k,h}(w^{s,k,h}) - I_j^{k,h}(w^{s,k,h}) + q(x^{k,j}) \\
&- \sqrt{\epsilon_s} d(w^{s,k,h}, w^s) - \sqrt{\delta_k} d(w^{s,k,h}, w^{s,k})) = 0, \ j = 1, \dots, l_k. \qquad (4.17)
\end{aligned}$$

The following lemma is a direct consequence of the last relations if we admit as the collections n various "single-point" sets in the form $\{x^1 \equiv x,\ \gamma^{1,1} = 1,\ w^1 \equiv v,\ p_1 = 1,\ r_p = 1\}$ and remember that the functions a, F are continuous in u and the set U^* is dense in U.

LEMMA 4.2 *There exists a vector* $\mu^{s,k,h} \in R^{l_k+1}$,

$$\sum^{l_k} \mu_j^{s,k,h} = 1, \quad \mu_j^{s,k,h} \geq 0, \quad j = 0, 1, \dots, l_k$$

satisfying the relations (4.17) such that

$$\int_\Omega \max_{v\in U}\{\mathcal{H}(x, z[w^{s,k,h}](x), v, \psi^{s,k,h}[w^{s,k,h}](x), \mu_0^{s,k,h})$$
$$- \mathcal{H}(x, z[w^{s,k,h}](x), w^{s,k,h}(x), \psi^{s,k,h}[w^{s,k,h}](x), \mu_0^{s,k,h})\}\,dx$$
$$\leq 2meas\,\Omega(\sqrt{\epsilon_s} + \sqrt{\delta_k} + \sqrt{\gamma_h}), \tag{4.18}$$

where $\psi^{s,k,h}[w^{s,k,h}] \equiv \mu_0^{s,k,h}\eta_0[w^{s,k,h}] + \sum_{j=1}^{l_k} \mu_j^{s,k,h}\eta_j^{k,h}[w^{s,k,h}]$, $\eta_0[w^{s,k,h}]$ *is the solution of the adjoint problem (3.1) for* $u = w^{s,k,h}$,

$$\psi(x) = -\nabla_z F(x, z[w^{s,k,h}](x), w^{s,k,h}(x)),$$

$\eta_j^{k,h}[w^{s,k,h}]$ *is the solution of the same adjoint problem for* $u = w^{s,k,h}$,

$$\psi(x) \equiv -1/meas\,(S_h(x_j^k) \cap \Omega)\chi_j^{k,h}(x)\nabla_z G(x, z[w^{s,k,h}](x)).$$

Define the Radon measure $\lambda^{s,k,h}$ with support in $\bigcup_{i=1}^{l_k} S_h(x^{k,i}) \cap \Omega$, by the equality

$$\lambda^{s,k,h}(E) \equiv \sum_{j=1}^{l_k}\int_E \mu_j^{s,k,h} 1/meas\,(S_h(x^{k,j}) \cap \Omega)\chi_j^{k,h}(x)\,dx,$$

where $E \subset \overline{\Omega}$ is a Borel set. In addition, obviously, the following equality is valid:

$$\mu_0^{s,k,h} + |\lambda^{s,k,h}| = 1. \tag{4.19}$$

Then, Lemma 4.2 may by rewritten in the following form.

LEMMA 4.3 *There exists a pair* $(\mu_0^{s,k,h}, \lambda^{s,k,h})$ *with* $\mu_0^{s,k,h} \geq 0$ *and a positive Radon measure* $\lambda^{s,k,h} \in M(\Omega)$, $\mu_0^{s,k,h} + |\lambda^{s,k,h}| = 1$, *having support in totality of those sets* $S_h(x^{k,j}) \cap \Omega$, $j = 1, \ldots, l_k$, *for which*

$$J^{s,k,h}(w^{s,k,h}) - I_j^{k,h}(w^{s,k,h}) + q(x^{k,j}) - \sqrt{\epsilon_s}d(w^{s,k,h}, w^s)$$
$$- \sqrt{\delta_k}d(w^{s,k,h}, w^{s,k}) = 0,$$

such that the inequality (4.18) holds with $\psi^{s,k,h}[w^{s,k,h}] \equiv \mu_0^{s,k,h}\eta_0[w^{s,k,h}] + \eta^{s,k,h}[w^{s,k,h}]$, *where* $\eta^{s,k,h}[w^{s,k,h}]$ *is the solution of the problem (3.3) with* $\mu = -\nabla_z G(\cdot, z[w^{s,k,h}](\cdot))\lambda^{s,k,h}$, $u = w^{s,k,h}$.

We can pass to the limit in the relations of Lemma 4.3, at first, by estimate (4.11) as $h \to 0$, then, by estimate (4.4) as $k \to \infty$. Naturally, these passages compactness of a unit ball of the space of Radon measures, positiveness of the measures $\lambda^{s,k,h}$ and a priori estimates of Lemmas 3.1, 3.4. We omit the details of these sufficiently cumbersome but entirely obvious passages to the limit. As a result, we have the following lemma.

LEMMA 4.4 *There exists a pair* (μ_0^s, λ^s) *with a positive Radon measure* $\lambda^s \in M(\Omega)$, $\mu_0^s \geq 0$, $\mu_0^s + |\lambda^s| = 1$, *having support in* $\{x \in X : J(w^s) - G(x, z[w^s](x)) + q(x) = 0\}$, *such that*

$$\int_\Omega \max_{v\in U}\{\mathcal{H}(x, z[w^s](x), v, \psi^s[w^s](x), \mu_0^s) \\ - \mathcal{H}(x, z[w^s](x), w^s(x), \psi^s[w^s](x), \mu_0^s)\}\, dx \leq 2 meas\, \Omega \sqrt{\epsilon_s},$$

where $\psi^s[w^s] \equiv \mu_0^s \eta_0[w^s] + \eta^s[w^s]$, $\eta^s[w^s]$ *is the solution of the adjoint problem (3.3) with* $u = w^s$, $\mu = -\nabla_z G(\cdot, z[w^s](\cdot))\lambda^s$.

Finally, the first of estimates (4.4) together with the a priori estimates of Lemmas 3.1, 3.3, 3.4 give a possibility to rewrite the last lemma in terms of the primal m.a.s. u^s, $s = 1, 2, \ldots$. We get all relations of the theorem being proved as a result of such entirely obvious but quite cumbersome rewriting. The theorem is proved.

5. Regularity, normality, Slater condition

In connection with Theorem 4.1 it is natural to introduce the following definitions, Sumin (1995, 1996, 1997a, 1997b), of stationary, normal stationary, regular stationary and abnormal stationary sequence in Problem (P_q).

DEFINITION 5.1 *A sequence* $u^s \in \mathcal{D}$, $s = 1, 2, \ldots$ *is called stationary in Problem* (P_q) *if there exist a sequence of numbers* $\gamma^s \geq 0$, $\gamma^s \to 0$, $s \to \infty$, $u^s \in \mathcal{D}_q^{\gamma^s}$, $s = 1, 2, \ldots$, *and a bounded sequence of pairs* (μ_0^s, λ^s), $\mu_0^s \geq 0$, $\lambda^s \in M(\Omega)$, $\mu_0^s + |\lambda^s| \neq 0$, *with a positive Radon measure* λ^s *such that* $\int_X (G(x, z[u^s](x)) - q(x))\lambda^s(dx) \geq -\gamma^s$ *and nonzero limit points* $(\mu_0, \lambda) \neq 0$ *only, such that the inequality (4.1) holds with* $\eta^s[u^s]$ *being the solution of the adjoint problem (3.3) with* $u = u^s$, $\mu = -\nabla_z G(\cdot, z[u^s](\cdot))\lambda^s$.

DEFINITION 5.2 *A stationary sequence* $u^s \in \mathcal{D}_q^{\gamma^s}$, $s = 1, 2, \ldots$, $\gamma^s \geq 0$, $\gamma^s \to 0$, $s \to \infty$, *in Problem* (P_q) *is called normal (regular, abnormal) if all (there exists, does not exist) the sequences* (μ_0^s, λ^s), $s = 1, 2, \ldots$, *corresponding to it, according to Definition 5.1, have (having, having) limit points* (μ_0, λ) *with the component* $\mu_0 > 0$ *only (with the component* $\mu_0 > 0$ *only, with the component* μ_0*). Problem* (P_q) *is called normal (abnormal) if all stationary sequences in this problem are normal (abnormal). Problem* (P_q) *is called regular if there exist*

Further, we shall represent sufficient condition of normality for the so-called "linear-convex" Problem (P_q). It generalizes the classical Slater condition in mathematical programming to the case of a suboptimal problem with state constraints. We shall call it Slater condition as well. To this end we first prove the following lemma.

LEMMA 5.1 *Let $z \in \mathring{W}_2^1(\Omega)$ be a solution of the linear boundary value problem*

$$\frac{\partial}{\partial x_i} a_{i,j}(x) z_{x_j} + b(x) z = f(x), \ z(x) = 0, \ x \in S, \tag{5.1}$$

with coefficients $a_{i,j}$ satisfying condition (ii) and with coefficients $b, f \in L_{q/2}(\Omega)$, $q > n$, $b(x) \leq 0$ for a.e. $x \in \Omega$. Then, for any measure $\mu \in M(\Omega)$ we have

$$\int_\Omega z(x)\, \mu(dx) = \int_\Omega \eta[\mu](x) f(x)\, dx, \tag{5.2}$$

where $\eta[\mu] \in \mathring{W}_\sigma^1(\Omega)$, $\sigma < n/(n-1)$, is a solution of the adjoint problem (3.2).

Proof. Recall first the well known fact that any Radon measure can be *weakly attained by some sequence (of even smooth functions) bounded in L_1. Therefore, since $\mu \in M(\Omega)$, there exists a sequence g^i, $i = 1, 2, \ldots$, $g^i \in C(\Omega)$ such that

$$\|g^i\|_{1,\Omega} = |\mu|, \ \lim_{i\to\infty} \int_\Omega g^i(x)\phi(x)\, dx = \int_\Omega \phi(x)\, \mu(dx) \ \forall \phi \in C_0(\Omega) \tag{5.3}$$

(the existence of the sequence g^i, $i = 1, 2, \ldots$, can be proved with the same kind of construction as in Giusti, 1984, Theorem 1.17). By Lemma 2 on representation of a linear functional on the set of solutions of linear Dirichlet boundary value problem of Sumin (1989), we have the equality

$$\int_\Omega z(x) g^i(x)\, dx = \int_\Omega \eta^i(x) f(x)\, dx, \tag{5.4}$$

where $\eta^i \in \mathring{W}_2^1(\Omega) \subset \mathring{W}_\sigma^1(\Omega)$, $\sigma \in [1, n/(n-1))$, is the solution of the adjoint problem (3.2) with $\mu = g^i$. Due to the first estimate of Lemma 3.4 and to the equality (5.3), we can write the estimate $\|\eta^i\|_{\sigma,\Omega}^{(1)} \leq K$ with a constant $K > 0$ independent of i. Hence, by the limit relation (5.3) and by the uniqueness of the solution η (see Lemma 3.4) we can write $\|\eta^i - \eta\|_{p,\Omega} \to 0$, $i \to \infty$, with $p < n\sigma/(n-\sigma)$. In turn, by this limit relation and by the limit relations (5.3), we pass to the limit in the equality (5.4) as $i \to \infty$. As a result, we obtain equality (5.2). The lemma is proved. ■

THEOREM 5.1 *Let the function a involved in (2.1) have a form $a(x, z, u) \equiv a_1(x) z + a_2(x, u)$ and the function G setting the state constraint be convex with respect to z for every x. Let also $u^0 \in \mathcal{D}$ be such a control that $G(x, z[u^0](x)) - q(x) \leq -\gamma \ \forall x \in X$, $\gamma > 0$, i.e., Problem (P_q) satisfies the Slater condition.*

Proof. Assume that the assertion of the theorem is not true. Let $u^s \in \mathcal{D}$, $s = 1, 2, \ldots$, be a stationary sequence in Problem (P_q) such that the corresponding sequence of pairs (μ_0^s, λ^s), $s = 1, 2, \ldots$, has a limit point (μ_0, λ) with $\mu_0 = 0$. Then, by positiveness of the measure λ^s, by convexity in z of the function G and by Lemma 5.1, we can write for any $u \in \mathcal{D}$

$$\begin{aligned}
&\int_\Omega (G(x, z[u](x)) - G(x, z[u^s](x)))\, \lambda^s(dx) \\
&\geq \int_\Omega \nabla_z G(x, z[u^s](x))(z[u](x) - z[u^s](x))\, \lambda^s(dx) \\
&= \int_\Omega \eta^s[u^s](x)(a(x, z[u^s](x), u^s(x)) - a(x, z[u^s](x), u(x)))\, dx \\
&= \int_\Omega (H(x, z[u^s](x), u^s(x), \eta^s[u^s](x)) \\
&- H(x, z[u^s](x), u(x), \eta^s[u^s](x)))\, dx
\end{aligned} \tag{5.5}$$

where $\eta^s[u^s]$ is the solution of the adjoint problem (3.3) with $u = u^s$, $\mu = -\nabla_z G(\cdot, z[u^s](\cdot)\lambda^s)$. In addition, obviously, the difference $z[u](x) - z[u^s](x)$ is a solution of the linearized boundary value problem

$$\frac{\partial}{\partial x_i} a_{i,j}(x) z_{x_j} + a_1(x) z = a_2(x, u(x)) - a_2(x, u^s(x)),\ z(x) = 0,\ x \in S.$$

By virtue of the limit relation $\mu_0^s \to 0$, $s \to \infty$, the inequality (5.5) and stationarity of the sequence u^s, $s = 1, 2, \ldots$, we obtain for some sequence of nonnegative numbers $\overline{\gamma}^s$. $s = 1, 2, \ldots$, convergent to zero, the following inequality with $\psi^s[u^s] \equiv \mu_0^s \eta_0[u^s] + \eta^s[u^s]$

$$\begin{aligned}
&\int_\Omega (G(x, z[u](x)) - G(x, z[u^s](x)))\, \lambda^s(dx) \\
&\geq \int_\Omega \{\mathcal{H}(x, z[u^s](x), u^s(x), \psi^s[u^s](x), \mu_0^s) \\
&- \mathcal{H}(x, z[u^s](x), u(x), \psi^s[u^s](x), \mu_0^s))\}\, dx \\
&+ \int_\Omega \{-\mu_0^s(H_0(x, z[u^s](x), u^s(x), \eta_0[u^s](x)) \\
&- H_0(x, z[u^s](x), u(x), \eta_0[u^s](x)))\}\, dx \geq -\overline{\gamma}^s.
\end{aligned}$$

At the same time, by the belonging to the support of the positive measure λ^s in $\{x \in X : |G(x, z[u^s](x)) - q(x)| \leq \gamma^s\}$, by nondegeneracy of any limit measure for sequence λ^s, $s = 1, 2, \ldots$, and by the Slater condition, we can write

$$\int_\Omega (G(x, z[u^0](x)) - G(x, z[u^s](x)))\, \lambda^s(dx) \leq -\alpha$$

for some number $\alpha > 0$. The contradiction obtained completes the proof of the

6. Approximation of the primal problem with the state constraint by the problems with functional constraints

Consider the sequence of the optimization problems depending on the finite-dimensional vector parameter $q^k \equiv (q_1^k, \dots, q_{l_k}^k) \in R^{l_k}$ and approximating the primal Problem (P_q)

$$I_0(u) \to \inf, \; I^k(u) \in \mathcal{M}^k + q^k, \; u \in \mathcal{D}, \; q^k \in R^{l_k}, \qquad (P^k_{q^k})$$

where $\mathcal{M}^k \equiv \{y^k \in R^{l_k} : y_1^k \le 0, \dots, y_{l_k}^k \le 0\}$, $I^k(u) \equiv (I_1^k(u), \dots, I_{l_k}^k(u))$, $I_i^k(u) \equiv G(x^{k,i}, z[u](x^{k,i}))$. Here the sets $\widehat{X}_k \equiv \{x^{k,1}, \dots, x^{k,l_k}\}, \widehat{X}$ are the same as in Section 4.

As in case of Problem (P_q) and according to Warga (1971), a sequence $u^i \in \mathcal{D}$, $i = 1, 2, \dots$, is called m.a.s. in Problem $(P^k_{q^k})$ if the following relations hold for the value function $\beta_k : R^{l_k} \to R^1 \cup \{+\infty\}$

$$I_0(u^i) \le \beta_k(q) + \gamma^i, \; u^i \in \mathcal{D}_q^{\epsilon^i}, \; \gamma^i, \; \epsilon^i \ge 0, \; \gamma^i, \; \epsilon^i \to 0, \; i \to \infty, \qquad (6.1)$$

where $\mathcal{D}^\epsilon_{q^k} \equiv \{u \in \mathcal{D} : I_j^k(u) - q_j^k) \le \epsilon, \; j = 1, \dots, l_k\}$, $\beta_k(q^k) \equiv \beta_{k,+0}(q^k) \equiv \lim_{\epsilon \to +0} \beta_{k,\epsilon}(q^k) \le \beta_{k,0}(q^k)$, $\beta_{k,\epsilon}(q^k) \equiv \inf_{u \in \mathcal{D}^\epsilon_{q^k}} I_0(u)$, $\beta_{k,\epsilon}(q^k) \equiv +\infty$, if $\mathcal{D}^\epsilon_{q^k} = \emptyset$.

LEMMA 6.1 *Let $\beta(q) < \infty$, $q \in C(X)$. Then, there exists a sequence of vectors $q^k \in R^{l_k}$, $k = 1, 2, \dots$, such that $\beta_k(q^k) \to \beta(q)$, $k \to \infty$. In particular, one of such sequences has the form $\overline{q}^k \equiv (\overline{q}_1^k, \dots, \overline{q}_{l_k}^k)$, $\overline{q}_i^k = q(x^{k,i})$, $i = 1, \dots, l_k$.*

Proof. Let u^s, $s = 1, 2, \dots$, be a sequence of controls such that $u^s \in \mathcal{D}_q^{\epsilon^s}$ and $I_0(u^s) \to \beta(q)$, $\epsilon^s \ge 0$, $\epsilon^s \to 0$, $s \to \infty$. Due to Lemma 3.1 and to the condition (iii) for the function G, functions of the family $I_1(u^s)$, $s = 1, 2, \dots$, are uniformly bounded and equicontinuous on $\overline{\Omega}$. For this reason, we shall suppose without loss of generality that

$$|I_1(u^s) - \widehat{q}|^{(0)}_{\overline{\Omega}} \to 0, \; s \to \infty, \; \widehat{q} \in C(\overline{\Omega}), \; \widehat{q}(x) \le q(x), \; x \in X,$$
$$\max\{0, I_1^j(u^s) - \overline{q}_1^j, \dots, I_{l_j}^j(u^s) - \overline{q}_{l_j}^j\} \to 0, \; s \to \infty, \; j = 1, 2, \dots.$$

Since $\widehat{X}_k \subset \widehat{X}_{k+1}$, $k = 1, 2, \dots$, it follows by the last limit relation and by definition of the function β_k that we may select such subsequence s_k, $k = 1, 2, \dots$, of the sequence $s = 1, 2, \dots$, for which $I_0(u^{s_k}) \ge \beta_k(\overline{q}^k) - \delta^k$, $\delta^k \ge 0$, $\delta^k \to 0$, $k \to \infty$. Hence, we may assert that $\limsup_{k \to \infty} \beta_k(\overline{q}^k) \le \beta(q)$. Simultaneously, we can show that the inequality $\liminf_{k \to \infty} \beta_k(\overline{q}^k) \ge \beta(q)$ holds. Obviously, the last will mean the end of the proof. Suppose that this inequality is not true. Then, it follows without loss of generality that $\lim_{k \to \infty} \beta_k(\overline{q}^k) = \alpha < \beta(q)$. This strict in-

$\max\{0, I_1^s(u^s) - \overline{q}_1^s, \ldots, I_{l_s}^s(u^s) - \overline{q}_{l_s}^s\} \to 0$, $I_0(u^s) \to \alpha$, $s \to \infty$. By equicontinuity on $\overline{\Omega}$ of the family $I_1(u^s)$, $s = 1, 2, \ldots$, the first of the last two limit relations leads us to the sequence of inclusions $u^s \in \mathcal{D}_q^{\epsilon^s}$ for some sequence of numbers $\epsilon^s \geq 0$, $\epsilon^s \to 0$, $s \to \infty$. These inclusions together with the second of limit relations mentioned above contradict the definition of the function $\beta(q)$ as the value function of Problem (P_q). The lemma is proved. ■

As in Sumin (1995, 1996, 1997a, 1997b) introduce the following definition of a stationary sequence in Problem $(P_{q^k}^k)$

DEFINITION 6.1 *A sequence of controls $u^i \in \mathcal{D}$, $i = 1, 2, \ldots$, is called a stationary sequence in Problem $(P_{q^k}^k)$ if there exist a sequence of numbers $\gamma^i \geq 0$, $i = 1, 2, \ldots$, $\gamma^i \geq 0$, $\gamma^i \to 0$, $i \to \infty$,*

$$u^i \in \mathcal{D}_{q^k}^{k,\gamma^i} \equiv \{u \in \mathcal{D} : I_j^k(u) - q_j^k \leq \gamma^i,\ j = 1, \ldots, l_k\},$$

and a bounded sequence of vectors $\mu^{k,i} \in R^{l_k+1}$, $i = 1, 2, \ldots$,

$$\sum_{j=0}^{l_k} \mu_j^{k,i} \neq 0,\ \mu_j^i \geq 0,\ j = 0, 1, \ldots, l_k,$$
$$\mu_j^i(I_j^k(u^i) - q_j^k) \geq -\gamma^i,\ j = 1, 2, \ldots, l_k,$$

such that

$$\int_\Omega \max_{v \in U}\{\mathcal{H}(x, z[u^i](x), v, \psi^{k,i}[u^i](x), \mu_0^i) - \mathcal{H}(x, z[u^i](x), u^i(x), \psi^{k,i}[u^i](x), \mu_0^i)\}\, dx \leq \gamma^i,$$

where $\psi^{k,i}[u^i] \equiv \mu_0^i \eta_0[u^i] + \sum_{j=1}^{l_k} \mu_j^{k,i} \eta_j^k[u^i]$, $\eta_0[u^i]$ is the solution of the problem (3.1) with $u = u^i$, $\psi(x) = -\nabla_z F(x, z[u^i](x), u^i(x))$, $\eta_j^k[u^i]$ is the solution of the problem

$$\frac{\partial}{\partial x_j} a_{i,j}(x) \eta_{x_i} + \nabla_z a(x, z[u](x), u(x))\eta = -\nabla_z G(x, z[u](x))\mu,$$
$$\eta(x) = 0,\ x \in S \tag{6.2}$$

with $u = u^i$, $\mu = \delta_{x^{k,j}}$, and moreover, the sequence $\mu^{k,i}$, $i = 1, 2, \ldots$, has only nonzero limit points.

As in Sumin (1995, 1996, 1997a, 1997b), together with this definition define the sets of Lagrange multipliers: $L_{q^k}^{k,\lambda} \equiv \{-\sum_{j=1}^{l_k} \mu_j^k e^j \in R^{l_k} : \mu^k \equiv (\mu_0^k, \mu_1^k, \ldots, \mu_{l_k}^k) \in R^{l_k+1}$, $\mu^k \neq 0$, $\mu_0^k = \lambda$, so that there exists a sequence of stationary controls vectors $\mu^{k,i}$, $i = 1, 2, \dots$, has the vector μ^k as its limit point}, $\lambda = 0, 1$; $M^{k,0}_{q^k} \equiv L^{k,0}_{q^k} \bigcup \{0\}$, $M^{k,1}_{q^k} \equiv L^{k,1}_{q^k}$.

Further, in complete analogy with Sumin (1996, 1997a) we may assert that the following theorem is valid for Problem $(P^k_{q^k})$ with a finite number of functional constraints.

THEOREM 6.1 *Let $\beta_k(q^k) < +\infty$. Then we have*

$$(\partial\beta(q) \cap M^{k,1}_{q^k}) \cup (\partial^\infty\beta(q) \cap M^{k,0}_{q^k}) \setminus \{0\} \neq \emptyset$$

and Clarke's generalized gradient of the value function β_k at q^k is equal to

$$\partial\beta_k(q^k) = \overline{conv}\{\partial\beta_k(q^k) \cap M^{k,1}_{q^k} + \partial^\infty\beta_k(q^k) \cap M^{k,0}_{q^k}\}.$$

COROLLARY 6.1 *Let all problems $P^k_{y^k}$ be normal in some neighborhood O_{q^k} of a point q^k, i.e., $M^{k,0}_{y^k} = \{0\}$, $y^k \in O_{q^k}$, and moreover, the sets $M^{k,1}_{y^k}$ uniformly boundend with respect to $y^k \in O_{q^k}$ by a constant K in some norm $\|\cdot\|$ (e.g., Euclidean norm $\|\cdot\| = |\cdot|$). Then, the value function β_k is Lipschitz in this norm on O_{q^k} with the same constant K.*

Proof. The assertion of the corollary follows immediately from the equality of Theorem 6.1, Theorem 2.3.7 (mean value theorem in the Lipschitz case) and Theorem 2.9.7 in Clarke (1983).

7. Lipschitz continuity of the value function, typical of regularity

First, we show in this section that the normality of Problem (P_q) (see Definition 5.2) implies Lipschitz continuity of its value function in the neighborhood of the point $q \in C(X)$. To this end, we show that the normality of Problem (P_q) implies the existence of $\delta > 0$ for which all sets $M^{k,1}_{y^k} \in R^{l_k}$, $k = 1, 2, \dots$ for y^k, $|y^k - \overline{q}^k|_\infty \leq \delta$ are uniformly bounded with respect to $k = 1, 2, \dots$ and y^k in c-norm $|\cdot|_\infty$ where we understand c-norm $|x|_\infty$ of a vector x as the quantity $\max\{|x_1|, \dots, |x_{l_k}|\}$.

Assume that it is not true. Then, there exist such sequences of vectors $\widetilde{y}^k \in R^{l_k}$, $\lambda^k \in M^{k,1}_{y^k}$, $k = 1, 2, \dots$, that

$$|\widetilde{y}^k - \overline{q}^k|_\infty \to 0, \ |\lambda^k|_\infty \to \infty, \ k \to \infty.$$

This means that for every $k = 1, 2, \dots$ there exist a sequence of controls $u^{k,i} \in \mathcal{D}$, $i = 1, 2, \dots$, a sequence of numbers $\gamma^{k,i} \geq 0$, $i = 1, 2, \dots$, $\gamma^{k,i} \geq 0$, $\gamma^{k,i} \to 0$, $i \to \infty$, and a bounded sequence of vectors $\mu^{k,i} \equiv (\mu_0^{k,i}, \widetilde{\mu}^{k,i}) \in R^{l_k+1}$, $i = 1, 2, \ldots$,

$$\sum_{j=0}^{l_k} \mu_j^{k,i} \neq 0, \ \mu_j^{k,i} \geq 0, \ j = 0, 1, \ldots, l_k,$$

$$\mu_j^{k,i}(I_j^k(u^{k,i}) - \widetilde{y}_j^k) \geq -\gamma^{k,i}, \ j = 1, 2, \ldots, l_k,$$

such that

$$\int_\Omega \max_{v \in U}\{\mathcal{H}(x, z[u^{k,i}](x), v, \psi^{k,i}[u^{k,i}](x), \mu_0^{k,i}) - \mathcal{H}(x, z[u^{k,i}](x), u^{k,i}(x), \psi^{k,i}[u^{k,i}](x), \mu_0^{k,i})\}\, dx \leq \gamma^{k,i},$$

where $\psi^{k,i} \equiv \mu_0^{k,i}\eta_0[u^{k,i}] + \sum_{j=1}^{l_k} \mu_j^{k,i}\eta_j^k[u^{k,i}]$, $\eta_0[u^{k,i}]$ is the solution of the problem (3.1) with $u = u^{k,i}$, $\psi(x) = -\nabla_z F(x, z[u^{k,i}](x), u^{k,i}(x))$, $\eta_j^k[u^{k,i}]$ is the solution of the problem (6.2) with $u = u^{k,i}$, $\mu = \delta_{x^{k,j}}$ and, moreover, the sequence $\mu^{k,i}$, $i = 1, 2, \ldots$, has the point $(1, \lambda^k)$ as its limit.

Let i_k, $k = 1, 2, \ldots$, be such sequence that

$$\gamma^{k,i_k} \to 0, \ \gamma^{k,i_k}/|\lambda^k|_\infty \equiv \overline{\gamma}^k \to 0, \ \mu_0^{k,i_k}/|\lambda^k|_\infty \equiv \overline{\mu}_0^k \to 0,$$
$$|\widetilde{\mu}^{k,i_k}/|\lambda^k|_\infty|_\infty \to 1, \ \widetilde{\mu}^{k,i_k}/|\lambda^k|_\infty \equiv \overline{\mu}^k = (\overline{\mu}_1^k, \ldots, \overline{\mu}_{l_k}^k).$$

Then, it is easy to observe that the sequence $u^k \equiv u^{k,i_k} \in \mathcal{D}_q^{\overline{\gamma}_1^k}$, $k = 1, 2, \ldots$, $\overline{\gamma}_1^k \to 0$, $k \to \infty$, satisfies the relations

$$\int_\Omega \max_{v \in U}\{\mathcal{H}(x, z[u^k](x), v, \psi^k[u^k](x), \overline{\mu}_0^k) - \mathcal{H}(x, z[u^k](x), u^k(x), \psi^k[u^k](x), \overline{\mu}_0^k)\}\, dx \leq \overline{\gamma}^k,$$

where $\psi^k \equiv \overline{\mu}_0^k\eta_0[u^k] + \eta^k[u^k]$, $\eta^k[u^k]$ is the solution of the adjoint problem (6.2) with $u = u^k$ and with $\mu = \overline{\mu}^k \equiv \sum_{j=1}^{l_k} \overline{\mu}_j^k \delta_{x^{k,j}}$ (here we preserve for the measure the same notation which we exploited above for the vector of multipliers). Moreover, obviously, $\overline{\mu}_0^k \to 0$, $|\overline{\mu}^k| \to 1$, $k \to \infty$, and the positive measure $\overline{\mu}^k \in M(\Omega)$ satisfies the inequality $\int_X (G(x, z[u^k](x) - q(x))\, \overline{\mu}^k(dx) \geq -\overline{\gamma}_2^k$, $\overline{\gamma}_2^k \to 0$, $k \to \infty$. Here and above $\overline{\gamma}_s^k$, $k = 1, 2, \ldots$, $s = 1, 2$, are some sequences of nonnegative numbers. Thus, the sequence u^k, $k = 1, 2, \ldots$, is a stationary one in Problem (P_q) ($\gamma^k = \max\{\overline{\gamma}^k, \overline{\gamma}_1^k, \overline{\gamma}_2^k\}$) but simultaneously it is not a normal stationary sequence. The contradiction obtained implies that the property of uniform boundedness of the sets $M_{y^k}^{k,1}$, mentioned above, is proved. By repeating the arguments of this proof, we may show also that all Problems $(P_{y^k}^k)$ for y^k satisfying the inequality $|y^k - \overline{q}^k|_\infty \leq \delta$ are, without loss of generality, normal ones.

So, according to Corollary 6.1, the value functions $\beta_k(y^k)$, $|y^k - \overline{q}^k|_\infty \leq$

$1, 2, \ldots$. It follows that the Lipschitz condition with the same constant in $\delta/2$-neighborhood of the point $q \in C(X)$ holds for the value function of primal Problem (P_q) as well. In fact, let $q^1, q^2 \in C(X)$, $\|q^i - q\|_X^{(0)} \leq \delta/2$, $i = 1, 2$. Then the inequalities $|\overline{q}^{i,k} - \overline{q}^k|_\infty \leq \delta/2$, $i = 1, 2$, hold also, where $\overline{q}^{i,k} \equiv (\overline{q}_1^{i,k}, \ldots, \overline{q}_{l_k}^{i,k})$, $\overline{q}_j^{i,k} = q^i(x^{k,j})$, $j = 1, \ldots, l_k$. But then, according to the fact proved above, we may write $|\beta_k(q^{1,k}) - \beta_k(q^{2,k})| \leq K|\overline{q}^{1,k} - \overline{q}^{2,k}|_\infty$, $k = 1, 2, \ldots$. Thanks to Lemma 6.1, passing to the limit in the last inequality as $k \to \infty$, we have $|\beta(q^1) - \beta(q^2)| \leq K\|q^1 - q^2\|_X^{(0)}$. Lipschitz continuity of the value function is proved.

Further, we will prove a result which is in some sense inverse. We will use the following definition.

DEFINITION 7.1 *A vector $n \in R^m$ is said to be a proximal normal or proximal normal vector to set $C \subset R^m$ at $x \in \overline{C}$ if there exist a vector $u \notin \overline{C}$ and a number $\lambda > 0$ such that $n = \lambda(u - x)$, $\|u - x\| = \rho(u, C)$, $\rho(u, C) \equiv \inf_{c \in C} |u - c|$. According to Clarke (1983), the vector $u - x$ is called perpendicular to C at x. The set of all proximal normals to C at $x \in \overline{C}$ is denoted by $PN_C(x)$ in Borwein and Strojwas (1986, 1987).*

THEOREM 7.1 *Let the value function β of Problem (P_q) be Lipschitz continuous in a neighborhood of $q \in C(X)$. Then, there exist regular m.a.s. for Problem (P_q) in a neighborhood of q.*

Proof. Due to space limitation, we give only the main idea of the proof. Consider the family of Problems $(\overline{P}_\nu) \equiv (P_{q+\nu\widetilde{q}})$ where $\widetilde{q} \equiv 1$, depending on parameter ν. Since function β is Lipschitz continuous in a neighborhood of q, it follows obviously that the function of one variable $\overline{\beta}(\nu) \equiv \beta(q + \nu\widetilde{q})$ is Lipschitz in a neighborhood of zero. Therefore, by proximal normal formula for Clarke's normal cone (see, e.g., Clarke, 1983, Proposition 2.5.7), there exist such sequences of numbers ν^i, ζ^i, η^i, $i = 1, 2, \ldots$, that

$$\nu^i \to 0,\ \overline{\beta}(\nu^i) \to \overline{\beta}(\nu),\ (\zeta^i, -\eta^i) \to (\zeta, -\eta) \neq 0,\ i \to \infty,\ \eta > 0,$$
$$(\zeta^i, -\eta^i) \in PN_{epi\,\overline{\beta}}(\nu^i, \overline{\beta}(\nu^i)).$$

By analogy with the proof of Lemma 4.2 in Sumin (1996) (see also Sumin, 1997a, Lemma 8) and by the above relations, we may conclude that if $u^{i,k}$, $k = 1, 2, \ldots$, is a m.a.s. in Problem $(\overline{P}_{\nu^i})$ in the sense of (2.2) ($\gamma^k = \gamma^{i,k}$, $\epsilon^k = \epsilon^{i,k}$) then the sequence $(u^{i,k}, \nu^i)$, $k = 1, 2, \ldots$, is the m.a.s., in the same sense, in the problem

$$I^i(u, \nu) \equiv \lambda^i\eta^i I_0(u) - \lambda^i\zeta^i\nu' + \frac{1}{2}\|(\nu', I_0(u)) - (\nu^i, \overline{\beta}(\nu^i))\|^2 \to \inf,$$
$$I_1(u) \in \mathcal{M} + q + \nu'\widetilde{q},\ \nu' \in S_P,\ u \in \mathcal{D}, \tag{7.1}$$

where $S_P \equiv (-P, P)$ is a segment of sufficiently large length such that $\nu^i \in S_P$, since the following inequalities hold

$$\lambda^i\eta^i I_0(u^{i,k}) - \lambda^i\zeta^i\nu^i + \frac{1}{2}\|(\nu^i, I_0(u^{i,k})) - (\nu^i, \overline{\beta}(\nu^i))\|^2$$
$$\leq \lambda^i\eta^i\overline{\beta}(\nu^i) - \lambda^i\zeta^i\nu^i + \lambda^i\eta^i\gamma^{i,k} + \frac{1}{2}(\gamma^{i,k})^2,$$
$$\rho(I_1(u^{i,k}) - q - \nu^i, \mathcal{M}) \leq \epsilon^{i,k},\ \gamma^{i,k}, \epsilon^{i,k} \to 0,\ k \to \infty,$$

and the lower bound in the problem (7.1) is equal to $\lambda^i\eta^i\beta(\nu^i) - \lambda^i\zeta^i\nu^i$. This fact may by proved with the same kind of arguments as in Sumin (1996) (see the proof of the equality (4.8)).

Further, we will write the maximum principle for the m.a.s. $(u^{i,k}, \nu^i)$, $k = 1, 2, \ldots$, in the problem (7.1). Since this may be realized according to scheme of proof of Theorem 4.1 (with the set of controls $\mathcal{D} \times \overline{S_P}$ instead of $\mathcal{D}$), we omit the proof.

LEMMA 7.1 *There exist a sequence of numbers $\gamma^{i,k} \geq 0$, $k = 1, 2, \ldots$, $\gamma^{i,k} \to 0$, $k \to \infty$ and a sequence of pairs $(\mu_0^{i,k}, \lambda^{i,k})$, $\mu_0^{i,k} \geq 0$, $\lambda^{i,k} \in M(\Omega)$, $\mu_0^{i,k} + |\lambda^{i,k}| = 1$, with a positive Radon measure $\lambda^{i,k}$ having a support in $\{x \in X : |G(x, z[u^{i,k}](x)) - q(x) - \nu^i| \leq \gamma^{i,k}\}$ such that*

$$\int_\Omega \max_{v\in U}\{\mathcal{H}(x, z[u^{i,k}](x), v, \psi^{i,k}[u^{i,k}](x), \mu_0^{i,k}\xi^{i,k})$$
$$- \mathcal{H}(x, z[u^{i,k}](x), u^{i,k}(x), \psi^{i,k}[u^{i,k}](x), \mu_0^{i,k}\xi^{i,k})\}\, dx \leq \gamma^{i,k},$$
$$\max_{\nu'\in\overline{S_P}}(-\mu_0^{i,k}\lambda^i\zeta^i - |\lambda^{i,k}|)(\nu' - \nu^i) \leq \gamma^{i,k}. \tag{7.2}$$

where $\xi^{i,k} \equiv \lambda^i\eta^i + I_0(u^{i,k}) - \overline{\beta}(\nu^i)$, $\psi^{i,k}[u^{i,k}] \equiv \mu_0^{i,k}\xi^{i,k}\eta_0[u^{i,k}] + \eta^{i,k}[u^{i,k}]$, $\eta^{i,k}[u^{i,k}] \equiv \eta[u^{i,k}, -\nabla_z G(\cdot, z[u^{i,k}](\cdot))\lambda^{i,k}]$.

We have $I_0(u^{i,k}) \to \overline{\beta}(\nu^i)$, $k \to \infty$, $\nu^i \to 0$, $i \to \infty$. Hence, there exists a subsequence k_i, $i = 1, 2, \ldots$ such that

$$\rho(I_1(u^i) - q, \mathcal{M}) \leq |\nu^i| + \rho(I_1(u^i) - q - \nu^i, \mathcal{M}) \leq |\nu^i| + \epsilon^{i,k_i},$$
$$\epsilon^{i,k_i} \to 0,\ \gamma^{i,k_i} \to 0,\ (I_0(u^i) - \overline{\beta}(\nu^i))/\lambda^i \to 0,$$
$$\gamma^{i,k_i}/\lambda^i \to 0,\ i \to \infty,\ u^i \equiv u^{i,k_i}.$$

Without loss of generality, by the second inequality (7.2), by the limit relation $\lambda^i \to 0$, $i \to \infty$, by the condition of normalization $\mu_0^{i,k} + |\lambda^{i,k}| = 1$, by the *weak compactness of a unit ball of the space of Radon measures, and by positiveness of the measures $\lambda^{i,k}$ we have simultaneously

$$\mu_0^{i,k_i} \to 1,\ |\lambda^{i,k_i}| \to 0,\ \lambda^{i,k_i}/\lambda^i \to \lambda \in M(\Omega),\ (\eta, \lambda) \neq 0,\ \zeta + |\lambda| = 0.$$

LEMMA 7.2 *There exist a sequence of numbers* $\gamma^i \geq 0$, $i = 1, 2, \ldots$, $\gamma^i \to 0$, $i \to \infty$, *an m.a.s.* $u^i \in \mathcal{D}_q^{\gamma^i}$, *and a sequence of pairs* (μ_0^i, λ^i), $\mu_0^i \geq 0$, $\lambda^i \in M(\Omega)$, $\mu_0^i + |\lambda^i| = 1$, *with a positive Radon measure* λ^i *having a support in* $\{x \in X : |G(x, z[u^i](x)) - q(x) - \nu^i| \leq \gamma^i\}$ *such that*

$$\int_\Omega \max_{v \in U}\{\mathcal{H}(x, z[u^i](x), v, \psi^i[u^i](x), \mu_0^i)$$
$$- \mathcal{H}(x, z[u^i](x), u^i(x), \psi^i[u^i](x), \mu_0^i)\}\, dx \leq \gamma^i,$$
$$\mu_0 = \eta > 0, \;\; \zeta + |\lambda| = 0.$$

where (μ_0, λ) *is an arbitrary *weak accumulation point of the sequence* (μ_0^i, λ^i), $i = 1, 2, \ldots$, $\psi^i[u^i] \equiv \mu_0^i \eta_0[u^i] + \eta^i[u^i]$, $\eta^i[u^i] \equiv \eta[u^i, -\nabla_z G(\cdot, z[u^i](\cdot))\lambda^i]$.

Thanks to Lemma 7.2, we can speak about regularity of Problem (P_q).

Naturally, in general situation Lipschitz continuity of the value function $\beta(q)$ does not take place. Nevertheless, the following common result is valid.

THEOREM 7.2 *A set of all points* $q \in dom\, \beta$, *for which there exist only regular m.a.s. in Problem* (P_q), *is dense in* $dom\, \beta$.

In practice, the proof of this theorem repeats the first part of the proof of Theorem 7.1 (see the part of the proof before Lemma 7.1). In fact, let $q \in dom\, \beta$, $\widetilde{q} \in C(X)$ be a positive function. Then $q + \nu\widetilde{q} \in dom\, \beta$, $\nu \in dom\, \overline{\beta}$ for $\nu \geq 0$ and the value function $\overline{\beta}(\nu) \equiv \beta(q + \nu\widetilde{q})$ from the proof of Theorem 7.1 is only a lower semicontinuous function of the number parameter $\nu \geq 0$ since the function $\beta(q)$ according to Lemma 3.2 is lower semicontinuous. It follows, by Borwein and Strojwas (1986, 1987), Theorem 7.1, that points ν where there exists a proximal normal $(\zeta, -\eta) \in PN_{epi\, \overline{\beta}}(\nu, \overline{\beta}(\nu))$ with $\eta > 0$, are dense in the segment $[0, \nu_0]$, $\nu_0 > 0$. As in the proof of Theorem 7.1, the last means that for some $\lambda > 0$ a m.a.s. $u^k \in \mathcal{D}$, $k = 1, 2, \ldots$, in Problem $(\overline{P}_\nu)$ is also m.a.s. (in pair with ν) in Problem (7.1) for $\lambda^i = \lambda$, $\eta^i = \eta$, $\zeta^i = \zeta$, $\nu^i = \nu$. Writing the maximum principle for this m.a.s. in the problem mentioned (see Lemma 7.1), we derive the regular maximum principle for the m.a.s. u^k, $k = 1, 2, \ldots$, in Problem $(P_{q+\nu\widetilde{q}})$. Since we may take the point ν arbitrarily near to zero, it follows that any neighborhood of the element $q \in dom\, \beta$ contains an element $q + \nu\widetilde{q}$ where Problem $(P_{q+\nu\widetilde{q}})$ is regular. Hence, the theorem is proved.

REMARK 7.1 *Using the normals in the sense of Mordukhovich (1988), Mordukhovich and Shao (1996), instead of the proximal normals in the sense of Definition 7.1, we may improve the result of Theorem 7.2. Namely, it may be proved that for any* $q \in dom\beta$ *and for any positive function* $\widetilde{q} \in C(X)$ *in*

8. Illustrative examples

In this section we will consider two illustrative examples in which usual optimal controls do not exist. At the same time, we can apply our results in their analysis.

EXAMPLE 8.1 Let us consider the problem with the well-known functional

$$\int_{\Omega}((z[u](x))^2 - u^2(x))\,dx \to \inf,\ z[u](x) \le q(x),\ x \equiv (x_1, x_2) \in \Omega,$$
$$q \in C(\overline{\Omega}),\ \Omega \equiv [0,1] \times [0,1],\ U \equiv [-1,1],\ n = 2,$$
$$\Delta z - z = u(x),\ x \in \Omega,\ z(x) = 0,\ x \in S.$$

It is easy to see that this "linear-convex" problem satisfies all conditions of Section 2. Since $z[u](x) \equiv 0$ for $u(x) \equiv 0$, then due to Theorem 5.1 and Section 7 the value function β is Lipschitz continuous in a neighborhood of any positive $q \in C(\overline{\Omega})$. The maximum principle of Theorem 4.1 for $q = 0$ may be written in this example in the following form:

$$\int_{\Omega} \max_{v \in [-1,1]} \{\mu_0^i (v^2 - (u^i(x))^2) + \psi^i[u^i](x)(u^i(x) - v)\}\,dx \le \gamma^i,$$

where $\gamma^i \to 0$, $i \to \infty$, $\mu_0^i \ge 0$, $\psi^i[u^i]$ is a solution of the problem

$$\Delta\psi - \psi = -\mu_0^i 2z[u^i](x) - \lambda^i,\ x \in \Omega,\ \psi(x) = 0,\ x \in S$$

with the positive measure $\lambda^i \in M(\Omega)$ having a support in $\{x \in \overline{\Omega} : |z[u^i](x)| \le \gamma^i\}$, $\mu_0^i + |\lambda^i| = 1$. An elementary analysis of these relations shows (see analogous examples in Sumin, 1986, 1996, 1997a, 1997b) that any sequence $u^i \in \mathcal{D}_0^{\gamma^i}$, $i = 1, 2, \ldots$, in our example will satisfy the maximum principle if the following relations hold: $\mu_0^i \to 1$, $\|\psi^i[u^i]\|_{1,\Omega} \to 0$, $\|u^i\|_{1,\Omega} \to 1$, $i \to \infty$. Obviously, one of such sequences has the form $u^i(x) \equiv \{1,\ 2k/2i \le x_1 < (2k+1)/2i,\ -1,\ (2k+1)/2i \le x_1 < (2k+2)/2i,\ k = 0, 1, \ldots, i-1\}$, $|\lambda^i| \to 0$, $i \to \infty$. It is easy to show that this sequence is really the m.a.s. in our example for $q = 0$. Finally, we note that the usual optimal control does not exist in this example for $q = 0$.

EXAMPLE 8.2 Let us consider now the same problem as in Example 8.1 but with the state constraint $z^2[u](x) \le q(x)$ instead of $z[u](x) \le q(x)$. It is easy to see that the control $u(x) \equiv 0$ is a unique control in this optimal control problem for $q = 0$ for which the state constraint holds in the proper classical sense. Therefore, we have the equality $\beta_0(0) = 0$. At the same time, it is easy to show with the same kind of arguments as in Example 8.1 (in this example the adjoint functions $\psi^i[u^i]$ satisfy the same adjoint equation but with the right-hand side $-\mu_0^i 2z[u^i] - 2z[u^i]\lambda^i$) that $\beta(0) = -1$. Thus, we have the strict inequality

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