

On the inverse problems of
Lyapunov theorem and Riccati equation

by

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Abstract: In this note, the inverse problem of Lyapunov theorem is reconsidered and the inverse problem of Riccati equation is introduced. Simple proofs are provided to guarantee the solution of such problems.

Keywords: linear systems theory, inverse problems.

1. Introduction

The inverse problem of the Lyapunov theorem has been proven, by a more complex mathematical proof shown in Sun (1998), but how to determine such equivalent system has not been provided in this reference. We will show, in this note, that the inverse problem of Lyapunov theorem is always true by using a simple proof. Furthermore, it will be proved that, for a given completely controllable system and any given positive definite Hermitian matrix, an equivalent systems always exists such that the given positive definite Hermitian definite matrix is actually the solution to the Riccati equation for the equivalent system.

2. Lyapunov theorem and its inverse problem

For convenience, we define some notations and abbreviations as follows:

$C^{m \times n}$:= the set of all complex m by n matrices,

$P > 0$ (res. $Q < 0$): P is a positive (res. negative) definite Hermitian matrix,

$P \geq 0$ (res. $Q \leq 0$): P is a positive (res. negative) semidefinite Hermitian matrix,

A^* := the conjugate transpose of the matrix A ,

A^{-*} := the conjugate transpose of the inverse of the nonsingular matrix A ,

H_c := the set of all matrices whose eigenvalues have negative real parts,

H_d := the set of all matrices whose eigenvalues have absolute values less than 1.

LEMMA 1 Let $P \in C^{n \times n}$ and $G \in C^{n \times n}$ be positive definite Hermitian matrices.

We will now present the last of our main results.

THEOREM 3 *Let $A \in C^{n \times n}$ and $B \in C^{n \times m}$. If (A, B) is completely controllable, then, for any given positive definite Hermitian matrix $P \in C^{n \times n}$, there exist nonsingular matrices $T \in C^{n \times n}$ and $Q \in C^{n \times n}$ such that*

- (i) $(T^{-1}AT)^*P(T^{-1}AT) - P$
 $- [B^*T^{-*}PT^{-1}AT]^*[I - B^*T^{-*}PT^{-1}B][B^*T^{-*}PT^{-1}AT] = -Q_1^*Q_1 < 0$;
- (ii) $(T^{-1}AT, T^{-1}B)$ is completely controllable;
- (iii) $(T^{-1}AT, Q_1)$ is completely observable.

Proof. By Lemma 3, there exists a matrix $G > 0$ satisfying

$$A^*GA - G - (A^*GB)[I + B^*GB]^{-1}(B^*GA) = -I.$$

By Lemma 1, there exists a nonsingular matrix T satisfying $G = T^{-*}PT^{-1}$. It can be readily obtained that

$$\begin{aligned} & A^*T^{-*}PT^{-1}A - T^{-*}PT^{-1} \\ & - (A^*T^{-*}PT^{-1})[I + B^*T^{-*}PT^{-1}B]^{-1}(B^*T^{-*}PT^{-1}A) = -I \\ & \Rightarrow T^*\{A^*T^{-*}PT^{-1}A - T^{-*}PT^{-1} \\ & - (A^*T^{-*}PT^{-1}B)[I + B^*T^{-*}PT^{-1}B]^{-1}(B^*T^{-*}PT^{-1}A)\}T = -T^*T < 0 \\ & \Rightarrow (T^{-1}AT)^*P(T^{-1}AT) - P \\ & - [B^*T^{-*}PT^{-1}AT]^*[I + B^*T^{-*}PT^{-1}B]^{-1}[B^*T^{-*}PT^{-1}AT] \\ & = -Q_1^*Q_1 < 0, \end{aligned}$$

if we define $Q_1 = T$. Clearly, it can be deduced that $(T^{-1}AT, T^{-1}B)$ is completely controllable and $(T^{-1}AT, Q_1)$ is completely observable in view of the PBH rank test from Chen (1984). This completes our proof. ■

4. Conclusions

In this note, the inverse problem of Lyapunov theorem has been reconsidered and the inverse problem of Riccati equation has been introduced. Simple proofs have been provided to guarantee the solution of such problems.

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