Control and Cybernetics

vol. 28 (1999) No. 4

On influence of the Taylor series remainder on unanticipated rates of return on fixed income bond portfolios

by

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Abstract: Changes in spot rates, unknown appriori to investors, induce unanticipated rates of return on all financial market instruments. In this paper we introduce and investigate a concept of the rest of a bond. The concept is related to the Taylor series remainder and gives a better approximation to an unanticipated rate of return of fixed income bonds and bond portfolios. It is shown that the rest of the portfolio composed of fixed income bonds is a convex combination of the rests of these bonds. A stronger version of the theorem on rates of return on fixed income bond portfolios is given.

Keywords: fixed income bonds, bond portfolios, rate of return, Taylor series, duration, convexity

1. Introduction

Changes in spot rates, unknown appriori to investors, induce unanticipated rates of return on all financial market instruments. For many years the duration and convexity of bonds have been used to investigate these changes (Brooks & Attinger, 1992; Dunetz & Mahoney, 1988; Fabozzi & Fabozzi, 1989; Mehran & Homaifar, 1993). Fabozzi & Fabozzi (1989) observed that the first three terms of the Taylor expansion series of bond investment value function can be used to approximate the changes in the bonds values:

$$\partial P = \frac{\partial P}{\partial K}(\partial K) + \frac{\partial^2 P}{\partial K^2} \cdot \frac{(\partial K)^2}{2!} + \frac{\partial^3 P}{\partial K^3} \cdot \frac{(\partial K)^3}{3!} + R_n \tag{1}$$

where K denotes the YTM (yield to maturity) value and R_n represents meaningless components.

By dividing both sides of above equation by P we get the duration (the first bond. Our aim in this paper is to introduce and investigate a notion of the *rest of bond*, giving a better approximation to an unanticipated rate of return for a percentage change in the rate.

Let bond A pay a coupon of C_t dollars t years from now, where $t \in T = \{t_0, t_1, \ldots, t_n\}$, with t_0 (t_n) being the maturity of a shortest (longest) bond. For a single bond some, or even most, of C_t are equal to zero. Treasury bills with maturity of less than one year are treated as zero-coupon bonds. Thus, bond A can be identified with a sequence of nonnegative coupons $C_{t_0}, C_{t_1}, \ldots, C_{t_n}$ (for simplicity the last payment is treated as a coupon), with each coupon C_t having its own investment value $C_t \cdot (1+y_t)^{-t}$, where y_t is the spot rate for the period of the nearest t years expressed at the annualized basis.

By the investment value of bond A one understands the sum of present values of all coupons generated by A, that is the amount:

$$P_A = P_A(y_{t_0}, y_{t_1}, \ldots, y_{t_n}) = P_A(\overline{y}) = \sum_{t=t_0}^{t_n} C_t \cdot (1+y_t)^{-t} \tag{2}$$

where $\overline{y} = (y_{t_0}, y_{t_1}, \ldots, y_{t_n})$ is called the *term structure* of interest rates. Value P_A can be thought of as a "fair" price of bond A, meaning that if a present price is below (above) P_A, then the bond is underpriced (overpriced) and then it pays to buy (sell) it.

With each coupon C_t of bond A its weight is associated:

$$x_t^A = \frac{C_t \cdot (1+y_t)^{-t}}{P_A} \quad \text{for } t \in \{t_0, t_1, \ldots, t_n\} \tag{3}$$

The sum of all weights of each bond is one. A question arises what would happen to P_A when the spot rates y_t are changed by a central bank's decision that is apriori unknown to the investors, i.e. $y_t \to y_t + h_t$, (or $\overline{y} \to \overline{y} + \overline{h}$ for short). We are interested here in the evaluation of an unanticipated rate of return $\frac{dP_A}{P_A(\overline{y})} = \frac{P_A(\overline{y}+\overline{h}) - P_A(\overline{y})}{P_A(\overline{y})}$ due to unknown changes in the spot rates.

2. Rests of fixed income bonds

Function P_A given by (2) may be expanded in the Taylor series. Hence, we obtain:

$$dP_A = \sum_{t=t_0}^{t_n} \left[\frac{\partial P_A(\overline{y})}{\partial y_t} \cdot h_t + \frac{1}{2} \cdot \frac{\partial^2 P_A(\overline{y})}{\partial y_t^2} \cdot h_t^2 + \frac{1}{6} \cdot \frac{\partial^3 P_A(\overline{y}+\theta)}{\partial y_t^3} \cdot h_t^3 \right] \tag{4}$$

for some $\theta = (\theta_{t_0}, \theta_{t_1}, \ldots, \theta_{t_n})$, $0 \le \theta_t \le h_t$ if $h_t > 0$ and $h_t \le \theta_t \le 0$ if $h_t < 0, t \in \{t_0, \ldots, t_n\}$. Thus,

$$dP_A = -\sum^{t_n} tC_t(1+y_t)^{-t} \cdot \frac{h_t}{1+y_t} +$$

$$+\frac{1}{2}\sum_{t=t_0}^{t_n} t(t+1)C_t(1+y_t)^{-t}\cdot\frac{h_t^2}{(1+y_t)^2}-\epsilon \tag{5}$$

where

$$\epsilon=\frac{1}{6}\sum_{t=t_0}^{t_n} t(t+1)(t+2)C_t(1+y_t+\theta_t)^{-t}\cdot\frac{h_t^3}{(1+y_t+\theta_t)^3},$$

$$\epsilon(\theta_{t_0},\theta_{t_1},\ldots,\theta_{t_n})=\frac{1}{6}\sum_{t=t_0}^{t_n} t(t+1)(t+2)\frac{C_t\cdot h_t^3}{(1+y_t+\theta_t)^{t+3}}.$$

By dividing equation (5) by P_A and using (3), we obtain:

$$\begin{aligned}\frac{dP_A}{P_A} &= -\sum_{t=t_0}^{t_n} tx_t^A\cdot\frac{h_t}{1+y_t}+\frac{1}{2}\sum_{t=t_0}^{t_n} t(t+1)x_t^A\cdot\frac{h_t^2}{(1+y_t)^2}\\ &\quad -\frac{1}{6}\sum_{t=t_0}^{t_n} t(t+1)(t+2)x_t^A\cdot\frac{h_t^3(1+y_t)^t}{(1+y_t+\theta_t)^{t+3}}\end{aligned} \tag{6}$$

where

$$x_t^A=\frac{C_t}{P_A(1+y)^t}.$$

Formula (6) is further analyzed for four cases.

Case I.

This is the most general case. Assume that shifts h_t satisfy the relationship:

$$\frac{h_t}{1+y_t}=g_t\cdot\frac{h_{t_0}}{1+y_{t_0}},\quad t\in\{t_0,t_1,\ldots,t_n\}. \tag{7}$$

The situation where an investor knows g_t rarely happens in practice and, therefore, formula (7) has a theoretical rather than a practical value. If, however, an investor is able to forecast g_t with a "high" degree of accuracy, then he/she is able to find an approximate value of $\frac{dP_A}{P_A}$. Note that

$$h_t=g_t\cdot\frac{1+y_t}{1+y_{t_0}}\cdot h_{t_0}. \tag{8}$$

Replacing in (6) h_t with (8), we obtain:

$$\begin{aligned}\frac{dP_A}{P_A} &= -\sum_{t=t_0}^{t_n} tx_t^A\cdot g_t\frac{h_{t_0}}{1+y_{t_0}}+\frac{1}{2}\sum_{t=t_0}^{t_n} t(t+1)x_t^A g_t^2\cdot\left(\frac{h_{t_0}}{1+y_{t_0}}\right)^2\\ &\quad -\frac{1}{6}\sum_{t=t_0}^{t_n} t(t+1)(t+2)x_t^A\cdot\frac{g_t^3\cdot(1+y_t)^{t+3}\cdot h_{t_0}^3}{(1+y_{t_0})^3(1+y_t+\theta_t)^{t+3}}\end{aligned}$$

$$= -\sum_{t=t_0}^{t_n} tx_t^A \cdot g_t \frac{h_{t_0}}{1+y_{t_0}} + \frac{1}{2}\sum_{t=t_0}^{t_n} t(t+1)x_t^A g_t^2 \cdot \left(\frac{h_{t_0}}{1+y_{t_0}}\right)^2$$
$$-\frac{h_{t_0}^3}{6\left(1+y_{t_0}\right)^3}\sum_{t=t_0}^{t_n} t(t+1)(t+2)x_t^A \cdot \frac{g_t^3 \cdot (1+y_t)^{t+3}}{(1+y_t+\theta_t)^{t+3}}.$$

Now, we can rewrite (6) as follows:

$$\frac{dP_A}{P_A} = -D_I(A)\frac{h_{t_0}}{1+y_{t_0}} + C_I(A)\left(\frac{h_{t_0}}{1+y_{t_0}}\right)^2 - R_I(A)\left(\frac{h_{t_0}}{1+y_{t_0}}\right)^3 \tag{9}$$

where

$$D_I(A) = \sum_{t=t_0}^{t_n} tx_t^A g_t$$

is the duration of bond A,

$$C_I(A) = \frac{1}{2}\sum_{t=t_0}^{t_n} t(t+1)x_t^A g_t^2$$

is the convexity of A, and

$$R_I(A) := \frac{1}{6}\sum_{t=t_0}^{t_n} t(t+1)(t+2)x_t^A \frac{g_t^3 \cdot (1+y_t)^{t+3}}{(1+y_t+\theta_t)^{t+3}} \tag{10}$$

is the *rest* of A. The rest of bond A gives a better approximation to the rate of return on bond A than the *velocity* (see Mehran & Homaifar, 1993).

Case II.
In this case, shifts h_t satisfy equation (7) with $g_t = L^{t-t_0}$, where L is a given (known) constant, $0 < L < 1$. From

$$\frac{h_t}{1+y_t} = L^{t-t_0} \cdot \frac{h_{t_0}}{1+y_{t_0}},\ t \in \{t_0, t_1, \ldots, t_n\}, \tag{11}$$

we obtain the following equation:

$$\frac{dP_A}{P_A} = -D_{II}(A)\frac{h_{t_0}}{1+y_{t_0}} + C_{II}(A)\left(\frac{h_{t_0}}{1+y_{t_0}}\right)^2 - R_{II}(A)\left(\frac{h_{t_0}}{1+y_{t_0}}\right)^3 \tag{12}$$

where $D_{II}(A)$ is the duration of A, $C_{II}(A)$ is the convexity of A, and

$$R_{II}(A) := \frac{1}{6}\sum_{t=t_0}^{t_n} t(t+1)(t+2)x_t^A \frac{(L^{t-t_0})^3(1+y_t)^{t+3}}{(1+y_t+\theta_t)^{t+3}} \tag{13}$$

Case III. (Proportional shifts in spot rates).
In this case, shifts h_t satisfy equation (11) with $L = 1$ and, from

$$\frac{h_t}{1+y_t} = \frac{h_{t_0}}{1+y_{t_0}},\ t \in \{t_0, t_1, \ldots, t_n\}. \tag{14}$$

we obtain that:

$$\frac{dP_A}{P_A} = -D_{III}(A)\frac{h_{t_0}}{1+y_{t_0}} + C_{III}(A)\left(\frac{h_{t_0}}{1+y_{t_0}}\right)^2 - R_{III}(A)\left(\frac{h_{t_0}}{1+y_{t_0}}\right)^3 \tag{15}$$

where $D_{III}(A)$ is the duration of bond A, $C_{III}(A)$ is the convexity of A, and

$$R_{III}(A) := \frac{1}{6}\sum_{t=t_0}^{t_n} t(t+1)(t+2)x_t^A \frac{(1+y_t)^{t+3}}{(1+y_t+\theta_t)^{t+3}} \tag{16}$$

is its *r*est.

Case IV.
This is the simplest case when all the spot rates are identical ($y_t \equiv y$) and $h_t \equiv h$. We have:

$$\frac{dP_A}{P_A} = -D_{IV}(A)\frac{h}{1+y} + C_{IV}(A)\left(\frac{h}{1+y}\right)^2 - R_{IV}(A)(h)^3 \tag{17}$$

where $D_{IV}(A)$ is the duration of A, $C_{IV}(A)$ is the convexity of A, and

$$R_{IV}(A) := \frac{1}{6}\sum_{t=t_0}^{t_n} t(t+1)(t+2)x_t^A \frac{(1+y)^t}{(1+y+\theta_t)^{t+3}} \tag{18}$$

is the *r*est.

3. Rests of bond portfolios

While considering bond portfolios we assume that shifts h_t satisfy equation (7). Therefore, we focus on Case I. However, since Case I (7) is the most general one, the statements presented below are valid for Cases II, III, IV, with appropriately modified coefficients g_t.

A bond portfolio $P = (O_1, O_2, \ldots, O_r)$ is a collection of single bonds O_k. Its investment value P^* is defined as the sum of the investment values of all bonds O_k ($k = 1, 2, \ldots, r$) present in P, that is,

$$P^* = \sum_{k=1}^{r} P_k, \quad P_k = a_k P_{O_k}(\overline{y}), \tag{19}$$

where a_k is the number of bonds O_k present in P.

The concept of duration (convexity) for bond portfolios was defined similarly portfolio $P = (O_1, \ldots, O_r)$ is a convex combination of durations (convexities) of bonds O_k $(k = 1, 2, \ldots, r)$, that is

$$D_I(P) = \sum_{k=1}^{r} x_k D_I(O_k), \quad x_k = \frac{P_k}{P^*}, \tag{20}$$

$$C_I(P) = \sum_{k=1}^{r} x_k C_I(O_k), \quad x_k = \frac{P_k}{P^*}, \tag{21}$$

where P_k is the investment value of bond O_k and $P^* = \sum_{k=1}^{r} P_k$.

Proceeding in the same way, we introduce the concept of the rest of a bond portfolio. The rest of portfolio P is given by:

$$R_I(P) := \frac{1}{6} \sum_{t=t_0}^{t_n} t(t+1)(t+2) x_t^P \frac{g_t^3 \cdot (1+y_t)^{t+3}}{(1+y_t+\theta_t)^{t+3}}$$

THEOREM 3.1 *If $P = (O_1, O_2, \ldots, O_r)$ is a bond portfolio, then its rest is a convex combination of the rests of bonds O_m, $m = 1, 2, \ldots, r$, that is*

$$R_I(P) = \sum_{k=1}^{r} x_k R_I(O_k), \quad x_k = \frac{P_k}{P^*} \tag{22}$$

where P_k is the investment value of bond O_k, while $P^* = \sum_{k=1}^{r} P_k$.

Proof. Let C_t^k denote the coupon of bond O_k payable at time t. Using (10), we obtain:

$$\begin{aligned} x_k \cdot R_I(O_k) &= \frac{P_k}{P^*} \cdot R_I(O_k) \\ &= \frac{P_k}{6P^*} \sum_{t=t_0}^{t_n} t(t+1)(t+2) x_t^k \frac{g_t^3 (1+y_t)^{t+3}}{(1+y_t+\theta_t^k)^{t+3}} \\ &= \frac{P_k}{6P^*} \sum_{t=t_0}^{t_n} t(t+1)(t+2) \frac{g_t^3 (1+y_t)^{t+3}}{(1+y_t+\theta_t^k)^{t+3}} \frac{C_t^k}{P_k (1+y_t)^t} \\ &= \frac{1}{6} \sum_{t=t_0}^{t_n} t(t+1)(t+2) \frac{g_t^3 (1+y_t)^3}{\left(1+y_t+\theta_t^k\right)^{t+3}} \cdot \frac{C_t^k}{P^*(1+y_t)^t} \end{aligned}$$

for $R_I(P)$, because $x_t^P = \sum_{k=1}^{r} \frac{C_t^k}{P^*(1+y_t)^t}$.

$$\begin{aligned} \sum_{k=1}^{r} & x_k \cdot R_I(O_k) \\ &= \sum_{k=1}^{r} \left[\frac{1}{6} \sum_{t=t_0}^{t_n} t(t+1)(t+2) \frac{g_t^3(1+y_t)^{t+3}}{(1+y_t+\theta_t^1)^{t+3}} \cdot \frac{C_t^k}{P^*(1+y_t)^t} \right] \\ &= \frac{1}{6} \sum_{t=t_0}^{t_n} t(t+1)(t+2) \cdot g_t^3 (1+y_t)^{t+3} \cdot \Psi_t(\theta_t) \end{aligned} \tag{23}$$

where

$$\begin{aligned} \Psi_t(\theta_t) &= \sum_{k=1}^{r} \frac{1}{(1+y_t+\theta_t)^{t+3}} \cdot \frac{C_t^k}{P^*(1+y_t)^t} \\ &= \frac{1}{(1+y_t+\theta_t)^{t+3}} \left[\frac{C_t^1}{P^*(1+y_t)^t} + \ldots + \frac{C_t^r}{P^*(1+y_t)^t} \right] \end{aligned}$$

Note that for any fixed $t \in \{t_0, \ldots, t_n\}$, $\Psi_t(\theta_t)$ is a continuous and differentiable function, and

$$\Psi_t'(\theta_t) = \frac{-(t+3)}{(1+y_t+\theta_t)^{t+4}} \cdot C$$

where $C = \frac{C_t^1}{P^*(1+y_t)^t} + \ldots + \frac{C_t^r}{P^*(1+y_t)^t}$ is a constant.

One may notice that Ψ_t is a decreasing function for $\theta_t \in (0, h_t)$ if $h_t > 0$ or for $\theta_t \in (h_t, 0)$ if $h_t < 0$. By the Darboux property there exists $\overline{\theta_t}$ such that:

$$\Psi_t(0) > \Psi_t(\overline{\theta_t}) > \Psi_t(h_t) \quad \text{for } h_t > 0$$

or

$$\Psi_t(h_t) > \Psi_t(\overline{\theta_t}) > \Psi_t(0) \quad \text{for } h_t < 0$$

and

$$\Psi_t(\overline{\theta_t}) = A$$

where

$$A = \frac{1}{(1+y_t+\theta_t^1)^{t+3}} \cdot \frac{C_t^1}{P^*(1+y_t)^t} + \ldots + \frac{1}{(1+y_t+\theta_t^r)^{t+3}} \cdot \frac{C_t^r}{P^*(1+y_t)^t}.$$

Therefore, for any $t \in \{t_0, \ldots, t_n\}$

$$\begin{aligned} \Psi_t(\overline{\theta_t}) &= \frac{1}{(1+y_t+\overline{\theta_t})^{t+3}} \left[\frac{C_t^1}{P^*(1+y_t)^t} + \ldots + \frac{C_t^r}{P^*(1+y_t)^t} \right] \\ &= \frac{1}{(1+y_t+\overline{\theta_t})^{t+3}} \cdot x_t^P \end{aligned} \tag{24}$$

Substituting (24) into (23) we get:

$$\sum_{k=1}^{r} x_k \cdot R_I(O_k) = \frac{1}{6}\sum_{t=t_0}^{t_n} t(t+1)(t+2) \cdot \frac{g_t^3(1+y_t)^{t+3}}{\left(1+y_t+\overline{\theta}_t\right)^{t+3}} \cdot x_t^P = R_I(P)$$

which completes the proof. ■

Theorem 3.1 can be used to extend the theorem on unanticipated rates of return on bond portfolios (Zaremba, 1995).

THEOREM 3.2 *If $P = (O_1, O_2, \ldots, O_r)$ is a bond portfolio, then the unanticipated rate of return on P due to shifts h_t in spot rates y_t that satisfy the relationship (7) is given by the formula:*

$$\frac{dP_P}{P_P} = -D_I(P) \cdot \frac{h_{t_0}}{1+y_{t_0}} + C_I(P) \cdot \left(\frac{h_{t_0}}{1+y_{t_0}}\right)^2 - R_I(P)\left(\frac{h_{t_0}}{1+y_{t_0}}\right)^3 \quad (25)$$

where P_P stands for the investment value of the bond portfolio P, while $D_I(P)$, $C_I(P)$, $R_I(P)$ denote the duration, convexity and rest of P, respectively.

Proof. From $P_P = \sum_{k=1}^{r} P_k$, where P_k is the investment value of O_k, we obtain $dP_P = \sum_{k=1}^{r} dP_k$. Further, because equation (9) holds for each bond O_k, the unanticipated rate of return on P satisfies the following:

$$\begin{aligned}
\frac{dP_P}{P_P} &= \sum_{k=1}^{r} \frac{dP_k}{P_k} \cdot \frac{P_k}{P_P} = \sum_{k=1}^{r} \frac{P_k}{P_P}\left(\frac{dP_k}{P_k}\right) \\
&= -\sum_{k=1}^{r} \frac{P_k}{P_P} \cdot D_I(O_k) \cdot \frac{h_{t_0}}{1+y_{t_0}} + \sum_{k=1}^{r} \frac{P_k}{P_P} \cdot C_I(O_k) \cdot \left(\frac{h_{t_0}}{1+y_{t_0}}\right)^2 \\
&\quad -\sum_{k=1}^{r} \frac{P_k}{P_P} \cdot R_I(O_k)\left(\frac{h_{t_0}}{1+y_{t_0}}\right)^3 \\
&= -D_I(P) \cdot \frac{h_{t_0}}{1+y_{t_0}} + C_I(P) \cdot \left(\frac{h_{t_0}}{1+y_{t_0}}\right)^2 - R_I(P)\left(\frac{h_{t_0}}{1+y_{t_0}}\right)^3
\end{aligned}$$

where $D_I(P)$ is the duration of P given by (20), and $C_I(P)$ is the convexity of P given by (21). ■

4. Conclusions

In this paper, we used the Taylor series remainder to introduce and analyze the notion of the rest of fixed income bonds and bond portfolios. First, we showed that the rest of a bond portfolio, being a collection of single bonds, is a convex combination of the rests of these bonds. Further, a stronger version of

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