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An occupational measure solution to a singularly perturbed optimal control problem

by

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Abstract: Employing limit occupational measures, we provide an explicit solution to a singularly perturbed optimal control problem for which the order reduction method does not apply.

Keywords: singular perturbations, optimal control, infinite horizon, limit occupational measures

1. Introduction

The main objective of the paper is to provide a complete solution to the following singularly perturbed optimal control problem.

$$\begin{aligned} &\text{maximize} \quad \int_0^1 |y_1(t) - 2y_2(t)|dt \\ &\text{subject to} \quad \varepsilon\frac{dy_1}{dt} = -y_1 + u \\ &\qquad\qquad\quad\ \varepsilon\frac{dy_2}{dt} = -2y_2 + u, \end{aligned} \tag{1.1}$$

with y_1 and y_2 scalars, and $u \in [-1, 1]$. To this end, we examine a general class of optimal control problems that (1.1) belongs to, namely,

$$\begin{aligned} &\text{maximize} \quad \int_0^1 c(y(t))dt \\ &\text{subject to} \quad \varepsilon\frac{dy}{dt} = g(y, u), \end{aligned} \tag{1.2}$$

with $y \in R^m$ and $u \in R^k$, and $c(y) : R^m \to R$ a continuous function.

In both the general case (1.2) and the particular case (1.1), we are interested in solving the problem for small $\varepsilon > 0$, namely, we wish to reveal the limit

A tool for the determination of the aforementioned limit behaviors is to identify a variational limit problem, namely, an optimal control problem whose value is the limit of the values of (1.2), and whose solutions can be used to generate solutions of (1.2) for small ε. We show in this paper that, under certain conditions, a specific infinite horizon problem is an appropriate variational limit of (1.2), and indicate how its solutions generate the near optimal solutions to (1.2). For the specific case of (1.1) we offer an explicit feedback solution.

We wish to point up front that the optimal control problem obtained by plugging $\varepsilon = 0$ in (1.1), referred to in the literature as the order reduction method, would not produce an appropriate variational limit. The order reduction method of solving singularly perturbed problems by addressing the case with $\varepsilon = 0$, has been proven useful in many circumstances, with remarkable applications, see Kokotovic and Khalil (1986), Kokotovic, Khalil and O'Reilly (1986), and references therein. However, for the method to be applied certain conditions have to be met, and without these conditions the method may provide wrong solutions. Veliov (1996, Example 5), stated problem (1.1), pointing out that the order reduction does not apply. Indeed, the system (1.1) with $\varepsilon = 0$ yields an optimal value equal to 0. Yet it is easy to see that a higher value can be obtained in the limit as $\varepsilon \to 0$. Several approaches to overcome the difficulty have been suggested, see Artstein (1999), (2000), Artstein and Gaitsgory (1997a), (1997b), Artstein and Vigodner (1996), Gaitsgory (1992), (1993), Gaitsgory and Leizarowitz (1999), Vigodner (1997). The particular structure of the problem (1.2) enables the use of ideas worked out in Artstein (1999), (2000), Artstein and Gaitsgory (1997a), Artstein and Vigodner (1996) and Vigodner (1997), and with some additional observations, an explicit solution can be reached.

In Section 2 of this paper we examine the general case (1.2). We introduce a solution concept, find how it is related to the limit occupational measures of the differential equation on the fast scale, and relate these to finitely optimal solutions on the infinite horizon. Section 3 is devoted to the examination of the concrete problem (1.1), resulting in the solution.

Acknowledgement. In my conference talk I presented a general account of the role of invariant measures in forming variational limits. Following my talk Vladimir Veliov presented to me problem (1.1), originally introduced in his paper (1996), and suggested that the techniques I mentioned may be applied. Vladimir's observation was correct, and I am indebted to him for offering the problem and for very helpful discussions.

2. The general case

In this section we examine a solution notion for the general case (1.2) and, under certain conditions, establish existence. The derivations employ several approaches available in the literature and, along with some new results, make

We start with some terminology and notations. When referring to a function we either use a dot in the argument, e.g. $y(\cdot)$, or use a boldface font, for example $\mathbf{y}$. The value and the solution of (1.2) may depend on an initial condition, say $y(0) = y_0$. An admissible trajectory of the differential equation in (1.2) is a pair $(y(\cdot), u(\cdot))$ defined on $[0,1]$ with $u(\cdot)$ measurable, satisfying the equation $\frac{dy}{dt} = g(y(t), u(t))$. When convenient, we refer to the state coordinate $y(\cdot)$ of an admissible trajectory as an admissible trajectory. Indeed, the payoff of the admissible trajectory $(\mathbf{y}, \mathbf{u})$ depends on $\mathbf{y}$ only, as determined by (1.2). We write

$$\text{payoff}(\mathbf{y}) = \int_0^1 c(y(t))\, dt. \tag{2.1}$$

Notice that $\mathbf{y}$ may be generated by more than one control.

For a fixed $\varepsilon > 0$ and a fixed initial condition y_0 we define

$$\text{val}\,(\varepsilon, y_0) = \sup\{\text{payoff}(\mathbf{y}) : (\mathbf{y}, \mathbf{u}) \text{ admissible and } y(0) = y_0\}. \tag{2.2}$$

We are interested in the limit of $\text{val}(\varepsilon, y_0)$ as $\varepsilon \to 0$, and we want to have a procedure to generate optimal or near optimal solutions for small ε.

As a tool in the analysis of (1.2), consider the differential equation

$$\frac{dy}{ds} = g(y, u) \tag{2.3}$$

obtained from (1.2) by the change of time scales $t = \varepsilon s$. Notice that solving (1.2) on the time interval $[0,1]$ is equivalent to maximizing

$$\varepsilon \int_0^{\varepsilon^{-1}} c(y(s))\, ds \tag{2.4}$$

subject to solving equation (2.3) on $[0, \varepsilon^{-1}]$. Thus, the limit as $\varepsilon \to 0$ of (1.2) is related to an infinite horizon problem. We wish to make this relation apparent.

CONVENTION 2.1 *We need to examine a given trajectory as a function of both time scales t and s. When a function is considered as a function of s we put a bar over it.*

The following notion reflects our version of a near optimal solution of (1.2).

DEFINITION 2.1 *A limiting solution of* (1.2) (*with initial condition* y_0) *is an admissible pair* $(\overline{\mathbf{y}}, \overline{\mathbf{u}})$ *defined on* $[0, \infty)$ *and satisfying* $\overline{y}(0) = y_0$, *such that the trajectories* $(\mathbf{y}_\varepsilon, \mathbf{u}_\varepsilon)$ *obtained from* $(\overline{\mathbf{y}}, \overline{\mathbf{u}})$ *by restricting it to* $[0, \varepsilon^{-1}]$ *and then applying the change of time scale* $t = \varepsilon s$, *satisfy* $\text{val}(\varepsilon, y_0) - \text{payoff}(\mathbf{y}_\varepsilon) \to 0$ *as*

In the sequel we relate the limit value and limiting solutions of (1.2) to limit occupational measures of the equation (2.3). We then establish, under a controllability condition, existence of limiting solutions. Finally, we consider an approach to detection of the limiting solutions and point out a relation to an infinite horizon optimization problem.

Trying to eliminate the less relevant complications, we work throughout under the following assumption:

HYPOTHESIS 2.1 *For a given initial condition $y(0) = y_0$ there exists a family of admissible trajectories $(\mathbf{y}_\varepsilon, \mathbf{u}_\varepsilon)$ such that* $\text{val}(\varepsilon, y_0) - \text{payoff}(\mathbf{y}_\varepsilon) \to 0$ *as* $\varepsilon \to 0$, *and such that the values $y_\varepsilon(t)$ for all t and all ε belong to a bounded closed set D in R^m.*

The preceding hypothesis assumes, in particular, that $y_0 \in D$. In the sequel we need only that with equation (2.3) the initial condition y_0 can be steered into D in a finite time (on the s scale). We leave out the details of this possibility.

Recall the following terminology. Let $\overline{\mathbf{y}} : [S_1, S_2] \to R^m$ be given. The occupational measure associated with $\overline{\mathbf{y}}$ is the probability measure $\mu(\overline{\mathbf{y}}, [S_1, S_2])$ on R^m given by

$$\mu(\overline{\mathbf{y}}, [S_1, S_2])(B) = \frac{1}{S_2 - S_1}\lambda\{s : y(s) \in B,\ S_1 \le s \le S_2\} \tag{2.5}$$

where λ is the Lebesgue measure on the real line. We need also the concept of weak convergence of probability measures, which we recall for completeness (see, e.g., Billinglsley, 1968). The sequence of probability measures μ_i on R^m converges weakly to μ_0 if

$$\int_{R^m} \gamma(y)\mu_i(dy) \to \int_{R^m} \gamma(y)\mu_0(dy) \tag{2.6}$$

for every bounded and continuous $\gamma : R^m \to R$.

DEFINITION 2.2 *A probability measure μ_0 on R^m is a limit occupational measure of (2.3) if there are admissible solutions $(\overline{\mathbf{y}}_i, \overline{\mathbf{u}}_i)$ to (2.3), defined, respectively, on intervals $[S_{i,1}, S_{i,2}]$, such that $S_{i,2} - S_{i,1} \to \infty$ as $i \to \infty$, and such that the corresponding occupational measures $\mu_i = \mu(\overline{\mathbf{y}}, [S_{i,1}, S_{i,2}])$ converge to μ_0. We say then that μ_0 is generated by the sequence $(\overline{\mathbf{y}}_i, \overline{\mathbf{u}}_i)$.*

With a limit occupational measure μ we associate the value

$$\text{val}(\mu) = \int_{R^m} c(y)\mu(dy). \tag{2.7}$$

Arguments similar to the following observation were used in the literature in even more general situations, see Artstein (1999), Artstein and Gaitsgory

PROPOSITION 2.1 *Suppose that Hypothesis* 2.1 *holds. Suppose that* μ^* *is a maximizer of* $\mathrm{val}(\mu)$ *among all limit occupational measures supported on* D *and generated by solutions* $\overline{\mathbf{y}}_i$ *with values in* D. *Let* $(\mathbf{y}_\varepsilon, \mathbf{u}_\varepsilon)$ *be admissible trajectories of* (1.2) *on* $[0,1]$ *with* $y_\varepsilon(t) \in D$ *for all* t *and all* ε. *Then* $\limsup \mathrm{val}(\varepsilon, y_\varepsilon(0)) \leq \mathrm{val}(\mu^*)$.

Proof (sketched). Let $\mathbf{y}_{\varepsilon_i}$ be a subsequence such that

$$\lim \mathrm{payoff}(\mathbf{y}_{\varepsilon_i}) = \limsup \mathrm{val}(\varepsilon, y_\varepsilon(0)). \tag{2.8}$$

The boundedness implies that a further subsequence exists, and we can assume it is the subsequence itself, which converges to a mapping $\nu(\cdot)$, which assigns to each $t \in [0,1]$ a probability measure on R^m, and the convergence is in the sense that

$$\int_0^1 \gamma(t, y_{\varepsilon_i}(t))dt \;\rightarrow\; \int_0^1 \int_{R^m} \gamma(t,y)\nu(t)(dy) \tag{2.9}$$

for every bounded and continuous real function $\gamma(\cdot,\cdot)$. (The probability measure valued maps are called Young measures, and the convergence is referred to as the narrow convergence, or statistical convergence, or convergence in the sense of Young measures.) The particular choice of $\gamma(t,y) = c(y)$ implies then that

$$\lim \mathrm{payoff}(\mathbf{y}_{\varepsilon_i}) = \int_0^1 \mathrm{val}(\nu(t))\, dt. \tag{2.10}$$

Now, the proof would be complete if we show that for almost every t the measure $\nu(t)$ is a limit occupational measure. Indeed, it is clear that $\nu(t)$ is supported on D, hence the right hand side of (2.10) is bounded by $\mathrm{val}(\mu^*)$. To verify that for a given $t_0 \in [0,1)$ the probability measure $\nu(t_0)$ is a limit occupational measure, we employ the change of time scales $s = \varepsilon^{-1}(t - t_0)$. We choose $S_\varepsilon \to \infty$ such that $\varepsilon S_\varepsilon \to 0$. On the one hand the occupational measures $\mu(\overline{\mathbf{y}}_{\varepsilon_i}, [0, S_{\varepsilon_i}])$ converge to the family of limit occupational measures, while on the other hand, they converge for almost every t_0 to $\nu(t_0)$. This completes the proof. ■

Under some conditions the value of the maximizer μ^* mentioned in Proposition 2.1 is related to limiting solutions and to the limit as $\varepsilon \to 0$ of $\mathrm{val}(\varepsilon, y_0)$ for every y_0, as follows:

DEFINITION 2.3 *Equation* (2.3) *has the finite controllability property in the region* $D \subseteq R^m$ *if there is a time* S' *such that for every initial condition* y_0 *and any terminal condition* y_1 *in* D, *there exists an admissible trajectory* $(\overline{\mathbf{y}}, \overline{\mathbf{u}})$ *of* (2.3), *defined on the interval* $[0, S']$ *such that* $\overline{y}(0) = y_0$, $\overline{y}(S') = y_1$ *and* $\overline{y}(s) \in D$ *for all* $0 \leq s \leq S'$. *We say then that* $\overline{\mathbf{y}}$ *steers* y_0 *to* y_1. (*See Remark 2.1 below for an extension.*)

PROPOSITION 2.2 *Suppose that Hypothesis* 2.1 *holds and that* (2.3) *has the finite controllability property on D. Then, there exists a probability measure μ^* which is a maximizer of* $\text{val}(\mu)$ *among all limit occupational measures supported on D and generated by solutions to* (2.3) *with values in D. Furthermore,*

$$\text{val}(\varepsilon, y_0) \to \text{val}(\mu^*) \tag{2.11}$$

for every $y_0 \in D$. Moreover, a limiting solution of (1.2) *exists.*

Proof. Existence of the maximizer μ^* follows from a simple compactness argument. Let $(\overline{\mathbf{y}}_i, \overline{\mathbf{u}}_i)$ be admissible trajectories of (2.3) generating μ^*, namely, they are defined, respectively, on intervals $[0, S_i]$, and satisfy $\overline{y}_i(s) \in D$ for every i and every $s \in [0, S_i]$. (Since the equation is time invariant we can assume that all the intervals start at $s = 0$.) By repeating, possibly, an index i several times, we may assume that $S_i(S_1 + \cdots + S_{i-1})^{-1} \to 0$ as $i \to \infty$.

Let $y_0 \in D$ be given. Consider the admissible trajectory $(\overline{\mathbf{y}}, \overline{\mathbf{u}})$ constructed by a concatenation procedure, as follows. On $[0, S']$ let the trajectory be the one steering y_0 to $y_1(0)$ as guaranteed by Definition 2.3. Denote $S' = s_1$. On $[s_1, s_1 + S_1]$ we define $(\overline{\mathbf{y}}, \overline{\mathbf{u}})$ to be the shift by s_1 of $(\overline{\mathbf{y}}_1, \overline{\mathbf{u}}_1)$. Inductively, on $[s_i, s_i + S_i]$ let $(\overline{\mathbf{y}}, \overline{\mathbf{u}})$ be the shift by s_i of $(\overline{\mathbf{y}}_i, \overline{\mathbf{u}}_i)$. On $[s_i + S_i, s_i + S_i + S']$ let $(\overline{\mathbf{y}}, \overline{\mathbf{u}})$ be the admissible trajectory that steers $\overline{y}(s_i + S_i)$ to $\overline{y}_{i+1}(0)$. Define $s_{i+1} = s_i + S_i + S'$. As $i \to \infty$ the procedure yields a well defined admissible trajectory on $[0, \infty)$.

Now, the condition $S_i(S_1 + \cdots + S_{i-1})^{-1} \to 0$ as $i \to \infty$, together with the weak convergence criterion, imply that

$$\frac{1}{S} \int_0^S c(y(s))ds \to \text{val}(\mu^*) \tag{2.12}$$

as $S \to \infty$. In view of Proposition 2.1 this implies that $(\overline{\mathbf{y}}, \overline{\mathbf{u}})$ is indeed a limiting solution, and that for every y_0 the number $\text{val}(\mu^*)$ is the limit of $\text{val}(\varepsilon, y_0)$ as $\varepsilon \to 0$. This concludes the proof. ∎

In view of the preceding result, a reasonable approach to finding a limiting solution to (1.2) would be to identify a limit occupational measure, say μ^*, which maximizes the value function in a set D given by Hypothesis 2.1, and then (if finite controllability holds on D) to use the concatenation of trajectories that generate μ^* to come up with a solution as given in the preceding proposition. A method to identify such a μ^* is to examine optimal solutions on long intervals, as follows:

DEFINITION 2.4 *An admissible trajectory $(\overline{\mathbf{y}}, \overline{\mathbf{u}})$ of equation* (2.3), *defined on $[S_1, S_2]$, is relatively optimal (with respect to the payoff function $c(y)$) if*

$$\int^{S_2} c(\overline{y}(s))\,ds \geq \int^{S_2} c(\overline{z}(s))\,ds \tag{2.13}$$

holds whenever $\overline{\mathbf{z}}$ *is an admissible trajectory on* $[S_1, S_2]$ *satisfying* $\overline{z}(S_1) = \overline{y}(S_1)$ *and* $\overline{z}(S_2) = \overline{y}(S_2)$.

PROPOSITION 2.3 *Under Hypothesis* 2.1, *and when equation* (2.3) *has the finite controllability property on the set* D, *let* $(\overline{\mathbf{y}}_i, \overline{\mathbf{u}}_i)$ *be relatively optimal trajectories of equation* (2.3), *defined on intervals* $[0, S_i]$ *such that* $S_i \to \infty$, *and such that the values* $\overline{y}_i(s) \in D$ *for all* i *and all* $s \in [0, S_i]$. *Suppose that* $\mu_i = \mu(\overline{\mathbf{y}}, [0, S_i])$ *converge to the probability measure* μ_0. *Then* $\mathrm{val}(\mu_0)$ *is maximal among the limit occupational measures in* D, *which are generated by admissible trajectories with values in* D.

Proof. Suppose that μ_0 is not a maximizer as described, and let μ^* be such a maximizer. Existence of μ^* was noted in Proposition 2.2. Let $\overline{\mathbf{z}}_j$ be the admissible trajectories of (2.3), defined, respectively, on intervals $[0, \sigma_j]$, such that $\overline{z}_j(s) \in D$ for every j and every $s \in [0, \sigma_j]$, and such that $\overline{\mathbf{z}}_j$ generate the limit occupational measure μ^*. As noted earlier, by repeating, possibly, an index j several times, we may assume that $\sigma_j(\sigma_1 + \ldots + \sigma_{j-1})^{-1} \to 0$ as $j \to \infty$.

The idea behind the proof is to replace on an interval $[0, S_{i_0}]$ for i_0 large enough, the trajectory $\overline{\mathbf{y}}_i$ by a trajectory related to $\overline{\mathbf{z}}_j$ for j large enough (gluing may be needed), maintaining the boundary conditions, yet improving the resulting payoff. This would contradict the relative optimality. The estimates are as follows:

Denote

$$\mathrm{val}(\mu^*) - \mathrm{val}(\mu_0) = \Delta_0 > 0. \tag{2.14}$$

Let $\eta > 0$ be a small number which will be determined later. For i large enough the inequality

$$|\mathrm{val}(\mu(\overline{\mathbf{y}}_i, [0, S_i])) - \mathrm{val}(\mu_0)| < \eta \tag{2.15}$$

holds. Let $\beta = \max\{|c(y) - c(z)| : y \in D, z \in D\}$, and let S' be determined by Definition 2.3. For j large enough both inequalities

$$|\mathrm{val}(\mu(\overline{\mathbf{z}}_j, [0, \sigma_j])) - \mathrm{val}(\mu^*)| < \eta \tag{2.16}$$

and

$$\beta S' \sigma_j < \eta \tag{2.17}$$

hold. We fix an index j_0 for which (2.16) and (2.17) are satisfied and then fix an i_0 large enough such that

$$\beta(\sigma_{j_0} + S')S_{i_0}^{-1} < \eta. \tag{2.18}$$

In particular, if k is defined by

then the inequality

$$\beta(S_{i_0} - k\sigma_{j_0})S_{i_0}^{-1} < \eta \tag{2.20}$$

holds. Now, we use the controllability condition to produce an admissible trajectory which steers $\overline{z}_{j_0}(\sigma_{j_0})$ to $\overline{z}_{j_0}(0)$ on the interval $[0, S']$ as guaranteed by Definition 2.3. When this trajectory is concatenated with $\overline{\mathbf{z}}_{j_0}$, we get a periodic trajectory on an interval of length $\sigma_{j_0} + S'$, which we denote by $\overline{\mathbf{z}}'$. It is clear from (2.16) and (2.17) that the following estimate holds for $\overline{\mathbf{z}}'$:

$$|\mathrm{val}(\mu(\overline{\mathbf{z}}', [0, \sigma_{j_0} + S'])) - \mathrm{val}(\mu^*)| < 2\eta. \tag{2.21}$$

The latter estimate clearly holds also for the trajectory obtained by iterating $\overline{\mathbf{z}}'$ for several periods. We consider $(k+1)$ such iterations, with k determined by (2.19). Denote the resulting periodic trajectory on $[0, (k+1)(\sigma_{j_0} + S')]$ by $\overline{\mathbf{z}}'$ also.

The last two modifications of $\overline{\mathbf{z}}'$ are to replace it on the interval $[0, S']$ by a trajectory which steers $\overline{y}(0)$ to $\overline{z}'(S')$ and replace it on the interval $[S_{i_0} - S', S_{i_0}]$ by a trajectory steering $\overline{z}'(S_{i_0} - S')$ to $\overline{y}(S_{i_0})$. In both cases the modification can be done with the values of the trajectories being in D. Both modifications are possible in view of the finite controllability property given in Definition 2.3. Denote the resulting trajectory on $[0, S_{i_0}]$ by $\overline{\mathbf{z}}''$.

The estimates (2.16), (2.17), (2.18), (2.20) and (2.21) imply that

$$\mathrm{val}(\mu^*) - \mathrm{val}(\mu(\overline{\mathbf{z}}'', [0, S_{i_0}])) < 5\eta. \tag{2.22}$$

The latter inequality together with (2.15) imply that if η is chosen such that $6\eta < \Delta_0$, when Δ_0 is given by (2.14), then the relative optimality of $\overline{y}$ on $[0, S_{i_0}]$ is refuted. This completes the proof. ∎

REMARK 2.1 *The controllability property given in Definition* 2.3 *requires the steering to hold on intervals of the prescribed length* S'. *Proposition* 2.2 *would be valid also if the steering is guaranteed only for an interval of length less than or equal to* S'. *If in addition we assume that the steering time is continuous as a function of the initial point, Proposition* 2.3 *would still be valid. We leave out the details.*

A relation of the preceding analysis to an infinite horizon optimization problem is as follows.

DEFINITION 2.5 *An admissible trajectory* $(\overline{\mathbf{y}}, \overline{\mathbf{u}})$ *of equation* (2.3), *defined on* $[0, \infty)$, *is finitely optimal (with respect to the payoff function* $c(y)$*) if it is relatively optimal on each subinterval* $[S_1, S_2]$.

THEOREM 2.1 *Under Hypothesis* 2.1, *and when equation* (2.3) *has the finite* *respect to the criterion* $c(y)$, *say* $(\overline{\mathbf{y}}, \overline{\mathbf{u}})$, *is a limiting solution to* (1.2). *Furthermore,*

$$\lim_{\varepsilon \to 0} \mathrm{val}(\varepsilon, y_0) = \lim_{S \to \infty} \frac{1}{S} \int_0^S c(\overline{y}(s))\, ds \tag{2.23}$$

for every $y_0 \in D$.

Proof. Let $\mathbf{y}_\varepsilon$ be obtained from $\overline{\mathbf{y}}$ by a restriction to $[0, \varepsilon^{-1}]$ and then the change of variables $t = \varepsilon s$. Let M be the family of limit occupational measures supported on D, which maximize the criterion $\mathrm{val}(\mu)$ and are generated as the narrow limit of admissible trajectories with values in D. It is clear that M is closed with respect to the weak convergence. In view of Proposition 2.3, a cluster point of $\mathbf{y}_\varepsilon$ in the narrow convergence (see (2.9)) is a measure valued map, say $\nu(\cdot)$, such that $\nu(t) \in M$ for almost every t. Since $\mathrm{payoff}(\mathbf{y}_\varepsilon)$ converges to $\int_0^1 \mathrm{val}(\nu(t))dt$, it follows from Proposition 2.1 that $(\overline{\mathbf{y}}, \overline{\mathbf{u}})$ is a limiting solution to (1.2). Equality (2.23) follows then from Proposition 2.3 (in fact, a more relaxed form follows, namely, the averaged integration in the right hand side may be taken over intervals $[S_1, S_2]$ as long as $S_2 - S_1$ tends to ∞). ■

The notion of finite optimality as given in Definition 2.5 and used in Theorem 2.1, is satisfied practically by all natural notions of optimality over an infinite interval, see Carlson et al. (1991). In particular, conditions guaranteeing existence of solutions of an overtaking nature imply existence of finitely optimal trajectories, see Carlson et al. (1991). For completeness, we now display a simple sufficient condition for the existence of finitely optimal trajectories. To this end denote by $v(\underline{y}, \widehat{y}, S)$ the maximal payoff obtained when the initial condition $\underline{y}$ is steered to the terminal condition $\widehat{y}$ along the interval $[0, S]$, with a trajectory whose values are in the set D.

PROPOSITION 2.4 *Assume that Hypothesis* 2.1 *holds and that* (2.3) *has the finite controllability property on* D. *Assume that on a finite time interval the set of admissible trajectories* $\mathbf{y}$ *of* (2.3) *is closed with respect to the sup norm. If for any fixed initial condition* $\underline{y} \in D$, *for* S *large enough, the mapping* $v(\underline{y}, \widehat{y}, S)$ *is continuous in the variable* $\widehat{y}$ *on the domain where* $v(\underline{y}, \widehat{y}, S)$ *is finite, then a finitely optimal trajectory of* (2.3) *(with respect to the criterion* $c(y)$*) exists.*

Proof. The closedness of the family of solutions implies that for every $(\underline{y}, \widehat{y}, S)$ an optimal solution to the steering problem exists. The continuity of the mapping $v(\underline{y}, \widehat{y}, S)$ in the variable $\widehat{y}$ implies that if a sequence of optimal solutions, say $\mathbf{y}_i$, satisfies $y(0) = \underline{y}$ and $y(S_i) = y_i$ and converges in the sup norm on, say, $[0, S]$, to a trajectory, say $\mathbf{y}_0$, then $\mathbf{y}_0$ is an optimal solution of the steering problem with terminal condition $\widehat{y} = y_0(S)$. Consider a sequence of such optimal solutions with a fixed initial condition $\underline{y}$, defined on intervals $[0, S_i]$ with $S_i \to \infty$. A subsequence converging uniformly on compact intervals exists and its limit is clearly a finitely optimal trajectory of (2.3). ■

3. Solving (1.1) explicitly

In this section we provide an explicit solution to the concrete problem (1.1) along the lines described in the previous section. In particular, we need to analyze on the infinite horizon $s \in [0, \infty)$ the equation

$$\begin{aligned} \frac{dy_1}{ds} &= -y_1 + u \\ \frac{dy_2}{ds} &= -2y_2 + u, \end{aligned} \tag{3.1}$$

with $y = (y_1, y_2) \in R^2$ and $u \in [-1, 1]$. This equation is the analogue of (2.3) in the case of (1.1). Since in this section practically all the derivations concern the fast time scale s, we do not use the bar convention of the previous section.

When $u(s) \equiv -1$ the trajectories converge to the equilibrium point $(-1, -\frac{1}{2})$, while when $u(s) \equiv 1$ the trajectories converge to the equilibrium point $(1, \frac{1}{2})$. Marked as dashes in Fig. 1 are the trajectories emanating from $y = (1, \frac{1}{2})$ and from $y = (-1, -\frac{1}{2})$, with controls, respectively, $u(s) \equiv -1$ and $u(s) \equiv 1$.

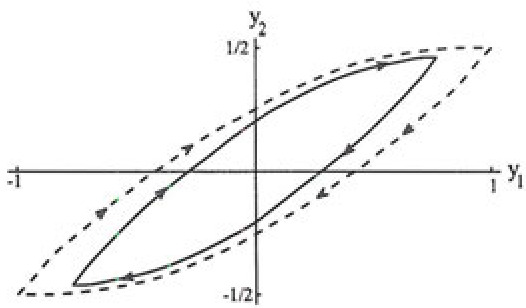


Figure 1.

OBSERVATION 3.1 *For any admissible trajectory* $(\mathbf{y}, \mathbf{u})$ *the function* $y(\cdot)$ *converges to the region encircled by the trajectories indicated in dashes as drawn in Fig.* 1.

Proof. Obvious. ∎

In particular, it is easy to see that Hypothesis 2.1 is satisfied with respect to the payoff criterion

$$c(y_1, y_2) = |y_1 - 2y_2| \tag{3.2}$$

on the region pointed out in the previous observation. This region would not be suitable for the analysis suggested in the previous section since equation (3.1) holds if we exclude the end points $(-1, -\frac{1}{2})$ and $(1, \frac{1}{2})$ and the boundary trajectories starting at them, but even then, finite controllability does not hold). This drawback can be fixed as follows:

PROPOSITION 3.1 *Let D be the region encircled by the two trajectories emanating from the points $(1-\delta, \frac{1}{2})$ and $(-1+\delta, -\frac{1}{2})$, using, respectively, the controls $u(s) \equiv -1$ and $u(s) \equiv 1$ (see the region encircled by the continuous line in Fig. 1). If $\delta > 0$ is small enough, then (3.1) with the criterion (3.2) satisfies both Hypothesis 2.1 and the finite controllability property on D.*

Proof. The finite controllability property on a region as encircled by the two trajectories portrayed in Fig. 1 is clear. We claim that Hypothesis 2.1 holds if δ is small enough. Indeed, as follows from (3.2), near the two points $(-1, -\frac{1}{2})$ and $(1, \frac{1}{2})$ the payoff function contributes values close to zero. The optimal value of the problem is positive; for instance, tracking the two trajectories which determine the boundary of the defined region yields a positive value. It is therefore clear that for ε small, a solution to (1.1), or equivalently, a solution to the analogous problem on the fast scale (see (2.3)–(2.4)), would not stay a long time near either of these points. Since starting at a point inside the region and reaching a close proximity of either of these extreme points would take a long time, it is clear that for small ε the solution stays out of a small neighborhood of either of the points $(-1, -\frac{1}{2})$ and $(1, \frac{1}{2})$. ■

As was pointed out after the statement of Hypothesis 2.1, there is no need to assume that the prescribed initial state be in D; it is enough that the initial state could be steered to D on a finite time. This property clearly holds for the set D identified in the preceding result.

Now we examine the structure of relatively optimal solutions to (3.1) with respect to the payoff criterion (3.2). Recall that a trajectory is relatively optimal if it is optimal given its initial and terminal states, see Definition 2.4. We consider relatively optimal trajectories within the set D described in Proposition 3.1.

LEMMA 3.1 *There is a bound $\overline{S}$ such that a relatively optimal trajectory in D does not stay on one side of the diagonal $y_1 = 2y_2$ for a time interval longer than $\overline{S}$ without crossing to the other side of the diagonal.*

Proof. An admissible trajectory $y(\cdot)$ of (3.1) which stays on one side of the mentioned diagonal must converge to the diagonal itself. On the diagonal the payoff criterion is zero. The optimal value of the problem is positive (as mentioned in the proof of the previous result, tracking, for instance, the trajectories in Fig. 1 yields a positive value). Hence, the finite controllability on D implies that a relatively optimal trajectory would not stay near the diagonal for too long. ■

LEMMA 3.2 *On one side of the diagonal $y_1 = 2y_2$, a relatively optimal solution of (3.3) is bang-bang (i.e., uses only the values $u = 1$ and $u = -1$), and the*

Proof. Notice that on one side of the diagonal the payoff function is linear in the state. Consider the region $y_1 \geq 2y_2$ (the reasoning for the other side is analogous). We add a third coordinate, denoted by y_3, to the payoff criterion and consider in R^3 the control equation

$$\frac{dy}{dt} = Ay + bu \tag{3.3}$$

where

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & -2 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \tag{3.4}$$

and with an initial condition $y_3(0) = 0$. With this choice, for a trajectory on $[0, S]$ the value $y_3(S)$ coincides with the payoff associated with the trajectory.

We follow now a technique demonstrated in, e.g., Pontryagin et al. (1962), Hermes and LaSalle (1969, Section 14). Given an initial condition $y(0) \in R^3$, the solution to (3.4) is given explicitly by

$$y(S) = e^{AS}y(0) + \int_0^S e^{A(S-s)}bu(s)ds \tag{3.5}$$

where

$$e^{A\sigma} = \begin{pmatrix} e^{-\sigma} & 0 & 0 \\ 0 & e^{-2\sigma} & 0 \\ 2 - e^{-\sigma} & e^{-2\sigma} & 1 \end{pmatrix}. \tag{3.6}$$

Consider a relatively optimal trajectory on $[0, S]$. Then $y_3(S)$ is maximal among the possible payoffs, hence $y(S)$ is on the boundary of the attainable set of trajectories on the interval $[0, S]$ emanating from $y(0)$. Consequently, a supporting vector $p = (p_1, p_2, p_3)$ exists, namely, the scalar product $p \cdot y(S)$ is maximal among all the attainable points. Furthermore, $p_3 > 0$. Resorting to the explicit formulas (3.5) and (3.6), we conclude that the optimal control, say $u^*(s)$, which is employed in generating the optimal path must maximize, point-wise almost everywhere, the expression

$$(2p_3 + (p_1 - p_3)e^{-(S-s)} + (p_2 + p_3)e^{-2(S-s)})u \tag{3.7}$$

subject to $-1 \leq u \leq 1$; this for $s \in [0, S]$. Hence, the values of $u^*(s)$ must be equal to either -1 or to 1, and a switch between the two values may occur only when the coefficient is equal to 0. Since the function multiplying u is the sum of two exponentials and the constant $2p_3 > 0$, it is clear that there is at most one zero of the coefficient. ∎

The following is a useful direct consequence. It could be checked (though

Corollary 3.1 *The steering time of an initial point to a terminal point with a trajectory on one side of the diagonal $y_1 = 2y_2$ is independent of the chosen path.*

Proof. Between two points on one side of the diagonal there are at most two trajectories meeting the specifications of Lemma 3.2. If in addition to the time consumed by the optimal trajectory there was another possible steering time, then there would be a continuum of possible steering times, and, consequently, there would exist relatively optimal trajectories with more than one switching point. ∎

Corollary 3.2 *A relatively optimal trajectory of* (3.1)–(3.2) *with the state values in D has the following structure. The trajectory alternates between the two sides of the diagonal $y_1 = 2y_2$, spending at most $\overline{S}$ units of time on each side, and in each side the optimal control has exactly one switch between the values* 1 *and* -1, *except possibly on the initial and terminal time segments where no switch may occur.*

Proof. An immediate consequence of Lemmas 3.1 and 3.2.

Next, we locate a maximizer of the value among the limit occupational measures in D. We say that a periodic trajectory in the state space is eye shaped if each period consists of exactly two segments, generated by $u \equiv 1$ and $u \equiv -1$, respectively. We say that it is symmetric if it is symmetric around $y = 0$. Any periodic trajectory generates a limit occupational measure supported on its state space trajectory. We say then that the limit occupational measure is eye shaped, or symmetric, if the associated periodic trajectory is eye shaped or symmetric, respectively. The solid line in Fig. 1 represents a limit occupational measure of symmetric eye shape. ∎

Lemma 3.3 *For any two points, say $\widehat{z}_1$ and $\widehat{z}_2$, on the diagonal $y_1 = 2y_2$ in D, there exists exactly one eye shaped periodic orbit. Let $\mu(\widehat{z}_1, \widehat{z}_2)$ be the associated eye shaped occupational measure. Then* $\mathrm{val}(\mu(\widehat{z}_1, \widehat{z}_2))$ *is a continuous function of* $(\widehat{z}_1, \widehat{z}_2)$.

Proof. Follows directly from the structure of the vector field. ∎

Lemma 3.4 *Let $\widehat{z}_1$ and $\widehat{z}_2$ be two points on the diagonal $y_1 = 2y_2$ in D. Let* $\mathbf{y}$ *be the trajectory steering $\widehat{z}_1$ to $\widehat{z}_2$ on one side of the diagonal, where the associated control has only one switch between* 1 *and* -1, *say on the interval* $[0, S]$. *Then* $\mathrm{val}(\mu(\mathbf{y}, [0, S]))$ *is the maximal value among occupational measures of trajectories steering $\widehat{z}_1$ to $\widehat{z}_2$ on one side of the diagonal.*

Proof. Follows directly from Lemma 3.2 and Corollary 3.1. ∎

THEOREM 3.1 *There exists a limit occupational measure μ^* of (3.2), which is symmetric and eye shaped, and which maximizes the criterion*

$$\mathrm{val}(\mu) = \int_{R^2} |y_1 - 2y_2|\mu(dy) \tag{3.8}$$

among all limit occupational measures generated by admissible trajectories in D.

Proof. We first verify that there is a limit occupational measure μ^* of (3.2), which is eye shaped, and which maximizes the criterion (3.8). Later on we show that it must be symmetric.

Denote by m^* the maximal value obtained in (3.8). The maximum is indeed obtained as noted in Proposition 2.2.

CASE 1. There exists a sequence, say μ_i, of the eye shaped occupational measures such that $\mathrm{val}(\mu_i) \to m^*$.

Since the family of the eye shaped occupational measures supported on D is clearly compact with respect to weak convergence, a cluster point of μ_i would verify the existence of an eye shaped maximum in this case.

CASE 2. There are admissible trajectories of arbitrary length in D, say $(\mathbf{z}_i, \mathbf{u}_i)$ defined on $[0, S_i]$ and $S_i \to \infty$, such that $\mathbf{z}_i = (y_{1,i}(\cdot), y_{2,i}(\cdot))$ does not intersect itself in R^2 during $[0, S_i]$, and the corresponding values satisfy

$$\mathrm{val}(\mu(\mathbf{z}_i, [0, S_i])) \to m^* \quad \text{as} \quad i \to \infty; \tag{3.9}$$

furthermore, each trajectory alternates between the two sides of the diagonal $y_1 = 2y_2$, where on each side the trajectory does not spend more that $\overline{S}$ units of time without crossing to the other side.

Let $\widehat{z}_i(j)$, $j = 1, \ldots, N(i)$, be the points of intersection of $\mathbf{z}_i$ with the diagonal $y_1 = 2y_2$ in D. Let $s_i(j)$ be the respective times of intersection. Since the time intervals $s_i(j+1) - s_i(j)$ are uniformly bounded, it follows that $N(i)$ grows *indefinitely* as $i \to \infty$. Due to the two-dimensional geometry, and since $\mathbf{z}_i$ is not self-intersecting, the sequences $\widehat{z}_i(2j+1)$ and $\widehat{z}_i(2j)$ of intersection points with odd and, respectively, even, indices are monotonic on the diagonal. In view of (3.9) the quantity $m^* - \mathrm{val}(\mu(\mathbf{z}_i, [0, S_i]))$ can be chosen arbitrarily small. On the other hand, $\mathrm{val}(\mu(\mathbf{z}_i, [0, S_i]))$ is the weighted average of the values $\mathrm{val}(\mu(\mathbf{z}_i, [s_i(j), s_i(j+2)]))$. Since the length of the latter intervals is uniformly bounded, it follows that for most of the indices j the values $\mathrm{val}(\mu(\mathbf{z}_i, [s_i(j), s_i(j+2)]))$ are close to (or greater than) m^*. Among the latter measures (since the diagonal is finite), there are such measures with $|\widehat{z}_i(j) - \widehat{z}_i(j+2)|$ small. It is clear, then, that if ν is the occupational measure generated by the eye shaped periodic orbit determined by the points $\widehat{z}_i(j)$ and $\widehat{z}_i(j+1)$, then, by Lemma 3.4, $\mathrm{val}(\mu)$ is either close to or greater or equal to $\mathrm{val}(\mu(\mathbf{z}_i, [s_i(j), s_i(j+2)]))$. This implies that the eye shaped occupational measures exist as described in Case 1, namely Case 2 implies Case 1 and the claim

CASE 3. Suppose that the condition in Case 1 does not hold.

Then a bound $\delta > 0$ exists such that

$$m^* - \mathrm{val}(\mu) \geq \delta \tag{3.10}$$

for every eye shaped occupational measure. Furthermore, since we showed that the condition in Case 2 implies the condition in Case 1, it follows from Proposition 2.3 that in Case 3 a bound $\widehat{S}$ exists, such that any relatively optimal trajectory of length greater than or equal to $\widehat{S}$ intersects itself.

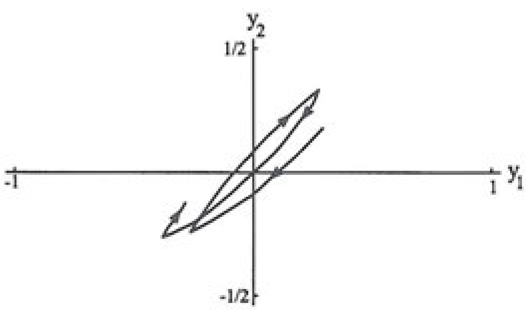
Let S_δ be such that whenever a relatively optimal trajectory $\mathbf{z} = (y_1(\cdot), y_2(\cdot))$ is defined on the interval $[0, S]$ with $S \geq S_\delta$, then

$$|m^* - \mathrm{val}(\mu(\mathbf{z}, [0, S]))| < \frac{1}{4}\delta. \tag{3.11}$$

The estimate S_δ exists in view of Proposition 2.3. We can assume that $S_\delta > \widehat{S}$ and $S_\delta > 2\overline{S}$, where $\widehat{S}$ and $\overline{S}$ are given, respectively, by the previous paragraph and by Corollary 3.2.

It is clear that for equation (3.1) with the payoff criterion (3.2) relatively optimal trajectories of an arbitrarily large length exist (see, e.g., Proposition 2.4). Let $\mathbf{z}$ be a relatively optimal trajectory defined on $[0, S]$ with $S > 4S_\delta$. Under the condition in Case 3 the trajectory $\mathbf{z}$ must intersect itself. Let z_I be the first intersection point in D, and let s_1 and σ_1 be the first two times where $z(s_1) = z(\sigma_1) = z_I$. The structure of relatively optimal trajectories described in Corollary 3.2 implies that on $[s_1, \sigma_1]$ the trajectory has either the structure of the curve of Fig. 2 or the structure of the curve of Fig. 3 (according to whether until time s_1, the trajectory was spiraling in or spiraling out, respectively). In particular $\sigma_1 - s_1 \leq 2\overline{S}$. In view of (3.10) and Lemma 3.4 it follows that

$$m^* - \mathrm{val}(\mu(\mathbf{z}, [s_1, \sigma_1])) \geq \delta. \tag{3.12}$$



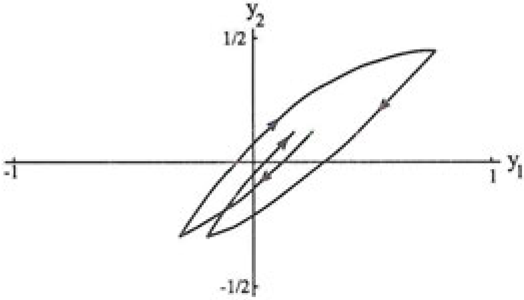


Figure 3.

In both cases let $\mathbf{z}_1$ be the trajectory obtained by skipping the part of $\mathbf{z}$ over $[s_1, \sigma_1]$. The trajectory $\mathbf{z}_1$ is defined on an interval of length $S-(\sigma_1 - s_1)$. Since $\mathrm{val}(\mu(\mathbf{z}, [0, S]))$ is a weighted average of $\mathrm{val}(\mu(\mathbf{z}, [s_1, \sigma_1]))$ and $\mathrm{val}(\mu(\mathbf{z}_1, [0, S - (\sigma_1 - s_1)]))$, it follows from (3.11) and (3.12) that

$$|m^* - \mathrm{val}(\mu(\mathbf{z}_1, [0, S - (\sigma_1 - s_1)]))| < \frac{1}{4}\delta. \tag{3.13}$$

Denote $S_1 = S - (\sigma_1 - s_1)$. Inductively, suppose that $\mathbf{z}_j$ is defined on $[0, S_j]$ such that the following inequalities are satisfied:

$$|m^* - \mathrm{val}(\mu(\mathbf{z}_j, [0, S_j]))| < \frac{1}{4}\delta \tag{3.14}$$

and

$$m^* - \mathrm{val}(\mu(\mathbf{z}_{j-1}, [s_j, \sigma_j])) \geq \delta. \tag{3.15}$$

If $S_j > S_\delta$, then $\mathbf{z}_j$ intersects itself. Although $\mathbf{z}_j$ may not be relatively optimal anymore, the process of eliminating a loop defined by the first self intersection of $\mathbf{z}_j$, as described earlier for $\mathbf{z}_1$, can be carried out, and again, the two possibilities of spiraling in and spiraling out may occur. The outcome would be a trajectory $\mathbf{z}_{j+1}$, for which (3.13) and (3.14) hold with the index $j+1$.

The process can go on until $S_\delta < S_j < 2S_\delta$. Say, this occurs when $j = j_0$. Now, the value in (3.11) is the weighted average of the value in (3.14) for the index $j = j_0$, and of the values in (3.15) for $j = 1, \ldots, j_0$. Since the sum of $\sigma_j - s_j$ for $j = 1, \ldots, j_0$ is $S - S_{j_0}$ we clearly get a contradiction when $S_{j_0} \leq \frac{1}{2}S$. The latter inequality will be reached due to the choice $S \geq S_\delta$. Hence, a contradiction has been established.

This contradiction implies that the condition in Case 3 does not occur, and the existence of an eye shaped occupational measure which maximizes the value is established. Suppose it is determined by the points $\widehat{z}_1$ and $\widehat{z}_2$ on the diagonal. Symmetry of the occupational measure means that $\widehat{z}_1 = -\widehat{z}_2$. If the latter does measure determined by $-\widehat{z}_1$ and $-\widehat{z}_2$ is also a maximizer of the value. Suppose $|\widehat{z}_1| > |\widehat{z}_2|$. A simple examination of the geometry reveals that the symmetric eye shaped occupational measure determined by $\widehat{z}_1$ has a value greater than the alleged maximizer. This verifies that $\widehat{z}_1 = -\widehat{z}_2$, namely that the eye shaped occupational measure at which the maximum is achieved is symmetric. ∎

THEOREM 3.2 *The occupational measure whose existence is established in Theorem* 3.1 *is the one associated with the eye shaped periodic solution through the points* $(0.63423, 0.31712)$ *and* $(-0.63423, -0.31712)$, *and the optimal value is* 0.29129 (*the numbers are given up to five significant digits*).

Proof. The numbers are a result of tedious derivations combined with numerical computations. We display here the main steps. We know that the occupational measure which maximizes the value is symmetric and eye shaped. Thus, the value is a function of its intersection point $(\eta, \frac{1}{2}\eta)$ on the diagonal, hence it is a function of η. Consider $\eta > 0$. We know that the optimal control generating the maximizer is $u \equiv -1$, say on an interval $[0, S_1]$, then switches to $u \equiv 1$, say on the interval $[S_1, S_1 + S_2]$, where the diagonal is reached again at the point $(-\eta, -\frac{1}{2}\eta)$. For the equations (3.1) the point $y(S_1)$ can be expressed analytically, namely

$$y(S_1) = \left(e^{-S_1}(\eta + 1) - 1, \frac{1}{2}e^{-2S_1}(\eta + 1) - \frac{1}{2}\right). \tag{3.16}$$

The same point should be reached starting at the initial condition $(-\eta, -\frac{1}{2}\eta)$ when employing the control $u \equiv 1$ in the reversed time direction, namely on the interval $[-S_2, 0]$. The result is

$$y(S_1) = \left(-e^{S_2}(\eta + 1) + 1,\ -\frac{1}{2}e^{2S_2}(\eta + 1) + \frac{1}{2}\right). \tag{3.17}$$

The equalities (3.16) and (3.17) provide two equations for S_1 and S_2, parameterized by η. They can be solved analytically, yielding

$$S_1 = -\log(1 - \eta^{\frac{1}{2}}) + \log(\eta + 1), \qquad S_2 = \log(1 + \eta^{\frac{1}{2}}) - \log(\eta + 1). \tag{3.18}$$

At this point, for each such eye shaped occupational measure, which we denote by $\mu(\eta)$, the average expressed in (3.8) as a space average can be calculated as the time average over the interval $[0, S_1 + S_2]$, of the quantity $y_1(s) - 2y_2(s)$, with respect to the initial condition $(\eta, \frac{1}{2}\eta)$ and the controls $u = -1$ on $[0, S_1]$ and $u = 1$ on $[S_1, S_2]$. This can still be carried out explicitly, yielding

$$\mathrm{val}(\mu(\eta)) = \frac{\eta}{\log(1 + \eta^{\frac{1}{2}}) - \log(1 - \eta^{\frac{1}{2}})}. \tag{3.19}$$

The optimal payoff is the maximum of the expression (3.19) for $0 \leq \eta \leq 1$, maximum. At this point I could not figure out the analytic expressions, hence I resorted to numerics. The numbers displayed in the statement of the theorem are a result of a simple computer calculation. (The computation shows also that the value is a concave function of η and that there is a unique maximum). ■

REMARK 3.1 *The closed solid line of Fig. 1 is a good approximation of the trajectory which generates the optimal occupational measure for* (1.1).

A limiting solution (see Definition 2.1) to (1.1) can now be constructed as follows. Given an initial condition (y_1, y_2), find a control which steers it to the point $(\eta, \frac{1}{2}\eta)$ with $\eta = 0.63423$. At this point use the control $u \equiv -1$ for a period of $S_1 = 2.08270$ units of time, and from thereon alternate between $u \equiv 1$ and $u \equiv -1$ on time intervals of period $S_1 + S_2 = 2.25676$; the length of the intervals is derived from (3.18) for $\eta = 0.63423$. A drawback of such a solution is that computational errors are accumulated. In addition, the intervals are in the s scale, and the adjustment to the slow scale t depends on the specific ε. The two drawbacks can be removed by using a feedback procedure, or a synthesized form, which applies directly to both time scales, and is independent on ε, as follows.

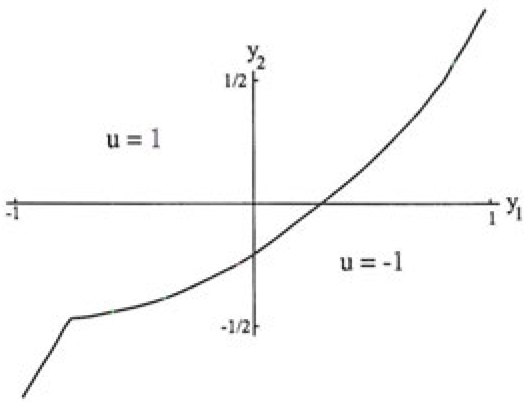


Figure 4.

THEOREM 3.3 *The line of Fig. 4 marks the part of the trajectory of* (3.1) *generated by the control $u \equiv -1$ and passing through the point* (0.63423, 0.31712), *and part of the trajectory generated by $u \equiv 1$ and passing through* (−0.63423, −0.31712). *Denote by R the region in R^2 to the right of the marked line, including the marked portion on the first trajectory. Denote by L the complement region. The feedback control defined by $u(y) = -1$ for $y \in R$ and $u(y) = 1$ for $y \in L$ constitutes a near optimal control for the singularly perturbed problem* (1.1), *namely, the resulting payoff is arbitrarily close to the value of the problem*

Proof. Obvious. ∎

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