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Modelling of distributed parameter nonlinear systems by differential Taylor method

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Abstract: Modelling, solution, control and even design of many ecological and engineering systems involve dealing with nonlinear partial differential equations of which analytic solutions are rarely available and numerical approach with or without linearization, or approximation is inevitable most of the time. In this paper the possibility of analysing such systems by using a fairly new method known as Differential Taylor (DT) Transform and its advantages are proved. The results obtained by this method are compared with the experimental results and shown to be within good agreement with them. It is emphasised that DT Transform is not effective for only filtration systems, but can also be used equally well for absorption, heat and mass transfer, convective diffusion and similar systems.

1. Introduction

Partial differential equations arise in connection with various physical and geometrical problems when the system space parameters are of distributed nature, only the simplest physical systems can be modelled under certain assumptions by ordinary differential equations, and in reality almost all physical systems such as fluid and solid mechanics, heat transfer, electromagnetic theory are full of problems that must be modelled by partial differential equations. Since the number of independent variables in these equations is more than one, usually the time and one, two or three of the space coordinates, the problems encountered for their solutions are much more than those for the ordinary differential equations. The scope of this paper is not to introduce all the features of distributed parameters systems described by linear or nonlinear partial differential equations and to investigate all the present analytic and /or numeric solutions methods, instead it is to emphasise the importance of a fairly new but not widely used method " Differential Taylor Transform Method" and to prove its simplicity and competitivity when compared with the others. Although the discussions are confined to magnetic filtration, the arrived conclusions are general for any distributed system.

Filtration of technological liquids and gases to clean them from tiny particles is one of the present problems of the modern word both from economical and ecological point of views. In recent years the use of magnetic filters (MF) is accepted as a perspective method for this purpose Watson (1973), Gerber (1994), Cuellar (1995). Unfortunately a general theory explaining the cleaning process in these filters has not been developed yet. The main reasons for this are that the physical events taking place in these filters can not be explained sufficiently and the determination of the parameters effecting the process is done empirically. The well known MF theory is valid for the stationary (time-independent) cases Watson (1973), Gerber (1994), Sandulyak (1988). Time dependence of the variation of the input-output concentrations of the particles occurring in liquids and gases has been obtained for the special cases and within certain approximations Cuellar (1995), Watson (1978), Akoto (1977). In addition to being approximate and valid for special cases, these methods have another important fault: They assume that the particles are only hold an gathered in filter matrix, whilst they are both captured meanwhile some of the hold particles get free and move along the filter matrix. It is mainly due to this reason that the above conventional filtration models are defective. In fact if the getting off process were considered, the differential equations used for the analysis would have come out to be nonlinear. Therefore many of the relations between the parameters of the MF have used to be obtained empirically.

In principle, the filtration regimes in MF are similar to those in classical filters. The difference appears in characters of the forces holding the particles in the pours of the filters. Mass equilibrium in filter matrix and kinetic equations of the saturation of the pours by the hold particles are the basis of the filtration theory; in classical filtration theory these parameters are indirect so that they are characterised by the accumulation and detachment coefficients Ives (1980), Adin (1989). In other words, both type of filters are dynamic systems characterised by these coefficients, and the filter performance is characterised by the ratios of the concentrations at the input and output of the filter.

Even in the simplest situations, the main equilibrium and kinetic equations used in the filter theory are two dimensional nonlinear partial differential equations which do not have an analytic solution in general case. Several of the common approaches to analyse such systems are the use of numerical solution methods, approximate analytical methods, linearization techniques, etc. In particular cases, the solutions are usually complex and not enough clear as to be used for practical calculations and design purposes Adin (1989). On the other hand, the equations describing practically very important technological regimes such as absorption, flotation, coagulation, separation etc. have similar properties to the general filtration equations. For this reason the solution of the general filtration equations is very important for the investigation, automatization and control of similar technological processes.

In this paper, the solution of partial differential equations describing the above type of nonlinear systems is considered by a new method which is known as the Differential Taylor (DT) Transform method Puhov (1986, 1990). This method transforms the mathematical model of the system into differential spectrum on which simple operations can be carried to derive and understand the system performance.

2. Basic interaction

2.1. Model filtration theory

In filtration theory, the mass equilibrium and kinetic equations are linear or nonlinear partial differential equations which can be classified for the following situations:

i) Nonlinear models for the filtration of the low velocity suspensions with high concentration and carrying rather large particles grater than micron size.

ii) Linear models for the filtration of the high velocity suspensions with low concentration and carrying particles of micron or smaller size.

iii) A general nonlinear filtration model which considers and is valid for both of the above cases.

In the first model, since the detachment (get away) of the hold particles is hardly possible, they are completely ignored. In the second model, the detachment of the hold particles is highly possible and should also be considered. And finally in the third model, both of the events, being captured and escaping are considered.

In the first model, the partial differential equations in the axial symmetric filter matrix are;

$$\rho_t + vc_x = 0; c(0,t) = c_0, \tag{1}$$

$$\rho_t = \beta v \left(1 - \frac{\rho}{\rho_n}\right) c; \rho(x,0) = 0. \tag{2}$$

For the second model

$$\rho_t + vc_x = 0; c(0,t) = c_0, \tag{3}$$

$$\rho_t = \beta vc - \alpha\rho; \rho(x,0) = 0. \tag{4}$$

And finally for the third model

$$\rho_t + vc_x = 0; c(0,t) = c_0, \tag{5}$$

$$\rho_t = \beta v (1 - \gamma\rho) c - \alpha\rho; \rho(x,0) = 0. \tag{6}$$

In the above equations $c(x,t)$ and $\rho(x,t)$ are the concentrations of the free and captured particles, respectively; c_0 is the initial concentration, ρ_n is the limiting value of the concentration of the hold particles; β and α are the accumulation and detachment coefficients, respectively, which should be properly determined according to the physical, chemical and geometric properties of the filtration system; finally v is the filtration velocity of the suspension.

2.2. Differential Taylor (DT) transform

It is well known that linear differential equations with constant coefficients can be transformed into algebraic equations and then easily solved in complex frequency domain by using Laplace and/or Fourier transformations Kreyszig (1988). For time varying systems, although the application of these techniques is possible by some modifications, it is not as easy and simple as the former case. For nonlinear differential systems, the problem gets more complex due to the frequency domain convolution integrals which result from the time domain products of dependent variables or their derivatives; therefore the use of ordinary transform techniques is impractical for nonlinear systems. Fortunately, the use of DT Transform override most of the mentioned difficulties and the convolution integrals are replaced by simple sums of algebraic terms.

DT Transform method converts the differential form mathematical model of a system in to its spectral form on which algebraic operations can be carried to derive and understand the system performance. For an analytic function $x(t)$ described by its Maclaurin series

$$x\left(t\right)=\sum_{k=0}^{\infty}\frac{1}{k!}\left.\frac{d^{k}x\left(t\right)}{dt^{k}}\right|_{t=0}t^{k}, \tag{7}$$

in the interval $t\in[0,T]$, the spectral model (or transform) is defined to be the discrete function

$$X\left(k\right)=\frac{T^{k}}{k!}\left.\frac{d^{k}x\left(t\right)}{dt^{k}}\right|_{t=0} \tag{8}$$

which is known to be the Differential Transform. Using this transform the Taylor series in (7) can be written as

$$x\left(t\right)=\sum_{k=0}^{\infty}X\left(k\right)\left(\frac{t}{T}\right)^{k} \tag{9}$$

which is now named as the Taylor Transform (Puhov, 1986, 1990). In this equation the interval length T is known to be the scale factor of the transform.

An important property of the DT Transform is its applicability to systems involving two independent variables and naturally partial differential equations. The DT Transform expressions of a function of two variables (generally space and time) are given by

$$C\left(p,k\right)=\frac{L^{p}T^{k}}{p!k!}\left.\frac{\partial^{p+k}c\left(x,t\right)}{\partial x^{p}\partial t^{k}}\right|_{x=0\ t=0}, \tag{10}$$

$$c(x,t)=\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}C(p,k)\left(\frac{x}{L}\right)^{p}\left(\frac{t}{T}\right)^{k} \tag{11}$$

where $p, k = 0, 1, 2, \ldots; L$ and T are the scale factors with respect to x and t, respectively. The use of DT Transform will be shown in the following section as applied to the mass equilibrium and kinetics equations of filtration models expressed in (1)-(6).

3. Solution of filtration equations by DT transform

Applying the well known rules about the DT Transform (Puhov, 1986, 1990) to the nonlinear partial differential equations in (1), we obtain

$$\frac{k+1}{T} R(p, k+1) + v \frac{p+1}{L} C(p+1, k) = 0; C(0, k) = c_0 \delta_k, \tag{12}$$

$$\frac{k+1}{T} R(p, k+1) = \beta v \left[C(p, k) - \frac{1}{\rho_n} R(p, k) \cdot C(p, k) \right]; R(p, 0) = 0, \tag{13}$$

where k is the Kronecker delta which is equal to 1 if $k = 0$, otherwise it is zero. Simplifying and rearranging, (12), (13) can be expressed for the spectrums of c and ρ as

$$C(p+1, k) = \frac{L}{(p+1)} \beta \left[\frac{1}{\rho_n} R(p, k) \cdot C(p, k) - C(p, k) \right]; C(0, k) = c_0 \delta_k, \tag{14}$$

$$R(p, k+1) = \frac{T}{k+1} \beta v \left[C(p, k) - \frac{1}{\rho_n} R(p, k) \cdot C(p, k) \right]; R(p, 0) = 0, \tag{15}$$

respectively. Where the algebraic convolution term $R \cdot C$ is

$$R(p, k) \cdot C(p, k) = \sum_{m=0}^{p} \sum_{n=0}^{k} R(p-m, k-n) C(m, n). \tag{16}$$

Starting from the given initial conditions and increasing the values of p and k from 0 on sequentially, the spectrums $C(p, k)$ and $R(p, k)$ can be computed upto any desired order (say P for p, K for k). Then knowing its spectrum, the output concentration can readily be computed by the inverse transform

$$c(x, t) \cong \sum_{p=0}^{P} \sum_{k=0}^{K} \left(\frac{x}{L} \right)^p \left(\frac{t}{T} \right)^k C(p, k). \tag{17}$$

In the applications the choice of P and K are made properly by considering the convergence properties of the series; however, in many cases P and K are chosen so that terms upto a certain order (say $S = P + K$) are sufficient and the summations in this equation are carried upto S for p and $S - p$ for k.

In a similar manner (3), (4) and (5), (6) can be transformed into spectral domain. For (3), (4) – (14), (15) take the following form

$$C(p+1, k) = -\frac{L}{v(p+1)} \left[\beta v C(p, k) - \alpha R(p, k) \right]; C(0, k) = c_0 \delta_k, \tag{18}$$

$$R(p, k+1) = \frac{T}{k+1}\left[\beta v C(p,k) - \alpha R(p,k)\right]; R(p,0) = 0. \tag{19}$$

As the first example to show the application of DT Transform, consider (1) and (2) with the numerical data: $v = 200$ m/h, $c_0 = 2 \times 10^{-3}$ kg/m^3, $\beta = 1.2$ m^{-1}, $\rho_n = 10$ kg/m^3. The explicit analytic solution of these equations for $c(x,t)$ can be derived to be

$$c(x,t) = \left[c_0 e^{\frac{c_0 \beta v t}{\rho_n}}\right] / \left[e^{\beta x} + e^{\frac{c_0 \beta v t}{\rho_n}} - 1\right]. \tag{20}$$

The filter quality factor defined by

$$F(x,t) = 0.8\left(1 - \frac{c(x,t)}{c_0}\right) \tag{21}$$

is computed at $x = 1m$ by using the explicit solution in (20) and plotted against time as in Fig. 1a. In the same figure the results of DT Transform solution with $L = 1$ m and $T = 5$ h are also shown for different values of $S = P + K$; $S = 10, 20, 30$. Obviously the results of DT Transform method converges to the actual solution rapidly and when $S = 30$ the maximum deviation becomes less than 1% at $t = 5$ h. The execution time for the solution for $S = 30$ is about only 15.27 s in a PC Pentium 133.

As the second example the magnetic filters which are used to clean the technological condensats used in thermopower stations from the magnetic particles of micron size are considered. The experimental change of quality factor of such a filter is shown in Fig. 1b. The system parameters are: $c_0 = 0.1 \times 10^{-3}$ kg/m^3, $L = 1$ m, $v = 200$ m/h, $\beta = 3.08$ m^{-1}; and $\alpha = 0.74$ h^{-1}. These parameters belong to a magnetic filter with filter elements of diameter $d = 0.005$ m, and an external magnetic field intensity of $H = 70$ kA/m Sandulyak (1988). For the case considered, Equations in (3), (4) are valid and their analytic solutions are not available. However when the equivalent Differential Transform equations in (18), (19) are solved and Taylor Transform in (17) is used for $S = 40$ to compute $c(x,t)$, (21) yields the quality factor plotted by the continuous curve. Obviously the curve follows the experimental data fairly well. The discrepancies are not due to the approximate nature of the DT Transform method (since sufficiently large S is checked to be chosen), but they are originating from the imperfectness of the original mathematical model.

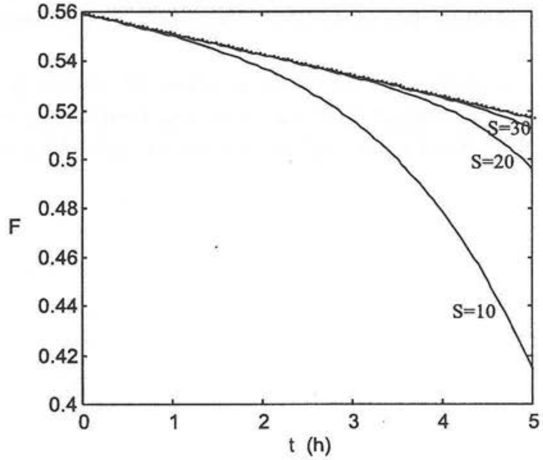
4. Conclusions

As a result of the observations in the proceeding sections and applications of DT Transform to many other systems, the following conclusions are arrived:

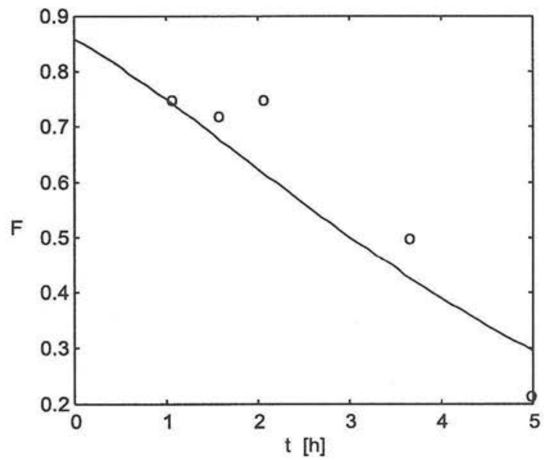
i) DT Transform can be used as a tool in modelling of physical systems of which many properties are described by nonlinear partial differential equations.

ii) Depending on the properties of the system, the number of spectrums can be chosen as to satisfy the desired accuracy. And within this accuracy the model of the system can be obtained both in analytic and/or discrete forms.

iii) DT Transform can equally be used in the analysis of similar nonlinear systems having vital importance in engineering applications, a few of which are absorption, fluid-solid mass and heat transfer, diffusion, etc.



(a)



(b)

Figure 1. Change of the quality factor of the analysed magnetic filter; a) bold curve is calculated by using the explicit solution and the others are calculated by the DT Transform for different values of S, b) continuous curve is calculated by the DT Transform, and o indicates the experimental data.

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