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Mixed finite element formulation and optimal design of thin composite laminates

by

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Abstract: A Hellinger–Reissner variational principle is introduced to derive the weak form equation of thin generally orthotropic laminates. It leads naturally to a mixed finite–element approximation that has the out–of–plane deflection and the bending and twisting moments as independent unknowns. A triangular element is derived that is used for both analysis and optimization purposes. Numerical simulations on example laminates of irregular geometry are presented to validate the theoretical framework.

Keywords: composites, mixed finite elements, optimal design.

1. Introduction

Despite the huge amount of literature which has appeared in the last decades on modeling and optimization of composite structures, most results are concerned with orthotropic laminates of rectangular geometry. This may be attributed to the fact that, under these conditions, semi–analytical methods such as that of trigonometric sequences, or basic numerical approaches such as the original Rayleigh–Ritz method may be successfully applied. Irregular geometries call for the adoption of finite–element approximations for which we refer to Reddy (1984). However, the displacement–based finite elements are usually very demanding as far as the computational burden is concerned since the fourth order operator that governs the displacement of the structure finds its natural functional space in H^2, the space of measurable functions of integrable square along with their derivatives up to second order. Therefore, polynomials of high order are necessary to assemble a convergent finite–element scheme in the case of compatible approximations. This motivates the choice of a mixed approximation to be developed hereafter for which the simplicity of the consequent finite element outweighs the increased number of equations to be solved. The paper is organized as follows. Classic relations governing thin generally orthotropic laminates are briefly reviewed and a Hellinger–Reissner variational principle is then introduced. The resulting functional drives toward the construction of a simple triangular element with linear shape functions having as nodal unknowns the displacement and the bending and twisting moments. Some optimality conditions for the design of this type of laminates are then recalled before presenting a few numerical results on the analysis and optimization of irregular plates.

2. Governing relations

2.1. Constitutive law, compatibility and equilibrium

Thin generally orthotropic laminates are considered complying with the hypotheses that allow decoupling between bending and in–plane actions. Therefore, the structural constitutive law reads

$$\left\{ \begin{array}{c} M_x \\ M_y \\ M_{xy} \end{array} \right\} = \left[\begin{array}{ccc} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{array} \right] \left\{ \begin{array}{c} \chi_x \\ \chi_y \\ \chi_{xy} \end{array} \right\} \tag{1}$$

where M_x and M_y are the bending moments and M_{xy} is the twisting moment. They are related to the curvatures χ_x, χ_y and χ_{xy} by the classic lamination coefficients D_{ij} that are computed as

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{NL} (\overline{Q}_{ij})_k \left(h_k^3 - h_{k-1}^3\right) \tag{2}$$

where the $\overline{Q}_{ij}$ are given as

$$\begin{aligned}
\overline{Q}_{11} &= Q_{11} \cos^4\theta + 2(Q_{12} + 2Q_{66}) \sin^2\theta \cos^2\theta + Q_{22} \sin^4\theta \\
\overline{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66}) \sin^2\theta \cos^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta) \\
\overline{Q}_{22} &= Q_{11} \sin^4\theta + 2(Q_{12} + 2Q_{66}) \sin^2\theta \cos^2\theta + Q_{22} \cos^4\theta \\
\overline{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66}) \sin\theta \cos^3\theta + (Q_{12} - Q_{22} + 2Q_{66})(\sin^3\theta \cos\theta) \\
\overline{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66}) \sin^3\theta \cos\theta + (Q_{12} - Q_{22} + 2Q_{66})(\sin\theta \cos^3\theta) \\
\overline{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \sin^2\theta \cos^2\theta + Q_{66}(\sin^4\theta + \cos^4\theta)
\end{aligned} \tag{3}$$

In the above, the local coefficients Q_{ij} are given as

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12} E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{66} = G \tag{4}$$

Under the Kirchhoff hypothesis, the compatibility equations may be written as

$$\left\{ \begin{array}{c} \chi_x \\ \chi_y \\ \chi_{xy} \end{array} \right\} = - \left\{ \begin{array}{c} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{array} \right\} \tag{5}$$

while equilibrium is governed by the well known relation

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0 \tag{6}$$

2.2. Variational principles

When classic compatible elements are used, reference is made to the stationarity of the functional

$$J_1(w) = \frac{1}{2}\int_\Omega \{M_x\chi_x + M_y\chi_y + M_{xy}\chi_{xy}\}\, d\Omega + \int_\Omega qw\; d\Omega + \int_{\partial\Omega}[\overline{T}_n w + \overline{M}_n\phi_n + \overline{M}_{ns}\phi_s]ds \tag{7}$$

in which moments and curvatures are expressed in terms of the out–of plane displacement w by means of (1) and (5). The resulting finite element formulation find its natural environment in $H^2(\Omega)$ defined as

$$H^2(\Omega) = \{v \text{ measurable}|\; v,\; Dv \text{ and } D^2v \in L^2(\Omega)\} \tag{8}$$

The downside of such an approach is represented by the high order that the polynomial shape functions are to be given to ensure a convergent behavior. A remedy is constituted by the adoption of a mixed formulation in which the displacement and the moments are discretized independently, thus increasing the number of equations to be solved and reducing the order of the problem. Toward the formulation of such an approach the constitutive law of equation (1) is inverted as

$$\left\{\begin{array}{c}\chi_x\\ \chi_y\\ \chi_{xy}\end{array}\right\} = \begin{bmatrix} V_{11} & V_{12} & V_{16}\\ V_{12} & V_{22} & V_{26}\\ V_{16} & V_{26} & V_{66}\end{bmatrix}\left\{\begin{array}{c}M_x\\ M_y\\ M_{xy}\end{array}\right\} \tag{9}$$

where the compliance matrix V reads

$$\mathrm{V} = \frac{1}{\mathrm{D}}\begin{bmatrix} D_{22}D_{66} - D_{26}^2 & D_{16}D_{26} - D_{12}D_{66} & D_{12}D_{26} - D_{16}D_{22}\\ D_{16}D_{26} - D_{16}D_{66} & D_{11}D_{66} - D_{16}^2 & D_{12}D_{16} - D_{11}D_{26}\\ D_{12}D_{26} - D_{16}D_{22} & D_{12}D_{16} - D_{11}D_{26} & D_{11}D_{22} - D_{12}^2 \end{bmatrix} \tag{10}$$

and $D = -D_{16}^2D_{22} + 2D_{12}D_{16}D_{26} - D_{11}D_{26}^2 - D_{12}^2D_{66} + D_{11}D_{22}D_{66}$. By expressing the elastic energy in terms of moments, imposing the equilibrium in weak form and after some algebra the following mixed functional may be derived

$$\begin{aligned} J(w, M_x, M_y, M_{xy}) = \frac{1}{2}\int_\Omega &[M_x(V_{11}M_x + V_{12}M_y + V_{13}M_{xy}) + \\ &M_y(V_{21}M_x + V_{22}M_y + V_{23}M_{xy}) + \\ &M_{xy}(V_{31}M_x + V_{32}M_y + V_{33}M_{xy}) + \\ &\frac{\partial w}{\partial x}\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}\right) - qw]\; d\Omega + \\ &\int_{\partial\Omega}\left[M_n\frac{\partial w}{\partial n} + \left(\frac{\partial M_{ns}}{\partial s} + T_n\right)w\right] ds \end{aligned} \tag{11}$$

The stationarity of the functional in (11) leads naturally to the finite element discretisation to be discussed next.

3. Finite element discretisation and optimal design

The weak form of the governing equations, i.e. the stationarity of the functional in (11), suggests the adoption of the linear shape functions for all the unknowns. In fact, the convergence analysis for such an element may be carried out in $H^1(\Omega)$ that means that piecewise linear shape functions guarantee convergence of the discrete solution to the actual one. As to the optimal de-

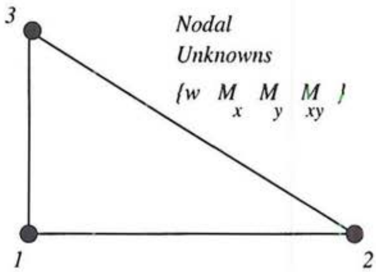

Figure 1. The triangular finite element used in the numerical examples

sign, several objectives may be pursued thanks to the flexibility offered by the finite–element approach. In Cinquini, Mariani and Venini (1995), a Rayleigh–Ritz method was applied to determine the influence of elastic boundaries on the eigenfrequencies of rectangular laminates. In Cinquini, Mariani and Venini (1996), the influence of uncertainties on the optimal solution to eigenvalue based objectives was assessed by means of a stochastic Rayleigh–Ritz approach. In this paper, attention will be focused on optimizing the first three eigenfrequencies of an irregular laminate variously constrained at the boundary. By denoting with λ_i the i–th eigenvalue, possible objective functions include

$$\begin{aligned} &\max\ [\min\ \lambda] = \max\ \lambda_1 \\ &\max\ [\lambda_{i+1} - \lambda_i]\,, \quad i = 1,2 \end{aligned} \tag{12}$$

The above objectives are known to be nondifferentiable and present peculiar problems when the multiplicity of an eigenvalue is greater than one. These topics are however left for future works. Herein a sequential quadratic programming (SQP) scheme is applied to find the extremum points of the first three eigenfrequencies in the case of no eigenvalue crossing and absence of repeated eigenvalues. A single lamina is the object of investigation and its lamination angle is the design variable. The section to come presents a few numerical results from which extreme points may be found by inspection. However, they were also detected by the SQP approach in very few iterations.

4. Numerical simulations

A graphite–epoxy fiber–reinforced lamina is considered. The elastic moduli of

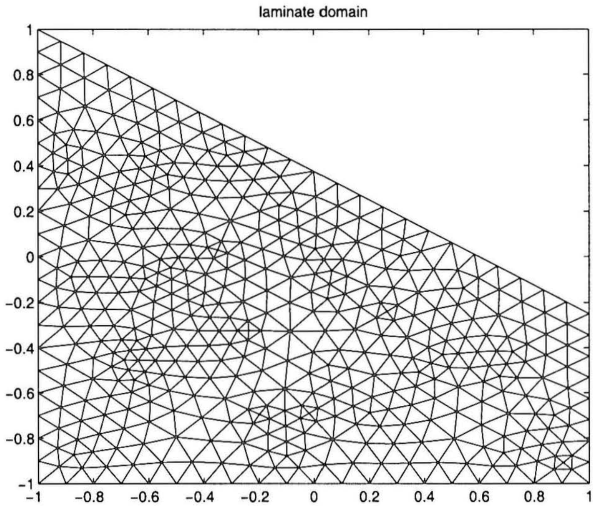

Figure 2. Domain of the laminates under investigation and relevant mesh

0.96×10^6 psi. A trapezoidal plate was considered, shown in Fig. 2 with the mesh used in the numerical study. In the first case, the lamina was considered clamped at all the edges. The results of the eigenanalysis are summarized in Figs. 3–8. In particular, Figs. 3, 4 and 5 present the first three modal shapes for various angles of lamination. The dominant role of the reinforcement for the vibrational behavior of the system is clearly enhanced. As to the optimal design, the variations of the first three frequencies are reported in Figs. 6, 7 and 8. It is interesting to note that the optimal solutions for rectangular plates, i.e. $\theta = 0°$, $\theta = 45°$ and $\theta = 90°$, see Cinquini, Mariani and Venini (1995), are no longer such in the case under investigation. One may – conversely – note that the optimal solution is approximately $\theta = 60°$, i.e. the direction normal to the inclined edge of the plate. The same analyses were then repeated by varying the boundary conditions. Starting from the horizontal edge, the four edges were, respectively, simply supported, simply supported, free and clamped. Figs. 9, 10 and 11 present the first three modal shapes of the plate for different angles of lamination. The pattern is significantly affected by the presence of a free edge, but still the importance of the reinforcement on the behavior of the structure is clearly visible. Figs. 12, 13 and 14 show the first three eigenfrequencies of the plate for this case. It is worth noting that the situation is reversed with respect to the previous simulation as far as the optimal design is concerned. In this new configuration, $\theta = 60°$ happens to be the worst design choice, while before it was the optimal one. This is due to the fact that the inclined edge is now unconstrained and therefore the internal reinforcement of the structure does not find an adequate partnership in the boundary constraints. This suggests to

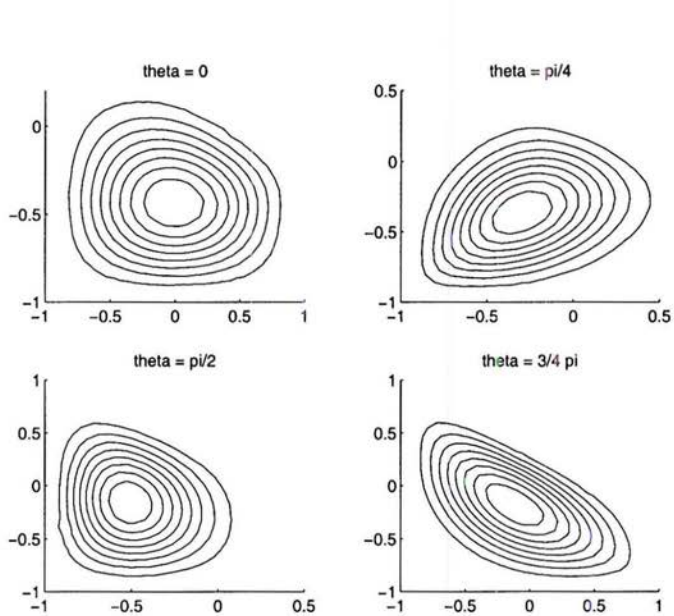

Figure 3. First eigenmode for different lamination angles (I case)

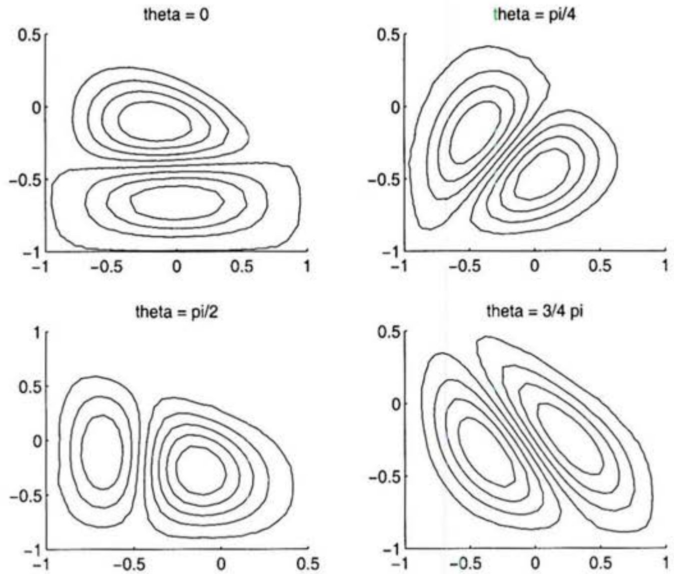

Figure 4. Second eigenmode for different lamination angles (I case)

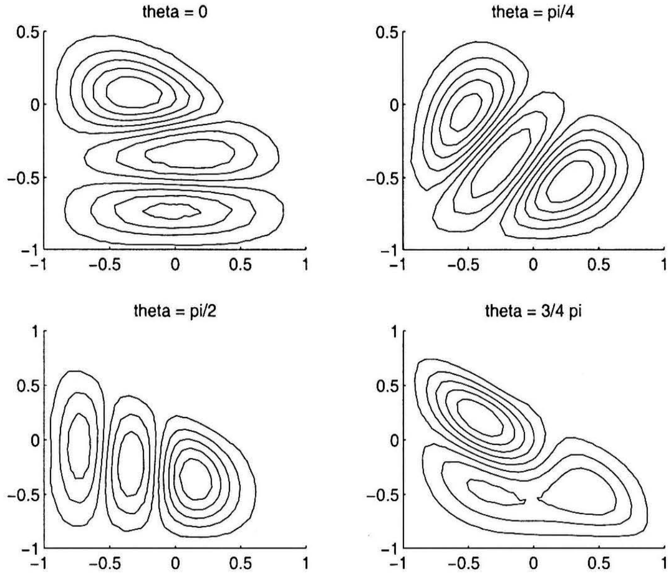

Figure 5. Third eigenmode for different lamination angles (I case)

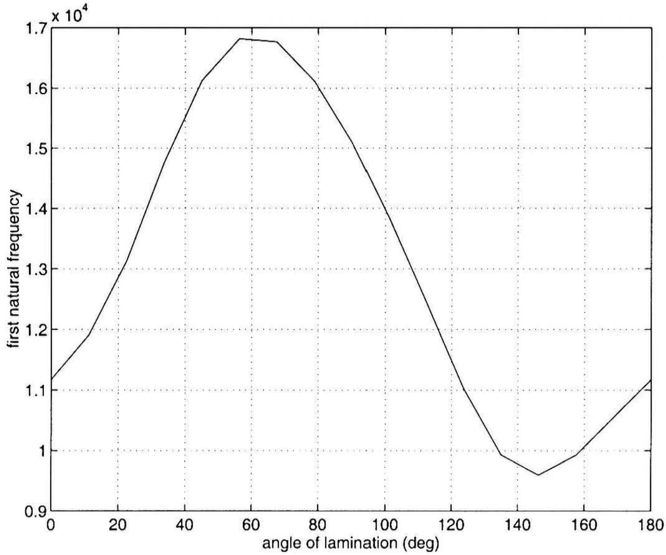

Figure 6. First natural frequency for different lamination angles (I case)

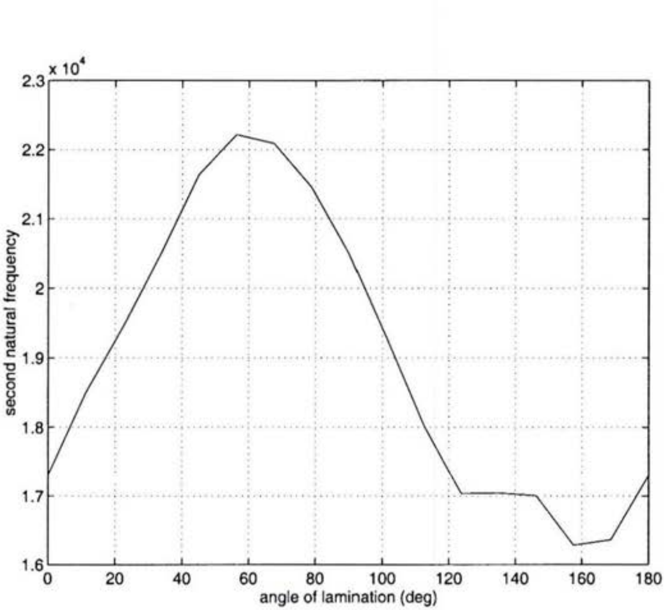

Figure 7. Second natural frequency for different lamination angles (I case)

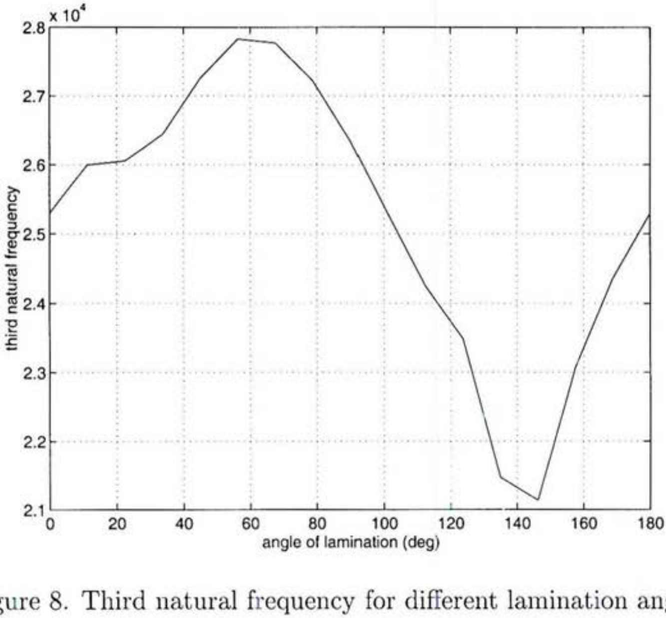

Figure 8. Third natural frequency for different lamination angles (I case)

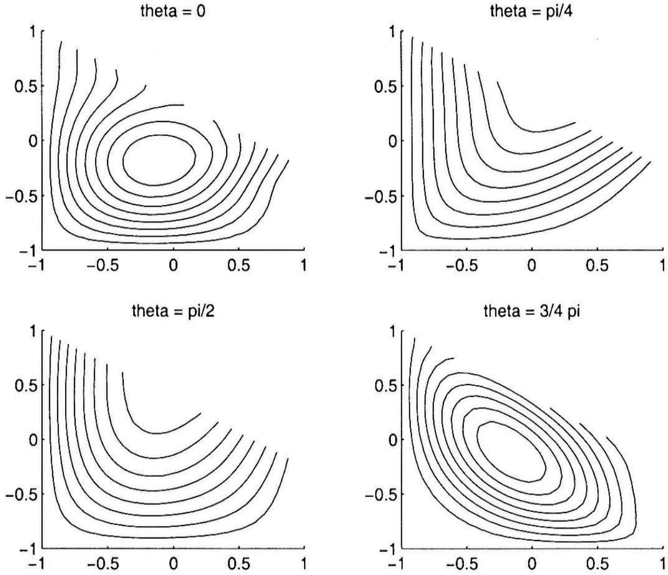

Figure 9. First eigenmode for different lamination angles (II case)

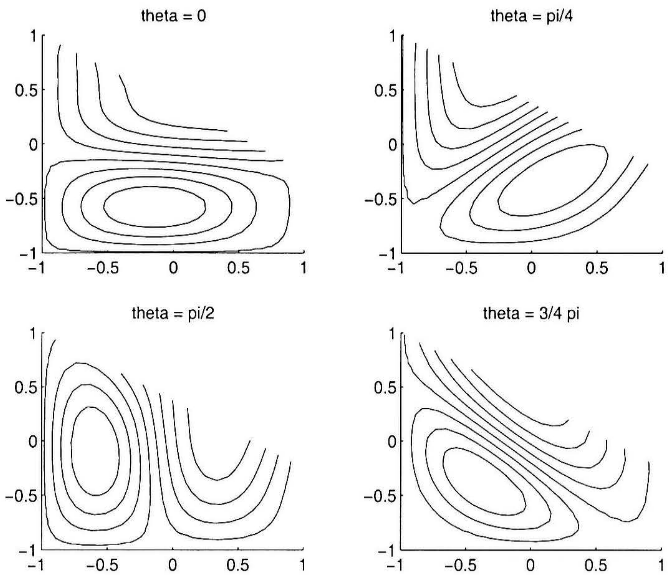

Figure 10. Second eigenmode for different lamination angles (II case)

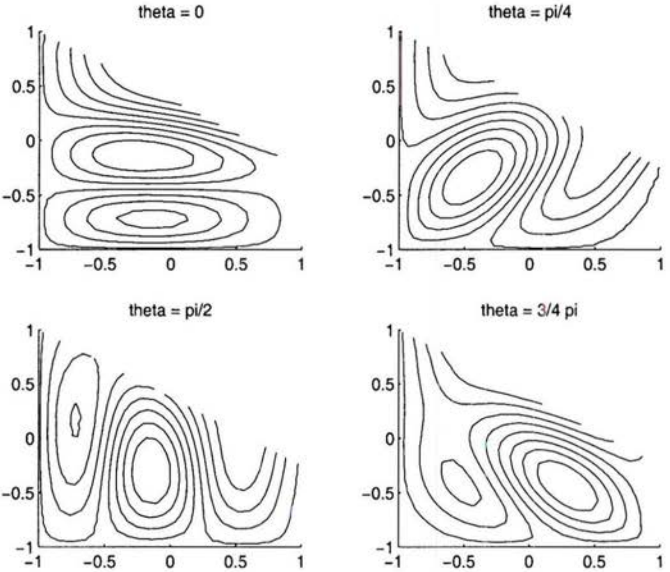

Figure 11. Third eigenmode for different lamination angles (II case)

include the boundaries themselves as optimization variables in a strategy where one chooses position and type of boundary.

5. Conclusions

A mixed–finite–element approximation for thin generally orthotropic laminates was presented. The main motivation behind this choice was the capability of solving a fourth order problem with simple linear shape functions. The discretized structure was the object of frequency–domain optimization where ply angles were chosen as design variables. Ongoing extensions include more complicated optimal design objectives and dynamic analyses of the system in the presence of an active controller.

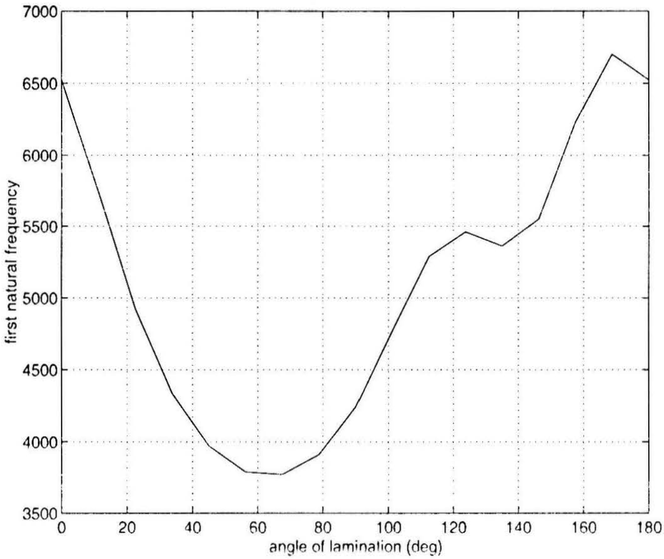

Figure 12. First natural frequency for different lamination angles (II case)

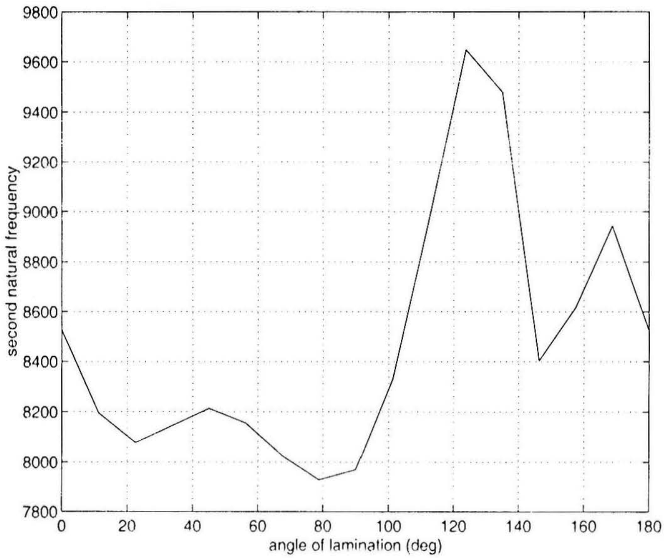

Figure 13. Second natural frequency for different lamination angles (II case)

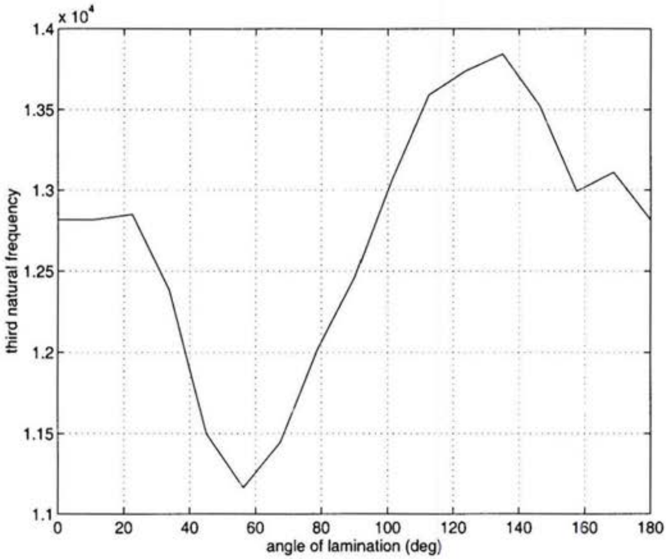

Figure 14. Third natural frequency for different lamination angles (II case)

References

BREZZI, F., BATHE, K.J., FORTIN, M. (1991) Mixed-interpolated elements for Reissner–Mindlin plates. *Int. J. Numer. Methods Engrg.*, **28**, 1787–1801.

CINQUINI, C., MARIANI, C. and VENINI, P. (1995) Rayleigh–Ritz Analysis of elastically constrained thin laminated plates on Winkler inhomogeneous foundations. *Computer Methods in Applied Mechanics and Engineering*, **123**, 263-286.

CINQUINI, C., MARIANI, C. and VENINI, P. (1996) Optimal robust design of novel materials: problems of stability and vibrations. *Engineering Optimization*, in press.

HERRMANN, L.R. (1967) Finite element bending analysis for plates. *J. Eng. Mech. Div. ASCE*, **98**, 5, 13–26.

REDDY, J.N. (1984) *Energy and Variational Methods in Applied Mechanics.* John Wiley & Sons.