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Minimisation of public debt servicing costs based on nonlinear mathematical programming approach

by

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Abstract: The paper presents formulations of optimisation problems for minimisation of servicing cost of government debt instruments (T-bills and T-bonds) issued in Poland and examples of their applications. The first of these problems allows for determining the structure of T-bills with different maturities — from 1 to 52 weeks (sold at multiprice auction), which minimise their servicing cost. This cost (criterion function) is based on T-bills profitability determined by compound rate of return (CRR). The constraints of the task express: • minimum receipts to the State Budget, • maximal profitability, • average maturity, and • minimal and maximal amount for each type of bills. The generalisation of this problem for some series of (consecutive) auctions is also discussed. The second task makes it possible to determine optimal structure of T-bonds using a similar (but more complex) form of the criterion function and more extensive set of constraints. The problems presented are of discrete and non-linear form and in result — rather difficult to solve in *real time* (during the auction). Therefore, some continuous approximations of these problems based on polynomials determined with the use of least squares method are suggested. Such an approach makes it possible to obtain an optimal solution in several minutes using Excel spreadsheet (package *solver*) and PC computer. Examples of actual problems for T-bills and T-bonds, based on data for the year 2000 are presented, too. The application of optimisation methods resulted in (relative) *gain* measured by a criterion function equal to about 1% (annual costs of the domestic debt equals about $3 \cdot 10^9$ US$).

Keywords: minimisation of servicing costs of public debt, optimisation approach in public debt management, profitability minimisation of debt instruments sold at auctions.

1. Introduction

A debt issuer has to determine the structure of debt instruments, especially their type (fixed-rate or variable rate), maturity (short-, middle- or long-term), duration, etc. The above features of the instruments influence their servicing cost, as well as the risk associated with the cost. Therefore, the essential problem of the issuer is to search for an optimal structure of the debt instruments, i.e. their types and maturity distribution. The main purpose is usually to minimise the servicing cost of the debt or risk associated with the cost or both these criteria together (multicriterial optimisation). Moreover, the structure of the debt issued ought to satisfy some constraints. If the aim is to minimise servicing cost, the constraints comprise: risk level, maturity structure, profitability level, etc. If the aim is to minimise the risk level, the constraints include the servicing cost. In case of multicriterial optimisation each of the factors (cost and risk) should be *associated* with the proper *weight*. The profitability and risk level should be expressed by appropriate measures (see Jajuga and Jajuga, 1997), e.g.: • internal rate of return — IRR or realised compound rate of return — CRR (profitability) and • duration, variances and covariances of interest rates volatility (risk). The search for the optimal solution can be performed in practice on the basis of *intuitive* or formalised — mathematical programming approach (see e.g. Grabowski, 1980). The *intuitive* approach may be efficient in case of simple problems, i.e., when the number of possible solutions is insignificant. Otherwise, such approach leads typically to a suboptimal solution or — in some cases — far from the optimal one. The mathematical programming approach can be applied if both the criterion and constraints (the feasible set) are expressed in the form of functions satisfying some assumptions. The feasible set of the problems under consideration is usually finite, because the value of debt issued is limited and nominal values of debt instruments are fixed. Therefore, an optimal solution always exists for such problems. The aim of this paper is to propose: • a formulation of two problems for minimisation of the servicing cost of the debt instruments — treasury bills and bonds — sold at the multiprice auctions, • methods for solving these problems, • directions of their modification and • empirical results.

The idea of application of optimisation methods for public debt problems is not a new one — it was applied e.g. in the papers by Barro (1995) and Cleassens et al. (1995), though on the *macroeconomic* level. The problems presented in this paper (see also Klukowski, 2000) correspond to the *microeconomic* level; their formulation is much simpler (than in Barro, 1995, and Cleassens et al., 1995) and therefore they can be applied in a day-to-day activity. The optimisation approach is (in authors' opinion) of a special importance for *emerging market* countries, because of significant inflation level, high volatility of interest rates, structural changes in financial markets, etc. However, it seems that these methods can be successfully applied in the well-developed countries as well.

The paper consists of four sections. The second section deals with an optimisation problem for T-bills — features of these instruments, rules of auctions (in Poland), a formulation of minimisation problem and a proposed method of its solving, an example of application as well as generalising remarks. In Section 3 — minimisation problems for T-bonds are presented — in a similar way.

The last section summarises the results obtained, suggests directions for further investigations in this area, and presents some general conclusions.

2. Minimisation of servicing cost for T-bills sold at auctions

The problem of minimisation of servicing cost of T-bills sold at a multiprice auction is presented in this section. We describe basic features of T-bills and auctions in Poland, give a mathematical formulation of the criterion function with a feasible set for one auction, an approximate method for its solving — using continuous polynomial approximation, an example of application, as well as generalising comments concerning some series of auctions.

2.1. Main features of T-bills and auctions in Poland

The main features of T-bills and auctions for these instruments in Poland are as follows:

1) T-bills are short-term, fixed-rate instruments with a discount paid at maturity, and maturities ranging from 1 to 52 weeks;
2) auctions are typically organised once a week;
3) during individual auction T-bills with different maturities (usually three or four types) can be sold — typically: 6-week, 8-week, 13-week, 26-week and 52-week bills;
4) the nominal value of one T-bill is 10 000 Polish zlotys (1 zloty $\approx$ 0.23 US\$), while the value of T-bills offered at the same auction ranges usually from 500–1000 mil. of Polish zlotys;
5) auctions are of multiprice type; the issuer announces the offered amount of T-bills of each type (before auction) and determines the minimum accepted price for each type — as a result of auction;
6) the set of participants is limited to *primary participants* (a kind of primary dealers), i.e. about 60–70 institutional investors, obliged to satisfy some requirements.

2.2. Formulation of minimisation problem for one auction

The problem of minimisation of T-bills servicing cost consists of: a criterion function of CRR type and constraints on: • minimal receipts, • maximal profitability, • average maturity, • minimal and maximal amount of bills for each maturity and • a share of T-bills with maturity i weeks in total amount of bills sold at the auction. It is clear that a sequence of optimal solutions for some series of auctions is, in general, not equivalent to the optimal solution for whole set of these auctions. Therefore, the generalisation of *one auction problem* for some series of auctions is also drafted — see point 2.5.

The following notation is introduced to present the *one auction problem*:
x_i — decision variables — the number of T-bills with maturity of i weeks, $i = 1, \dots, 52$;
$d^{(i)}(x_i)$ — the average discount for one T-bill corresponding to x_i bills sold at the auction, assuming that the bids are ranked according to increasing value of discount;
r_i, s_i — constants defined in the following way: $r_i = \text{ent}(52/i)$, $s_i = 52/i$, ent(.) — integer part of the argument;
N — the nominal value of one T-bill (now 10,000 Polish zlotys);
I — the set of types (maturities) of T-bills sold at the current auction, $I \subseteq \{1, \dots, 52\}$;
$\mathbf{x}$ — vector of decision variables x_i, $i \in I$;
$\varphi^{(i)}(x_i)$ — a function providing comparability of discount of T-bills with different maturities of the form:

$$\varphi^{(i)}(x_i) = \begin{cases} \prod\limits_{k=0}^{r_1-1} (1+\gamma_k^{(i)}) - 1; \ s_i = r_i \\ (1+(\gamma_{r_i}^{(i)}))^{s_i-r_i} \prod_{k=0}^{r_i-1}(1+\gamma_k^{(i)}) - 1; \ s_i \neq r_i \end{cases} \qquad (i = 1, \dots, 52)$$

where:
$\gamma_0^{(i)} = d^{(i)}(x_i)/(N - d^{(i)}(x_i))$ — profitability of T-bill with maturity of i weeks corresponding to x_i bills sold at the current auction;
$\gamma_k^{(i)}$ $(k = 1, \dots, r_i - 1)$ — forecast of profitability of T-bills with i weeks maturity sold at future auctions (k is the number of a consecutive auction).

The optimisation problem can be written using above notation in the following way: to minimise the function

$$\Phi(x) = \sum_{i \in I} x_i (N - d^{(i)}(x_i)) \varphi^{(i)}(x_i), \tag{1}$$

under the constraints:

$$\sum_{i \in I} x_i (N - d^{(i)}(x_i)) \geq a \tag{2}$$

(constraint on minimal level of receipts from the auction);

$$d^{(i)}(x_i)/(N - d^{(i)}(x_i)) \leq b_i \ (i \in I) \tag{3}$$

(constraint on maximal profitability of T-bills with maturity i weeks);

$$e \leq \sum_{i \in I} i * x_i / \sum_{i \in I} x_i \leq f \tag{4}$$

(constraint on average maturity);

$$c_{i\,\min} \leq N * x_i \leq c_{i\,\max} \ (i \in I) \tag{5}$$

(constraint on minimal and maximal nominal value of T-bills with maturity i weeks);

$$g_i \leq x_i / \sum_{i \in I} x_i \leq k_i \ (i \in I) \tag{6}$$

(constraint on the share of T-bills with maturity i weeks sold at the auction);

$$x_i \ (i \in I) \text{ — integer values.} \tag{7}$$

Each component $x_i(N - d^{(i)}(x_i))\varphi^{(i)}(x_i)$ of the criterion function $\Phi(\mathbf{x})$ is the product of three factors: decision variable x_i (number of T-bills with maturity i weeks), average capital of (one) T-bill corresponding to x_i bonds sold (nominal value N minus average discount $d^{(i)}(x_i)$) and function $\varphi^{(i)}(x_i)$ expressing one year profitability of T-bill with maturity i weeks (for x_i bills sold). Therefore, the criterion function expresses the servicing cost of the entire set of T-bills sold at the auction for one-year period.

Let us notice that the variables x_i can assume integer values only, and that the criterion function (1) and some of constraints (especially (2)) are non-linear empirical functions. Therefore the problem (1)–(7) is a non-linear discrete minimisation problem.

2.3. Approximate method of solving of the minimisation problem

The minimisation problem (1)–(7) is not easy to solve in *real time* (during the auction — about 15 minutes), because of its discrete and non-linear form. Therefore, it is rational to determine some *adequate* approximation of the actual problem and to solve the approximated problem. In this paper a very simple, but efficient *polynomial least square* method of approximation is suggested. The approach proposed concerns two aspects:

a) replacement of the discrete problem with its continuous *equivalent* (let us notice that the nominal value of one T-bill is approximately equal to 0.001%–0.002% of the total value of bills sold at one auction) and
b) approximation of the criterion function components of the form $x_i(N - d^{(i)}(x_i))\varphi^{(i)}(x_i)$ and the non-linear constraint (2) (of the form $x_i(N - d^{(i)}(x_i))$ with polynomials of a possibly low order.

The following method of approximation of the criterion function (1) is suggested:

1) each component $\Gamma^{(i)}(x_i) = x_i(N - d^{(i)}(x_i))\varphi^{(i)}(x_i)$ of the function (1) is approximated with the polynomial $\Gamma_a^{(i)}(x_i)$ of the form:
$$\Gamma_a^{(i)}(x_i) = \beta_0^{(i)} + \beta_1^{(i)} x_i^1 + \beta_2^{(i)} x_i^2 + \ldots + \beta_{n_i}^{(i)} x_i^{n_i}, \tag{8}$$
where:
   - parameters $\beta_k^{(i)}$ $(k = 1, \ldots, n_i)$ are estimated with the use of (ordinary) least squares method (OLS) based on the values $\langle \Gamma^{(i)}(x_{ik}), x_{ik} \rangle$, where $x_{ik} = w_{i1} + \ldots + w_{ik}$, $(k = 1, \ldots, L_i)$, w_{ik} — number of T-bills with maturity i weeks in k-th bid (bids are ranked according to increasing value of discount), L_i — number of bids submitted at the auction,

- order n_i of the polynomial (8) is determined on the basis of adequacy analysis of the approximated and actual functions (i.e. $\Gamma_a^{(i)}(x_i)$ and $\Gamma^{(i)}(x_i)$); the following measures of adequacy are used: determination coefficient R^2, standard deviation of residual terms: $e^{(i)}(x_i) = \Gamma^{(i)}(x_i) - \Gamma_a^{(i)}(x_i)$, and tests for significance of individual parameters $\beta_k^{(i)}$ (t-Student's test) and whole set of parameters $\beta_1^{(i)}, \ldots, \beta_{n_i}^{(i)}$ (F-Snedecor's test);

2) the criterion function $\sum_{i\in I} \Gamma^{(i)}(x_i)$ is approximated with the function $\sum_{i\in I} \Gamma_a^{(i)}(x_i)$.

The approximation $\sum_{i\in I} \xi_a^{(i)}(x_i)$ of the function $\sum_{i\in I} x_i(N - d^{(i)}(x_i))$ (constraint (2)) is performed in a similar way, i.e. each component $\xi^{(i)}(x_i) = x_i(N - d^{(i)}(x_i))$ is approximated with the polynomial:

$$\xi_a^{(i)}(x_i) = \alpha_0^{(i)} + \alpha_1^{(i)} x_i^1 + \alpha_2^{(i)} x_i^2 + \ldots + \alpha_{m_i}^{(i)} x_i^{m_i}. \tag{9}$$

The approximated continuous problem with criterion function $\sum_{i\in I} \Gamma_a^{(i)}(x_i)$ and constraint (2) of the form $\sum_{i\in I} \xi_a^{(i)}(x_i) \geq a$ can be solved with the use of the *solver* package from the Excel'97 spreadsheet. The *continuous* optimal solution is rounded to integer numbers. The precision of the approximation is discussed in point 2.4.

The criterion function (1) and constraint (2) can be approximated with the use of more sophisticated approach, e.g. optimal systems of polynomials, exponential functions, etc. (see e.g. Ralston, 1965). However, the approach proposed in this paper provides sufficient precision for problems under consideration and can be easily implemented in practice — using a PC computer and a *non-specialised* software (OLS estimation algorithm and conjugated gradient algorithm from *solver* package). Moreover, computation time is relatively short, because:

— the polynomials (8) and (9) assume (typically) low order (n_i and m_i from the range 7–11 for the number of bids L_i at the auction from the range 50—300);

— the OLS estimation of the coefficients and determination of the optimal solution of continuous *polynomial* problem is not *time consuming.*

2.4. Example of empirical results

The empirical results presented below are based on data (results of an auction) from the first half of 2000. The parameters of the optimisation problem were determined in the following way (the derivatives $d\Gamma_a^{(i)}(x_i)/d(x_i)$ of approximated components $\gamma_a^{(i)}(x_i)$ of the criterion function (1) and residuals $e^{(i)}(x_i)$ for some degrees of approximation polynomials are presented in Fig. 1):

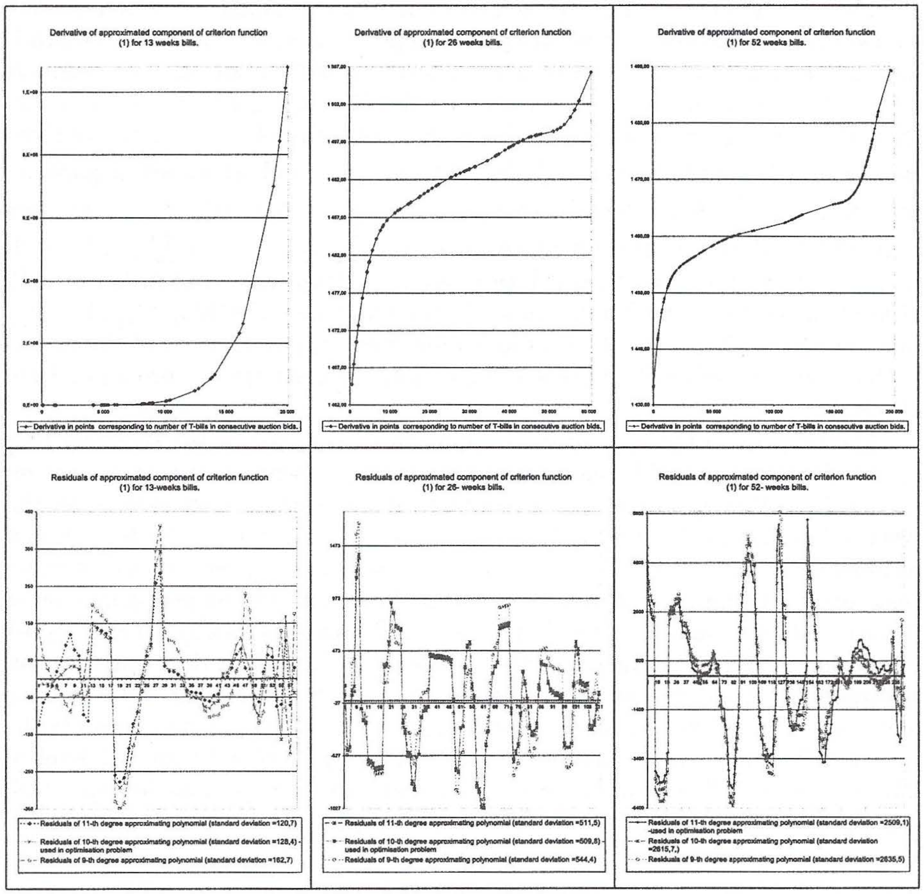

Figure 1. Derivatives and residuals of approximated components of criterion function for T-bills

— the approximation of the criterion function (1) was determined on the basis of auction bids and *forecasts* of future interest rates;
— the parameters of constraints (2)–(6) were determined according to current budgetary needs and the situation on the financial market;
— degrees of polynomials in (approximated) criterion function (1) and non-linear constraint (2) were determined on the basis of the analysis of results of OLS estimation (typically both polynomials assume the same order); the (absolute) values of parameters $\alpha_k^{(i)}$ and $\beta_k^{(i)}$ in functions (8) and (9) ranged between 10^{-60} and 10^4.

The parameters determining the precision of the obtained approximation assumed the *satisfactory* — from the practical point of view — values: • determination coefficient R^2 was greater than 0.9999, • standard deviation of residuals assumed values from the range of $\kappa * 10^1 \div \lambda * 10^4$ ($\kappa, \lambda \in (0,1)$), i.e. were less than the nominal value of one T-bill, • significance levels of tests for verification of individual parameters $\beta_k^{(i)}$ (t-Student's test) and whole set of parameters $\beta_1^{(i)}, \ldots, \beta_{n_i}^{(i)}$ (F-Snedecor's test) assume values below 0.01 (very often less than 10^{-5}). Moreover, the differences $|\eta(\mathbf{x}^*)| = |\sum_{i \in I} \Gamma^{(i)}(x_i^*) - \sum_{i \in I} \Gamma_a^{(i)}(x_i^*)|$, where $\mathbf{x}^*$ is vector of optimal solution, assumed (typically) values below 25% of nominal value of one T-bill, while the relative value of $\eta(\mathbf{x}^*)/\sum_{i \in I} \Gamma^{(i)}(x_i^*)$ was less than 0.001% (!). The computation time (approximation of the actual problem and its solving) equals several minutes (PC computer, processor Intel 266 MHz).

The purpose of the optimisation approach is to minimise the servicing cost of the T-bills issued. Therefore, it is important to evaluate the *gain* level resulting from this approach. This *gain* can be determined • ex ante — as the (relative) difference between criterion function corresponding to *traditional* and optimal solution or/and • ex post, as the difference between actual servicing cost corresponding to *traditional* and optimal solution (the ex post gain can be determined after maturity time of T-bills under consideration). The ex post evaluation is not available currently — because of the fact that optimisation approach has been started in the first quarter of the year 2000. The ex ante assessments show that the optimisation approach provides (relative) gain in the criterion function value ranging between 0.1% and 1.8% (the annual level of interest rates: 16%–18%). These facts indicate that the optimisation approach can provide significant reduction of the servicing costs.

2.5. Formulation of the problem for the case of sequence of auctions

The problem presented in points 2.2–2.4 can be easily generalised for the case of a series of auctions under the assumption that the time horizon of these auctions is shorter than maturity of each T-bill sold (because there does not exist *feed back* effect). The basic problem (1)–(7) can be modified in the following way:

- the set of decision variables x_i, $1 \le i \le 52$ should be replaced with the set x_{ij}, $1 \le i \le 52$, $j = 1, \ldots, T$, T — number of auctions in the series (variables x_{ij}, for $j > 1$ can be assumed as the supply levels for future auctions);
- the forms of the functions $d^{(ij)}(x_{ij})$ and $\varphi^{(ij)}(x_{ij})$ have to be predicted for the future auctions ($j > 1$);
- the constraints (2)–(7) should be formulated not only for each individual auction, but also for the entire set of the auctions and its subsets.

The resulting optimisation problem is more complex, in particular, it requires the forecasts of some parameters and functions, but is more general.

3. Minimisation of servicing cost of T-bonds sold at multiprice auction

The problem of minimisation of the T-bonds servicing cost is more complex than that corresponding to T-bills, because of the following reasons:

a) the *construction* of individual bonds (currently four types — for institutional investors) is more non-homogenous in comparison with the T-bills, involving bonds that are: • zero-coupon, fixed-rate and variable rate • with maturity from two to ten years (therefore it is rational to take into account the risk associated with interest rate variability);
b) each of these bonds is sold at a different auction; only two of these auctions are performed in the same time (two-years zero-coupon and five-years fixed rate).

The optimisation problem is formulated, first, for two- and five-year fixed rate bonds sold at the same time and then generalised for the entire set of T-bonds.

3.1. Main features of T-bonds and auctions for these instruments in Poland

The set of T-bonds sold in Poland consists of four bonds:

1) two-year zero coupon bond sold with a discount paid at maturity,
2) five-year fixed-rate bond with the coupon paid annually and a discount paid at maturity,
3) ten-year floating-rate bond; the coupon (paid annually) is equal to the average profitability of 52-weeks T-bill (from some number of auctions) plus one percentage point, with discount paid at maturity,
4) ten-year fix-rate bond with the coupon paid annually and a discount paid at maturity.

All these bonds are sold at multiprice auctions. Auctions for two-year and five-year bonds are held once a month at the same time. Ten-year bonds (fixed and floating rate) are sold every two-months — alternatively. The organisational rules of these auctions are similar to the T-bills case, with the difference that participants are not limited to *primary participants.* The typical amount offered at individual auction is in the range of 0.2–10 billion (10^9) of Polish zlotys (1 zloty $\approx$ 0.25 US$).

3.2. Formulation of the minimisation problem for two-year (zero coupon) and five-year (fixed rate) bonds

The minimisation problem formulated for two-year and five-year T-bonds is similar to that for T-bills, but takes into account additional constraints for duration and risk level — in the form of variance and covariance matrix of future interest rates (see e.g. Elton and Gruber, 1991).

Let us introduce the following notation:
x_1 — the number of two-year bonds,
x_2 — the number of five-year bonds (x_i — decision variables),
$\Delta^{(i)}(x_i)$ $(i = 1, 2)$ — the average discount (for one bond) — corresponding to x_i — bond sold,
R — coupon of five-year bonds (currently 8.5%),
$r_{i,t+\iota}$ $(i = 1, 2;\ \iota = 1, \ldots, 5)$ — annual interest rates corresponding to i-th bond (*adjusted* to the payments date) in the years: $t + \iota$, t — current year,
δ_i $(i = 1, 2)$ — average (for the auction) duration of i-th bond,
s_1^2 — variance of interest rate of two-year bond during maturity period,
s_2^2 — variance of interest rate of five-year bond during maturity period,
ρ_{12} — correlation coefficient of interest rates volatility (two-year and five-year bonds),
Ω — variance and covariance (or semivariance and semicovariance) matrix of the form:

$$\Omega = \begin{bmatrix} s_1^2 & \rho_{12} \\ \rho_{21} & s_2^2 \end{bmatrix},$$

N — nominal value of one T-bond (currently 1000 Polish zlotys),
$\tau_1 = \nu_1/365$; ν_1 — number of days between the date of payment for two-year (zero-coupon) bonds and maturity date of the five-year bonds,
$\tau_2 = \nu_2/365$; ν_2 — number of days between the date of payment and maturity time of the five-year bonds,
C_2 — accrued interests of five-year bonds paid at the auction (they are not the same for the bonds from the same *series*, i.e. with the same maturity time, but sold at different auctions) — component of *full price*; (this component does not exist in zero coupon bond, $C_1 \equiv 0$).

Under these notations the minimisation problem can be stated as follows. Minimise the function:

$$(N - \Delta^{(1)}(x_1))x_1\psi_1(x_1) + (N - \Delta^{(2)}(x_2) + C_2)x_2\psi_2(x_2) \tag{10}$$

where:

$$\psi_1(x_1) = \Big((N/(N - \Delta^{(1)}))\prod_{j=3}^{5}(1 + r_{1j})\Big)^{1/\tau_1} - 1,$$

$$\psi_2(x_2) = \Big(\Big(N * R\sum_{k=1}^{4}\prod_{j=k+1}^{5}(1 + r_{2j})$$

$$+ N(1 + R)\Big)\Big/(N - \Delta^{(2)}(x_2) + C_2)\Big)^{1/\tau_2} - 1,$$

under the constraints:

$$\sum_i x_i(N - \Delta^{(i)}(x_i) + C_i) \geq a \tag{11}$$

(*minimal receipts* constraint),

$$b_i \leq Nx_i \leq c_i \; (i = 1, 2) \tag{12}$$

(minimal and maximal nominal amount for each type of bond),

$$e \leq (2x_1 + 5x_2)/\sum_i x_i \leq f \tag{13}$$

(average maturity of bonds sold),

$$g \leq (\delta_1 x_1 + \delta_2 x_2)/\sum_i x_i \leq h \tag{14}$$

(average duration of bonds sold),

$$\Delta^{(i)}(x_i)/(N - \Delta^{(i)}(x_i)) \leq D_i \; (i = 1, 2) \tag{15}$$

(maximal discount to capital ratio of bonds sold),

$$u \leq x_1/(x_1 + x_2) \leq w \tag{16}$$

(share of the two-year T-bonds in total bonds sold),

$$\mathbf{z}^T \Omega \mathbf{z} \leq \Re, \text{ where } \mathbf{z}^T = [z_1, z_2], \; z_i = x_i/\sum_i x_i \; (i = 1, 2) \tag{17}$$

(maximal risk level constraint for bonds sold),

$$x_i \; (i = 1, 2) \text{ integer numbers.} \tag{18}$$

Each component of the criterion function is the product of: decision variable x_i, full price $(N - \Delta^{(i)}(x_i) + C_i)$, and the function $\psi_i(x_i)$, which is a special case of compound rate of return function (CRR($\cdot$)) for five years investment horizon (see Jajuga and Jajuga, 1997). The CRR($\cdot$) function can be written in general in the form:

$$\mathrm{CRR}(\cdot) = \Big[\Big(\sum_{i=t}^{n} S_i \prod_{j=i+\iota}^{n} (1 + r_j) + N)/I_0\Big]^{1/n} - 1,$$

where:
S_t — cash flow (coupon of bond) in time t;
r_j — (annual) interest rate in period j;
N — nominal value paid at redemption;
I_0— investment value (full price of bond).

The function $\psi_1(x_1)$ is a special case of CRR function for one cash flow only — nominal value N paid at maturity, while the price of the bond is equal to $(N - \Delta^{(i)}(x_i))$. The function $\psi_2(x_2)$ is a case of CRR function for five coupons — each equal to $N * R$, nominal value N paid at maturity and (full) price equal to $(N - \Delta^{(2)}(x_2) + C_2)$. Therefore, the criterion function (10) expresses the annual servicing cost of the bonds under consideration in the assumed investment horizon.

The form of the optimisation problem for T-bonds is more complex in comparison with the T-bills case, but it can be also solved using the Excel spreadsheet.

3.3. Minimisation problem for the entire set of T-bonds

This minimisation problem for the whole set of T-bonds can be formulated in the same way as (10)–(18), but it cannot be applied during the auction (because only two of these auctions are performed at the same time). Therefore, such an optimisation can be applied only *ex post* i.e. on the basis of data from auctions held during some period of time (e.g. a quarter). The result of the optimisation can be applied — under the assumption that the form of the *demand function* is stable in some time interval — for determining *supply level* and its structure. Let us also notice that the T-bonds optimisation problem can be solved for different time horizons, e.g. five or ten years; the optimal solution may be different for various individual horizons. In this paper we formulate the problem with time horizon corresponding to the instrument with the longest maturity (ten year floating or variable rate). The form of such a problem is — in general — similar to (10)–(18), with the following differences:

a) the criterion function comprises four components – each for one type of a bond,
b) the constraints may include some additional conditions — e.g. share of fixed rate bonds in the total amount,
c) some constraints include a subset of bonds, e.g. the constraint for average duration is formulated for the fixed rate bonds only.

Using CRR as a criterion function and the notation:
x_1 — number of two-year bonds,
x_2 — number of five-year bonds,
x_3 — number of fixed-rate ten-year bonds,
x_4 — number of floating-rate ten-year bonds,
R_i — coupon of i-th ($i = 2, 3$) fixed-rate bonds,
R_{4j} — coupon of floating-rate ten-year bonds, in the j-th year of its life,
$r_{i,t+\iota}$ ($i = 1, \ldots, 4$; $\iota = 1, \ldots, 10$) — annual interest rate in the year $t + \iota$ (*adjusted* to the payments dates — coupons and principal value — of i-th bond),
t — current year,
$\tau_i = \nu_i/365$; ν_i ($i = 1, \ldots, 4$) — number of days between the date of payment for i-th bonds and maturity date of ten year bonds (with *latest* maturity time),
C_i ($i = 2, \ldots, 4$) — accrued interests of i-th bond paid at the auction (as in the point 3.2), the criterion function can be written in the form:

$$\begin{aligned} &(N - \Delta^{(1)}(x_1))x_1\psi_1(x_1) + (N - \Delta^{(2)}(x_2) + C_2)x_2\psi_2(x_2) \\ &+ (N - \Delta^{(3)}(x_3) + C_3)x_3\psi_3(x_3) \\ &+ (N - \Delta^{(4)}(x_4) + C_4)x_4\psi_4(x_4) \rightarrow \min \end{aligned} \tag{19}$$

where:

$$\psi_1(x_1) = \Big((N/(n - \Delta^{(1)}(x_1))) \prod_{j=3}^{10}(1 + r_{ij})\Big)^{1/\tau_1} - 1,$$

$$\psi_2(x_2) = \bigg(\Big(N * R_2 \sum_{k=1}^{5} \prod_{j=k+1}^{10} (1 + r_{2j})$$

$$+ N \prod_{j=6}^{10}(1 + r_{2j})\Big) \Big/ (N - \Delta^{(2)}(x_2) + C_2)\bigg)^{1/\tau_2} - 1,$$

$$\psi_3(x_3) = \bigg(\Big(N * R_3 \sum_{k=1}^{9} \prod_{j=k+1}^{10} (1 + r_{3j})$$

$$+ N(R_3 + 1)\Big) \Big/ (N - \Delta^{(3)}(x_3) + C_3)\bigg)^{1/\tau_3} - 1,$$

$$\psi_4(x_4) = \bigg(\Big(N \sum_{k=1}^{9} \prod_{j=k+1}^{10} R_{4,j-1}(1 + r_{4j})$$

$$+ N(R_{4,10} + 1)\Big) \Big/ (N - \Delta^{(4)}(x_4) + C_4)\bigg)^{1/\tau_4} - 1.$$

The interpretation of criterion function (19) is similar, as in Section 3.2. The constraints to this problem are analogous to (10)–(18), but for the entire set of T-bonds or/and its subsets.

3.4. Approximate method of solving the minimisation problems for T-bonds and an example of empirical results

The optimisation problems for T-bonds (for two bonds sold at the same time and for the whole set of bonds) can be solved in the same way, as the problem for T-bills (see point 2.3). The form of these problems was determined in the following way:

- the forecasts of future interest rates necessary for the criterion function and some constraints have been determined on the basis of official (government) programs and documents;
- the form of constraints have been assumed on the basis of budgetary needs and maturity profile required (the constraint on risk level has been not used).

The optimisation problem for *simultaneous* auctions (two- and five-year bonds) was solved for the five years horizon, while the problem for the entire set of bonds — for the ten years horizon.

The T-bond problems have been approximated in the same way as in the T-bills case. The results of the optimisation can be summarised in the following way (the derivatives of approximated components of the criterion function (10) and the residuals for some degrees of approximation polynomials are presented in Fig. 2):

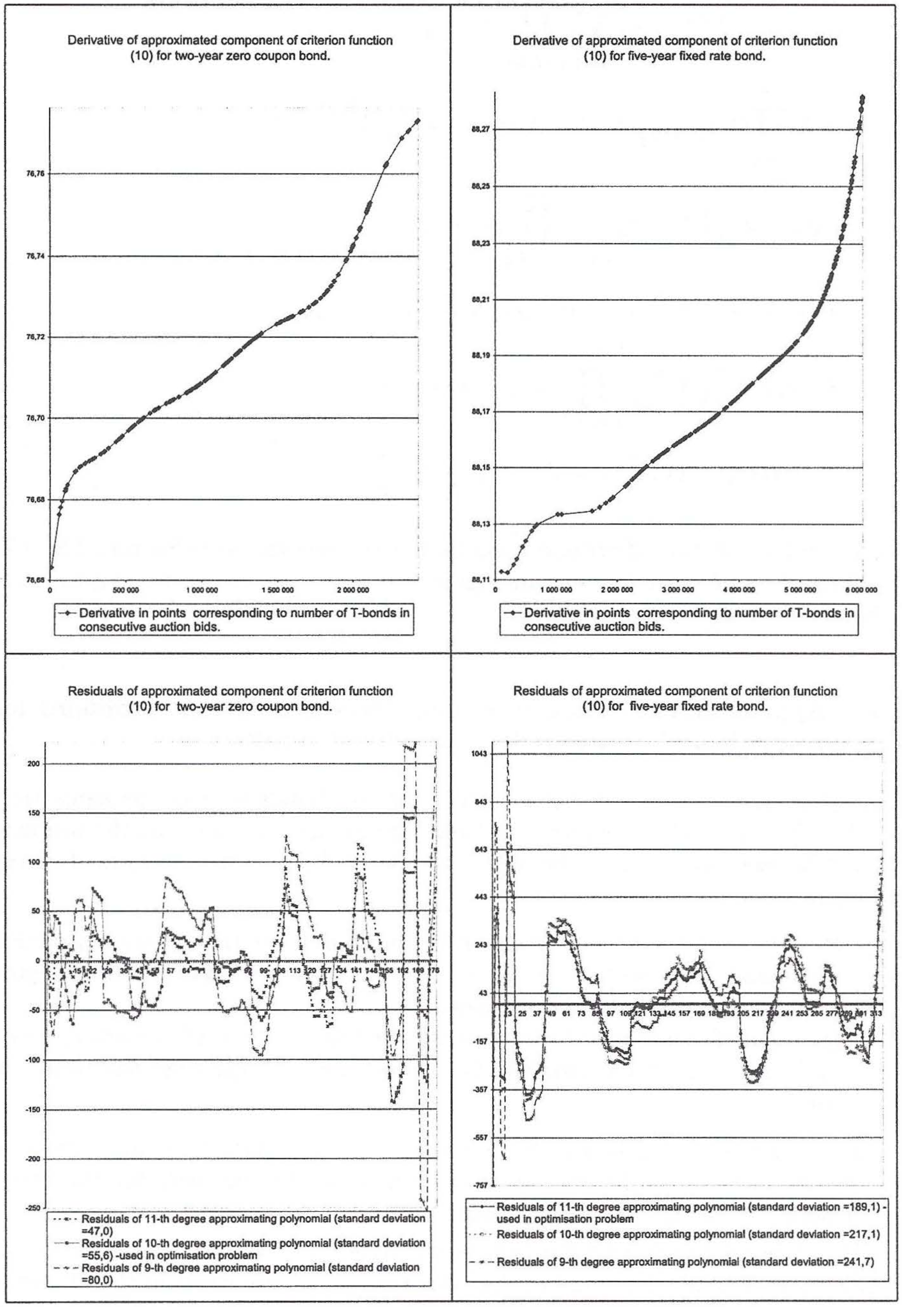

Figure 2. Derivatives and residuals of approximated components of criterion function for T-bonds

1) the precision level of approximation obtained for T-bonds problems was lower — in comparison with T-bills case, but sufficient for practical purposes;
2) the time necessary for solving the problems (10)–(18) (and with the criterion function (19)) is slightly higher in comparison with the T-bills task;
3) the (ex ante) relative *gain* from the optimisation approach was in the range 0.5%–1%;
4) the results of optimisation for the five years horizon were in some sense opposite to those from the ten years horizon; in the first case the two-year bonds were *cheaper* than five-year, while in the second – vice versa.

4. Conclusions

The results presented in the paper are the basis for some general conclusions:

1. The optimisation problems formulated in the paper allow to minimise the servicing cost of debt instruments issued by government sold at multiprice auctions. The results of optimisation can also be used in the further formal analysis concerning the results of auctions.
2. The form of these problems reflects in an adequate way the actual conditions of the decision making process; in particular, they take into account various types of constraints and a precise structure of future interest rates (up to one day). It is not possible to obtain such results on the basis of experience or an intuitive approach only. The main role of the decision-maker in this approach is to determine realistic assumptions and *requirements*, while computer solves the computational side of the problem.
3. The problems and methods under consideration can be used not only to support decision-making process during the auctions (*in real time*), but also for determining the offer level and its structure in future periods. By analysing the optimal solutions obtained under different assumptions one can evaluate the influence of these assumptions, especially those about future interest rates and the optimisation horizon, on results.
4. The actual minimisation problems are replaced in this paper with their polynomial approximation. The suggested approximation method is very simple, but time efficient and can be performed with the use of *popular* software (solver from Excel'97); moreover, precision of the approximation is sufficient from the practical point of view. Such an approach can be implemented in a day-to-day work, not only by specialists in the optimisation area. Of course, it is possible to work out a more sophisticated method of approximation.
5. The optimisation tasks presented in the paper can be developed and generalised in many ways, especially it is possible to formulate problems for:

— the entire debt (not only for individual instruments), taking into account such factors as budgetary requirements, buy-back before maturity, foreign debt instruments, etc.;

— *issuing horizon* of strategic length — longer than one year.

The mathematical form of such problems may be not much more complicated than those presented in this paper, but determination of the parameters for such problems, especially of a broad set of *nontrivial* forecasts, may be not easy.

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