Control and Cybernetics

vol. **31** (2002) No. 1

The method of solution for hydrodynamic lubrication by synovial fluid flow in human joint gap

by

Krzysztof Wierzcholski

Department of Mechanical Engineering, Maritime Academy of Gdynia, Morska 83, PL-81225 Gdynia, Poland, E-mail: wierzch@wsm.gdynia.pl

Abstract: The paper shows a method of solution of systems of partial differential non-linear, second order equations for synovial axial-symmetrical and unsymmetrical fluid flow in curvilinear orthogonal coordinates between two bone surfaces in human joint gap. Theorems are formulated, describing unification method of solutions of partial differential non-linear equations of synovial fluid flow.
Keywords: non-Newtonian lubrication, human joints, deformable cartilage.

1. Introduction

This paper presents modelling and simulation for synovial fluid flow occurring in gap between two co-operating bone surfaces in human joint. The present paper gives an analysis of solutions of systems of non-linear, partial, differential equations for synovial fluid flow in human joint gap. Fig. 1 shows the geometry of various human joints.

In the hip joint the spherical rotational bone and the pelvis bone create a spherical gap (Fig. 1). In this gap between two co-operating bones, synovial fluid flows, see Mow (1969), Mow et al. (1984, 1990, 1991, 1994, 1998), Maquet (1984), Ungethüm et al. (1990), Wiślicki (1980), Wierzcholski et al. (1993, 1994, 1995),Wierzcholski (1993), Pytko et al. (1995), Dowson (1990).The flow of this fluid is caused by the motion of the bone head. The theoretical considerations of the synovial non-Newtonian fluid flow in thin joint gap, taking into account boundary layer simplifications, have practical applications in theory of lubrication in medicine, see Dowson et al. (1998), Gruca (1993), Winters et al. (1990). The considerations in the present paper enable to find synovial fluid flow parameters and carrying capacity force in joint gap between two co-operating bone surfaces in curvilinear orthogonal co-ordinates.

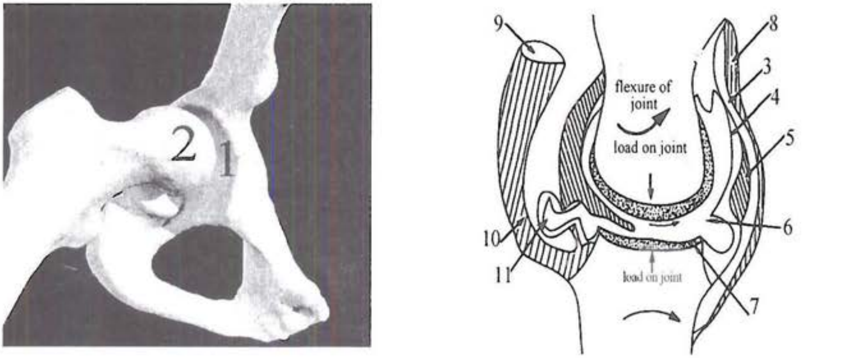

Figure 1. Geometry of human joint: 1—biobearing gap, 2—spherical rotational bone head, 3—fibrous capsule, 4—synovial membrane, 5—fat pad, 6—articular cavity containing synovial fluid, 7—articular cartilage, 8—ligament, 9—muscle, 10—tendon,

In opposition to the results of Mow (1969), Mow et al. (1990, 1998), the ones obtained in this paper show the estimation of synovial fluid flow equations and by virtue of present results we can apply various geometry of bone surfaces in the human joint analyses.

As contrasted with papers by Batchelor et al. (1996), Hayes et al. (1993), Maurel et al. (1998), Mow et al. (1984, 1991, 1994, 1998), Yao (1993), the present paper shows a unification and analytical method of solutions of the lubrication problem in human joint gap for various joints, for example with spherical, parabolic or hyperbolic bone surfaces and for the non-Newtonian micropolar experimental properties of synovial fluid.

2. Formulation of the problem

The aim of the paper is:

2.1 To present the mathematical estimation of the terms of basic partial differential non-linear equations of second order, for the fluid flow in the thin joint gap between two bone surfaces for various geometry.

2.2 To formulate the theorems which describe the unification and the analytical method of solution of partial differential non-linear equations for axial-symmetrical and unsymmetrical flow of synovial fluid in human joint gap.

3. Basic equations

In this section we show the basic equations describing the synovial fluid flow in joint gap. Equations of conservation of momentum and continuity equation have for the stationary flow of synovial compressible fluid between two non-rotational surfaces in the curvilinear orthogonal co-ordinates, the following form, see Wierzcholski (1993):

$$\rho\left(\operatorname{grad}\frac{1}{2}\boldsymbol{v}\boldsymbol{v}-\boldsymbol{v}\times\operatorname{rot}\boldsymbol{v}\right)=\operatorname{Div}\boldsymbol{S}, \tag{1}$$

$$\operatorname{div}\boldsymbol{v}=0, \tag{2}$$

where constitutive equations are as follows:

$$\boldsymbol{S}=-p\boldsymbol{I}+2\eta_p\boldsymbol{T_d}, \tag{3}$$

whereas the components of stress tensor $\boldsymbol{S}$, unit tensor $\boldsymbol{I}$, strain tensor $\boldsymbol{T_d}$, are respectively: τ_{ij}, δ_{ij}, θ_{ij}. We use the following notations:

δ_{ij} — Kronecker delta,
ρ — fluid density,
η_p — dynamic viscosity of the synovial non-Newtonian fluid, strain component dependent,
$\boldsymbol{v}$ — fluid velocity vector with components v_i,
p — pressure,
θ_{ij} — components of strain tensor.

Geometrical dependencies between strain components and fluid velocity components in orthogonal curvilinear coordinates have the form, see Truesdell (1972):

$$\theta_{ij}=\frac{1}{2}\left[\frac{h_i}{h_j}\frac{\partial}{\partial\alpha_j}\left(\frac{v_i}{h_i}\right)+\frac{h_j}{h_i}\frac{\partial}{\partial\alpha_i}\left(\frac{v_j}{h_j}\right)+2\delta_{ij}\sum_{k=1}^{3}\frac{v_k}{h_ih_k}\frac{\partial h_i}{\partial\alpha_k}\right], \tag{4}$$

moreover, h_1, h_2, h_3 are the Lamé coefficients, α_1, α_2, α_3—curvilinear orthogonal coordinates; $i,j=1,2,3$. We introduce equation (3) into right hand of equation (1). Thus we have:

$$(\operatorname{Div}\boldsymbol{S})_i\equiv\frac{1}{h_i}\frac{\partial p}{\partial\alpha_i}$$

$$+\frac{1}{gh_i}\left\{\sum_{j=1}^{3}\frac{\partial}{\partial\alpha_j}\left(2\eta_p\frac{gh_i\theta_{ij}}{h_j}\right)-\frac{1}{2}\sum_{j=1}^{3}\left[2\eta_p\frac{g\theta_{ij}}{h_j^2}\frac{\partial(h_j)^2}{\partial\alpha_i}\right]\right\}, \tag{5}$$

$$0=\operatorname{div}\boldsymbol{v}\equiv\frac{1}{g}\sum_{k=1}^{3}\frac{\partial}{\partial\alpha_k}\left(\frac{gv_k}{h_k}\right), \tag{6}$$

where $g\equiv h_1h_2h_3$ and $i=1,2,3$. Expanding the left hand of equation (1) we obtain finally:

$$\rho\left(\operatorname{grad}\frac{\boldsymbol{v}\boldsymbol{v}}{2}-\boldsymbol{v}\times\operatorname{rot}\boldsymbol{v}\right)_i$$

$$=\sum_{j=1}^{3}\left\{\left[\frac{1}{h_j}\frac{\partial v_i}{\partial\alpha_j}-\frac{v_j}{h_ih_j}\frac{\partial h_j}{\partial\alpha_i}+\delta_{ij}\frac{1}{h_j}\sum_{k=1}^{3}\left(\frac{v_k}{h_k}\frac{\partial h_j}{\partial\alpha_k}\right)\right]v_j\right\}. \tag{7}$$

4. Flow simulation for thin gap between two non-rotational bone surfaces

CASE OF FLOW 4.1. *When the orthogonal curvilinear co-ordinates α_1, α_2, α_3 coincide with curvature lines of the thin layer resting on non-rotational bone surface in human joint gap, where α_1—length direction, α_2—perpendicular direction to the bone surface (in gap height direction), α_3—width surface direction, then the Lamé coefficients for the thin layer surface with non-monotonic curvatures are as follows:*

$$h_1 \equiv h_1(\alpha_1, \alpha_3), \qquad h_2 \equiv 1, \qquad h_3 \equiv h_3(\alpha_1, \alpha_3), \tag{8}$$

Sketch of proof. Let the vector equation of the surface have the following form (Fig. 2):

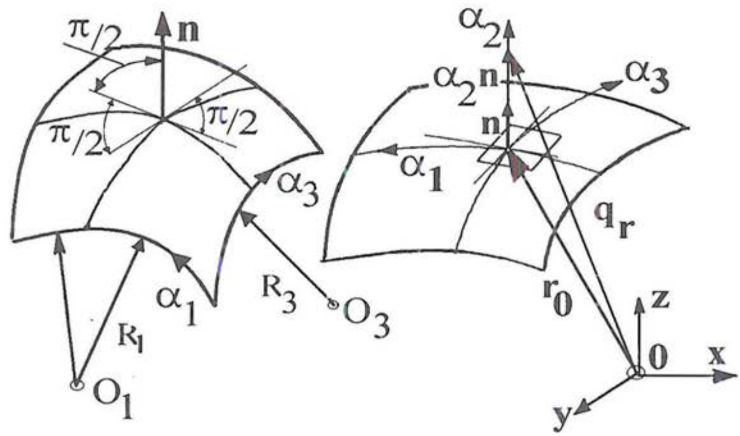

Figure 2. Non rotational bone surface.

$$\boldsymbol{r}_o \equiv \boldsymbol{r}_o(\alpha_1, \alpha_3). \tag{9}$$

The position of any point in the space along the normal vector $\boldsymbol{n}$ in relation to the surface is determined as follows:

$$\boldsymbol{q}_r \equiv \boldsymbol{r}_o(\alpha_1, \alpha_3) + \alpha_2 \boldsymbol{n}(\alpha_1, \alpha_3). \tag{10}$$

The square of the element of length is given by:

$$(ds)^2 \equiv (d\boldsymbol{q_r})^2 = \left(\frac{\partial \boldsymbol{r}_o}{\partial \alpha_1} + \alpha_2 \frac{\partial \boldsymbol{n}}{\partial \alpha_1}\right)^2 (d\alpha_1)^2 + (d\alpha_2)^2$$

$$+\left(\frac{\partial \boldsymbol{r}_o}{\partial \alpha_3}+\alpha_2\frac{\partial \boldsymbol{n}}{\partial \alpha_3}\right)^2 (d\alpha_3)^2. \tag{11}$$

From the Rodrigues Law we have:

$$\frac{\partial \boldsymbol{n}}{\partial \alpha_1}=\frac{1}{R_1}\frac{\partial \boldsymbol{r}_o}{\partial \alpha_1}, \qquad \frac{\partial \boldsymbol{n}}{\partial \alpha_3}=\frac{1}{R_3}\frac{\partial \boldsymbol{r}_o}{\partial \alpha_3}, \tag{12}$$

where R_1, R_2 denote the radii of curvature in directions α_1 and α_3. We substitute equation (12) into equation (11) and we take into account the thin layer simplifications, i.e. $\alpha_2/R_1 \Rightarrow 0$, $\alpha_2/R_2 \Rightarrow 0$, and then we obtain:

$$h_1=\left|\frac{\partial \boldsymbol{r}_o(\alpha_1,\alpha_3)}{\partial \alpha_1}\right|, \qquad h_2=1, \qquad h_3=\left|\frac{\partial \boldsymbol{r}_o(\alpha_1,\alpha_3)}{\partial \alpha_3}\right|. \tag{13}$$

This remark completes the case of flow 4.1. ∎

5. Flow in thin gap between two rotational surfaces

CASE OF FLOW 5.1. *If orthogonal curvilinear coordinates α_1, α_2, α_3 are curvature lines of a thin layer of synovial fluid resting on the rotational bone surface in human joint gap, where α_1—circumference direction, α_3—generating line of rotational bone direction, α_2—gap height direction, then Lamé coefficients for thin layer with non-monotone generating line are as follows:*

$$h_1 \equiv h_1(\alpha_3), \qquad h_2 \equiv 1, \qquad h_3 \equiv h_3(\alpha_3), \tag{14}$$

Sketch of proof. For the rotational surface the radius vector has the form:

$$\boldsymbol{r}_o \equiv \boldsymbol{i}R(\alpha_3)\cos\alpha_1+\boldsymbol{j}R(\alpha_3)\sin\alpha_1+\boldsymbol{k}Z(\alpha_3), \tag{15}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are the unit vectors in the Cartesian system and the projections Z and R of the vector $\boldsymbol{r}_o$ indicated in Figs. 3 and 4 are the functions of α_3 only. We put equation (15) into equation (13), hence we obtain the dependencies (14), which completes the proof of case of flow 5.1. ∎

CASE OF FLOW 5.2. *For monotone generating line of rotational surface (see Fig. 4) the Lamé coefficients have the following form:*

$$h_1 \equiv h_1(\alpha_3), \qquad h_2 \equiv 1, \qquad h_3 \equiv 1, \tag{16}$$

Sketch of proof. For the monotone generating line of rotational surface (see Fig. 4) the dependencies (16) are true by virtue of basic theory of differential geometry, which completes the proof of the case of flow 5.2. ∎

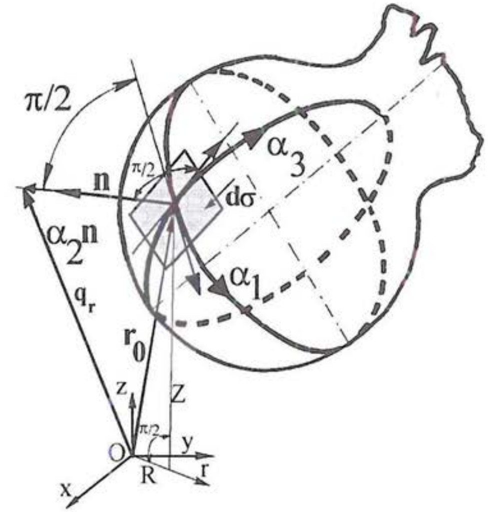

Figure 3. Rotational spherical bone head surface with non-monotone generating line α_3.

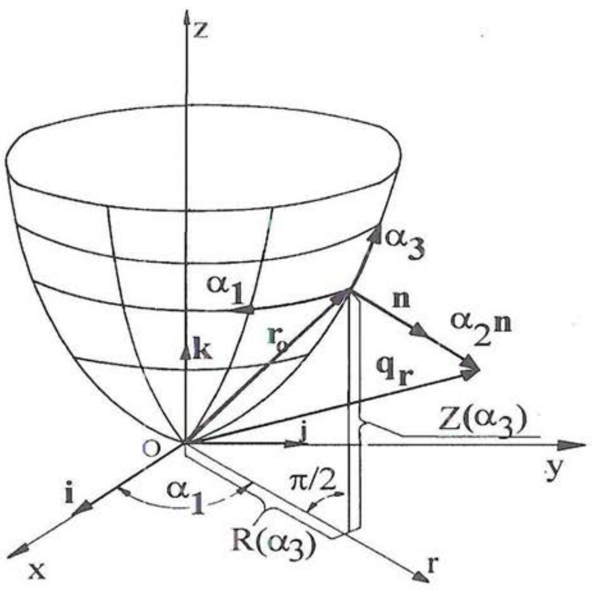

Figure 4. Rotational bone surface with monotone generating line α_3.

LEMMA 5.1. *Equations of conservation of momentum and continuity equation for incompressible, stationary synovial fluid flow in thin fluid layer in human joint resting on rotational bone surface with non-monotone generating line and for orthogonal, curvilinear coordinates* $(\alpha_1, \alpha_2, \alpha_3)$ *have the following form:*

$$\rho\left(\frac{v_1}{h_1}\frac{\partial v_1}{\partial \alpha_1} + v_2\frac{\partial v_1}{\partial \alpha_2} + \frac{v_3}{h_3}\frac{\partial v_1}{\partial \alpha_3} + \frac{v_1 v_3}{h_1 h_3}\frac{\partial h_1}{\partial \alpha_3}\right)$$
$$= -h_1\frac{\partial p}{\partial \alpha_1} + \frac{2}{h_1^2}\frac{\partial}{\partial \alpha_1}\left[\eta_p\left(\frac{\partial v_1}{\partial \alpha_1} + \frac{v_3}{h_3}\frac{\partial h_1}{\partial \alpha_3}\right)\right] + \frac{\partial}{\partial \alpha_2}\left(\eta_p\frac{\partial v_1}{\partial \alpha_2}\right)$$
$$+\frac{1}{h_1}\frac{\partial}{\partial \alpha_2}\left(\eta_p\frac{\partial v_2}{\partial \alpha_1}\right) + \frac{1}{h_1^2 h_3}\frac{\partial}{\partial \alpha_3}\left\{h_1^2\eta_p\left[\frac{h_1}{h_3}\frac{\partial}{\partial \alpha_3}\left(\frac{v_1}{h_1}\right) + \frac{1}{h_1}\frac{\partial v_3}{\partial \alpha_1}\right]\right\}, \quad (17)$$

$$\rho\left(\frac{v_1}{h_1}\frac{\partial v_2}{\partial \alpha_1} + v_2\frac{\partial v_2}{\partial \alpha_2} + \frac{v_3}{h_3}\frac{\partial v_2}{\partial \alpha_3}\right) = -\frac{\partial p}{\partial \alpha_2}$$
$$+\frac{1}{h_1}\frac{\partial}{\partial \alpha_1}\left[\eta_p\left(\frac{\partial v_1}{\partial \alpha_2} + \frac{1}{h_1}\frac{\partial v_2}{\partial \alpha_1}\right)\right] + 2\frac{\partial}{\partial \alpha_2}\left[\eta_p\left(\frac{\partial v_2}{\partial \alpha_2}\right)\right]$$
$$+\frac{1}{h_1 h_3}\frac{\partial}{\partial \alpha_3}\left[h_1\eta_p\left(\frac{1}{h_3}\frac{\partial v_2}{\partial \alpha_3} + \frac{\partial v_3}{\partial \alpha_2}\right)\right], \quad (18)$$

$$\rho\left(\frac{v_1}{h_1}\frac{\partial v_3}{\partial \alpha_1} + v_2\frac{\partial v_3}{\partial \alpha_2} + \frac{v_3}{h_3}\frac{\partial v_3}{\partial \alpha_3} - \frac{v_1^2}{h_1 h_3}\frac{\partial h_1}{\partial \alpha_3}\right) = -\frac{1}{h_3}\frac{\partial p}{\partial \alpha_3}$$
$$+\frac{1}{h_1}\frac{\partial}{\partial \alpha_1}\left\{\eta_p\left[\frac{h_1}{h_3}\frac{\partial}{\partial \alpha_3}\left(\frac{v_1}{h_1}\right) + \frac{1}{h_1}\frac{\partial v_3}{\partial \alpha_1}\right]\right\}$$
$$+\frac{\partial}{\partial \alpha_2}\left[\eta_p\left(\frac{1}{h_3}\frac{\partial v_2}{\partial \alpha_3} + \frac{\partial v_3}{\partial \alpha_2}\right)\right] + \frac{2}{h_1 h_3^2}\frac{\partial}{\partial \alpha_3}\left[h_1\eta_p\left(\frac{\partial v_3}{\partial \alpha_3}\right)\right]$$
$$-\frac{2}{h_1^2 h_3}\eta_p\frac{\partial h_1}{\partial \alpha_3}\left(\frac{\partial v_1}{\partial \alpha_1} + \frac{v_3}{h_3}\frac{\partial h_1}{\partial \alpha_3}\right) - \frac{2\eta_p}{h_3^3}\frac{\partial h_3}{\partial \alpha_3}\left(\frac{\partial v_3}{\partial \alpha_3}\right), \quad (19)$$

$$h_3\frac{\partial v_1}{\partial \alpha_1} + h_1 h_3\frac{\partial v_2}{\partial \alpha_2} + \frac{\partial}{\partial \alpha_3}(h_1 v_3) = 0. \quad (20)$$

We have in the direction of the length $0 \le \alpha_1 \le 2\pi$, *in the direction of the width* $b_m \le \alpha_3 \le b_s$ *and in the direction of gap height* $0 \le \alpha_2 \le \varepsilon(\alpha_1, \alpha_3)$, *whereas* b_m, b_s *are constant limits of lubrication in directions* α_1, α_3.

Proof. We put equations (3), (4), (5), (6), (7) and (14) in equations (1), (2), thus the conservation of momentum equations and the continuity equation for incompressible, stationary, synovial fluid flow in the thin layer resting on rotational surface with non-monotone generating line and in orthogonal, curvilinear coordinates α_1, α_2, α_3 have the form (17), (18), (19), (20). This remark completes the proof of Lemma 5.1. ■

LEMMA 5.2. *Estimation of dimensionless terms with exactness between* 0.10000 *and* 1.00000 *with respect to the neglected terms of order* 0.0010, *for equations of conservation of momentum and continuity equation (17)–(20), in curvilinear orthogonal coordinates* $(\alpha_1, \alpha_2, \alpha_3)$, *for incompressible, stationary, and unsymmetrical synovial fluid flow in thin layer resting on rotational bone surface of human joint gap, with non-monotone generating line, lead to following basic equations*

$$0 = -\frac{1}{h_1}\frac{\partial p}{\partial \alpha_1} + \frac{\partial}{\partial \alpha_2}\left(\eta_p \frac{\partial v_1}{\partial \alpha_2}\right), \tag{21}$$

$$0 = \frac{\partial p}{\partial \alpha_2}, \tag{22}$$

$$-\frac{\rho v_1^2}{h_1 h_3}\frac{\partial h_1}{\partial \alpha_3} = -\frac{1}{h_3}\frac{\partial p}{\partial \alpha_3} + \frac{\partial}{\partial \alpha_2}\left[\eta_p \left(\frac{\partial v_3}{\partial \alpha_2}\right)\right], \tag{23}$$

$$0 = h_3 \frac{\partial v_1}{\partial \alpha_1} + h_1 h_3 \frac{\partial v_2}{\partial \alpha_2} + \frac{\partial}{\partial \alpha_3}(h_1 v_3), \tag{24}$$

where in the length, width and gap height directions, we have, respectively:

$$0 \le \alpha_1 \le 2\pi, \qquad b_m \le \alpha_3 \le b_s, \qquad 0 \le \alpha_2 \le \varepsilon. \tag{25}$$

Proof. The system (21)–(24) describes four unknowns, namely three components of synovial fluid velocity $v_i(\alpha_1, \alpha_2, \alpha_3)$ for $i = 1, 2, 3$ and pressure $p = (\alpha_1, \alpha_3)$. If generating line of rotational thin layer surface in particular case is a monotone function, then in (21)–(24) we have $h_3 \equiv 1$. Now we are taking into account the axial unsymmetrical synovial fluid flow. We assume the following dimensionless values of the Lamé coefficients h_{11}, h_{31}; values of curvilinear coordinates: α_{11}, α_{21}, α_{31}; values of vector velocity components: v_{11}, v_{21}, v_{31}; pressure p_1 and dynamic viscosity η_1. Dimension values have then the following form:

$$h_1 \equiv R h_{11}, \quad h_3 \equiv h_{31}, \quad \alpha_1 \equiv \alpha_{11}, \quad \alpha_2 \equiv \Psi R \alpha_{21}, \quad \alpha_3 \equiv R^* \alpha_{31}, \tag{26}$$

$$v_1 \equiv U v_{11}, \qquad v_2 \equiv \Psi U v_{21}, \qquad \alpha_3 \equiv W v_{31},$$

$$p \equiv \frac{p_0^*}{\Psi^2} p_1, \qquad \eta \equiv \eta_0 \eta_1. \tag{27}$$

The following notations are used:

- R — radius R_1 of the curvature in α_1 direction or radius of the rotational surface,
- R^* — radius of the curvature in α_3 direction or bearing length,
- Ψ — dimensionless radial clearance $= \varepsilon/R \approx 10^{-4}$,
- ε — gap height,
- U — surface linear dimension velocity in α_1 direction,
- W — surface linear dimension velocity in α_3 direction,

p_o^* — estimated value of dimension pressure,
η_o — value of dimensional dynamic viscosity of synovial fluid.
We insert dependencies (26), (27) in equations (17)–(20), hence we have:

$$\operatorname{Re}\Psi\left(\frac{v_{11}}{h_11}\frac{\partial v_{11}}{\partial\alpha_{11}}+v_{21}\frac{\partial v_{11}}{\partial\alpha_{21}}+\frac{RW}{R^*U}\frac{v_{31}}{h_{31}}\frac{\partial v_{11}}{\partial\alpha_{31}}+\frac{RW}{R^*U}\frac{v_{11}v_{31}}{h_{11}h_{31}}\frac{\partial h_{11}}{\partial\alpha_{31}}\right)$$

$$=-\frac{\operatorname{Eu}\operatorname{Re}}{\Psi}\frac{1}{h_{11}}\frac{\partial p_1}{\partial\alpha_{11}}+\frac{2\Psi^2}{h_{11}^2}\frac{\partial}{\partial\alpha_{11}}\left[\eta_1\left(\frac{\partial v_{11}}{\partial\alpha_{11}}+\frac{RW}{R^*U}\frac{v_{31}}{h_31}\frac{\partial h_{11}}{\partial\alpha_{31}}\right)\right]$$

$$+\frac{\partial}{\partial\alpha_{21}}\left(\eta_1\frac{\partial v_{11}}{\partial\alpha_{21}}\right)+\Psi^2\frac{1}{h_{11}}\frac{\partial}{\partial\alpha_{21}}\left(\eta_1\frac{\partial v_{21}}{\partial\alpha_{11}}\right)$$

$$+\frac{R}{R^*}\Psi^2\frac{1}{h_{11}^2h_{31}}\frac{\partial}{\partial\alpha_{31}}\left\{h_{11}^2\eta_1\left[\frac{h_{11}}{h_{31}}\frac{\partial}{\partial\alpha_{31}}\left(\frac{v_{11}}{h_{11}}\right)+\frac{W}{U}\frac{1}{h_{11}}\frac{\partial v_{31}}{\partial\alpha_{11}}\right]\right\},\tag{28}$$

$$O(\operatorname{Re}\Psi^3)=-\frac{\operatorname{Eu}\operatorname{Re}}{\Psi}\frac{\partial p_1}{\partial\alpha_{21}}+O\left(\Psi^2\right)\tag{29}$$

$$\operatorname{Re}\Psi\left(\frac{v_{11}}{h_{11}}\frac{\partial v_{31}}{\partial\alpha_{11}}+v_{21}\frac{\partial v_{31}}{\partial\alpha_{21}}+\frac{WR}{UR^*}\frac{v_{31}}{h_{31}}\frac{\partial v_{31}}{\partial\alpha_{31}}-\frac{UR}{WR^*}\frac{v_{11}^2}{h_{11}h_{31}}\frac{\partial h_{11}}{\partial\alpha_{31}}\right)$$

$$=-\frac{\operatorname{Eu}\operatorname{Re}}{\Psi}\frac{R}{R^*}\frac{1}{h_{31}}\frac{\partial p_1}{\partial\alpha_{31}}$$

$$+\Psi^2\frac{1}{h_{11}}\frac{\partial}{\partial\alpha_{11}}\left\{\eta_1\left[\frac{UR}{WR^*}\frac{h_{11}}{h_{31}}\frac{\partial}{\partial\alpha_{31}}\left(\frac{v_{11}}{h_{11}}\right)+\frac{1}{h_{11}}\frac{\partial v_{31}}{\partial\alpha_{11}}\right]\right\}$$

$$+\Psi^2\frac{UR}{WR^*}\frac{\partial}{\partial\alpha_{21}}\left[\eta_1\left(\frac{1}{h_{31}}\frac{\partial v_{21}}{\partial\alpha_{31}}\right)\right]+\frac{\partial}{\partial\alpha_{21}}\left(\eta_1\frac{\partial v_{31}}{\partial\alpha_{21}}\right)$$

$$+2\Psi^2\left(\frac{R}{R^*}\right)^2\frac{1}{h_{11}h_{31}^2}\frac{\partial}{\partial\alpha_{31}}\left[h_{11}\eta_1\left(\frac{\partial v_{31}}{\partial\alpha_{31}}\right)\right]$$

$$-\frac{2\Psi^2}{h_{11}^2h_{31}}\frac{R}{R^*}\eta_1\frac{\partial h_{11}}{\partial\alpha_{31}}\left(\frac{U}{W}\frac{\partial v_{11}}{\partial\alpha_{11}}+\frac{R}{R^*}\frac{v_{31}}{h_{31}}\frac{\partial h_{11}}{\partial\alpha_{31}}\right)$$

$$-2\Psi^2\left(\frac{R}{R^*}\right)^2\frac{1}{h_{31}^3}\frac{\partial h_{31}}{\partial\alpha_{31}}\eta_1\left(\frac{\partial v_{31}}{\partial\alpha_{31}}\right),\tag{30}$$

$$h_{31}\frac{\partial v_{11}}{\partial\alpha_{11}}+h_{11}h_{31}\frac{\partial v_{21}}{\partial\alpha_{21}}+\frac{UR}{WR^*}\frac{\partial}{\partial\alpha_{31}}(h_{11}v_{31})=0,\tag{31}$$

where Reynolds and Euler numbers have the form:

$$\operatorname{Re}\equiv\frac{\rho U\varepsilon}{\eta_o},\qquad \operatorname{Eu}\equiv\frac{p_o^*}{U^2\rho},\tag{32}$$

and

$$\Psi = O(10^{-3}), \qquad 0 < \mathrm{Re} < 1, \qquad \Psi\frac{U}{W} = O(1), \tag{33}$$

because $W \ll U$. Now we have two possibilities.

$$\mathrm{Eu} = \left(\frac{\Psi}{\mathrm{Re}}\right)^2, \qquad \frac{\mathrm{Eu}\,\mathrm{Re}}{\Psi} = \sqrt{\mathrm{Eu}} = O\left(\frac{1}{10}\right),$$
$$\text{if} \quad p_o^* \equiv \frac{\eta_o^2}{\rho R^2}, \quad p_o \equiv \frac{p_o^*}{\Psi^2} = \frac{\eta_o^2}{\rho\varepsilon^2}, \tag{34}$$

and

$$\mathrm{Eu} = \frac{\Psi}{\mathrm{Re}}, \qquad \frac{\mathrm{Eu}\,\mathrm{Re}}{\Psi} = 1, \qquad \text{if} \quad p_o^* \equiv \omega\eta, \quad p_o \equiv \frac{p_o^*}{\Psi^2} = \frac{\omega\eta R^2}{\varepsilon^2}. \tag{35}$$

The terms of inertia forces in equations (28)–(30) are multiplied by the factor $\mathrm{Re}\,\Psi$. We neglect inertia forces terms and other terms (multiplied by the factor $\mathrm{Re}\,\Psi$ or Ψ^n for $n \geq 1$, $\Psi \cong 10^{-3}$) which are of order $(10^{-3})^n \leq 0.001$ as compared to the terms of order 1.000 or 0.100. Thus, the system of equations (28)–(31) for axial-unsymmetrical, isothermal, stationary synovial fluid flow in the film between two rotational surfaces with non-monotonic generating line has in the curvilinear, orthogonal co-ordinates α_1, α_2, α_3 the dimension form (21), (22), (23), (24). The term of centrifugal acceleration of order $\mathrm{Re}\,\Psi U/W$, occurring in equation (30), can be not negligibly small, because $W \ll U$. This term $-\dfrac{\rho v_1^2}{h_1 h_3}\dfrac{\partial h_1}{\partial \alpha_3}$ exists on the left hand of equation (23). Gap height may be a function of both variables α_1, α_3, i.e.: $\varepsilon_1 = \varepsilon_1(\alpha_1, \alpha_3)$ where $\varepsilon \approx 2 \cdot 10^{-5}$ m. This remark completes proof of Lemma 5.2. ■

6. Boundary conditions

The boundary assumptions for the pressure function $p(\alpha_1, \alpha_3)$ in the human gap joint, as shown in Fig. 5, follow Zimmermann (1995):

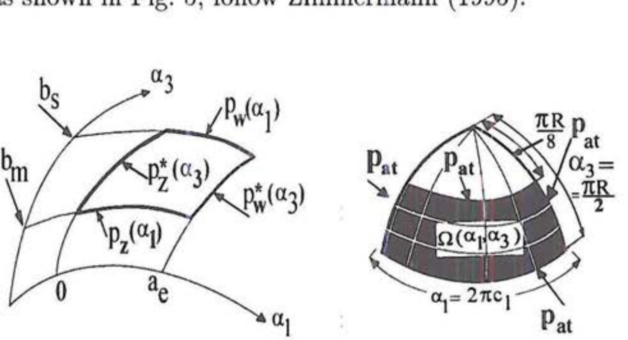

Figure 5. Boundary ranges of hydrodynamic pressure region on bone head in human

$$p(\alpha_1,\, \alpha_3 = b_m) = p_z(\alpha_1), \qquad p(\alpha_1,\, \alpha_3 = b_s) = p_w(\alpha_1),$$
$$p(\alpha_1 = 0,\, \alpha_3) = p_z^*(\alpha_3), \qquad p(\alpha_1 = 0,\, \alpha_3) = p_w^*(\alpha_3) \tag{36}$$

where: $p_z(\alpha_1 = a_e) = p_w^*(\alpha_3 = b_m)$, $p_z(\alpha_1 = 0) = p_z^*(\alpha_3 = b_m)$, $p_w^*(\alpha_3 = b_s) = p_w(\alpha_1 = a_e)$, $p_z^*(\alpha_3 = b_s) = p_w(\alpha_1 = 0)$, and $p_z(\alpha_1)$, $p_z^*(\alpha_3)$ means pressure value at the inlet of the articulation gap in directions α_1, α_3 respectively, $p_w(\alpha_1)$, $p_w^*(\alpha_3)$—pressure value at the outlet of the gap in directions α_1, α_3 respectively.

The boundary conditions for the synovial fluid velocity components, related to the rotational motion in α_1 direction of the surface, have the following form:

$$v_1 = \omega h_1, \qquad v_2 = 0, \qquad v_3 = 0 \qquad \text{for } \alpha_2 = 0, \tag{37}$$

where ω is the angular velocity of the bone. The cartilage of human joint is motionless, therefore:

$$v_1 = 0, \qquad v_2 = 0, \qquad v_3 = 0 \qquad \text{for } \alpha_2 = \varepsilon. \tag{38}$$

Approximation formulae for the dynamic values for various shear rates have the following form:

$$\eta_p \equiv \eta_\infty + \frac{\eta_o - \eta_\infty}{1 + A\Theta} \approx \eta_o - (\eta_o - \eta_\infty)\Theta A + \ldots \quad \text{for } 0 < \Theta^2 B \ll 1, \tag{39}$$

for other cases

$$\eta_p \equiv \eta_\infty + \frac{\eta_o - \eta_\infty}{1 + A\Theta + B\Theta^2} \approx \eta_o - (\eta_o - \eta_\infty)\Theta A - (\eta_o - \eta_\infty)\Theta^2 B + \ldots \tag{40}$$

where η_∞ and η_o, expressed in Pas, mean the dynamic viscosity value of synovial fluid for large and small shear rate values in s^{-1}. Symbols A and B denote the coefficients, which were obtained by Wierzcholski (1993) by virtue of the Cooke's and D. Dowson's (1990) experiments (see Fig. 6).

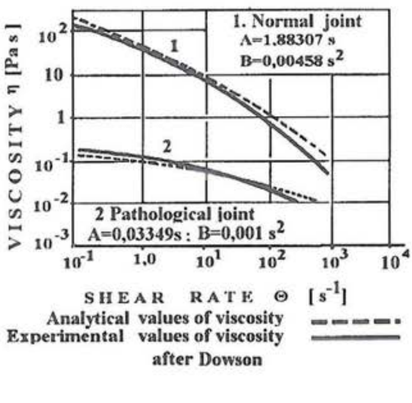

Figure 6. Viscosity of normal and pathological human synovial fluid versus shear rate (after D. Dowson).

We have obtained $A = 1.88307$ s and $B = 0.00458\ \mathrm{s}^2$ for the normal human joint and also $A = 0.03349$ s and $B = 0.00131\ \mathrm{s}^2$ for the pathological human joint. The shear rate has the form:

$$\Theta_0 \cong O\left(\frac{V_0}{\varepsilon}\right), \qquad \Theta \equiv \frac{\partial v_1}{\partial \alpha_2}. \tag{41}$$

7. Properties of analytical solutions

LEMMA 7.1. *The involved solutions of the non linear partial differential system of second order (21), (22), (23), (24), for synovial fluid flow velocity components and pressure in human joint gap resting on rotational bone surfaces with non monotone generating line for the boundary conditions (36), (37), (38) and variable dynamic viscosity function (39) obtained from experiments, have the following form:*

$$v_1(\alpha_1,\alpha_2,\alpha_3) = -\frac{1}{4\eta_\infty h_1}\frac{\partial p}{\partial \alpha_1}\alpha_2\varepsilon\left(1-\frac{\alpha_2}{\varepsilon}\right) + \omega h_1\left(1-\frac{\alpha_2}{\varepsilon}\right)$$

$$+\frac{1}{2A\eta_\infty}\Gamma(\alpha_2,p,C_1,A), \tag{42}$$

$$v_2(\alpha_1,\alpha_2,\alpha_3) = -\frac{1}{h_1}\int_0^{\alpha_2}\frac{\partial v_1}{\partial \alpha_1}\,d\alpha_2 - \frac{1}{h_1 h_3}\int_0^{\alpha_2}\frac{\partial}{\partial \alpha_3}(h_1 v_3)\,d\alpha_2, \tag{43}$$

$$v_3(\alpha_1,\alpha_2,\alpha_3) = \frac{1}{\eta_\infty h_3}\frac{\partial p}{\partial \alpha_3}$$

$$\times\left[\int_0^{\alpha_2}\alpha_2\frac{1}{\eta_{p1}(\Delta)}\,d\alpha_2 - \int_0^{\alpha_2}\frac{1}{\eta_{p1}(\Delta)}\,d\alpha_2\frac{\int_0^{\varepsilon}\alpha_2\frac{1}{\eta_{p1}(\Delta)}\,d\alpha_2}{\int_0^{\varepsilon}\frac{1}{\eta_{p1}(\Delta)}\,d\alpha_2}\right], \tag{44}$$

where the pressure function p satisfies equation:

$$\frac{\partial}{\partial \alpha_1}\int_0^{\varepsilon} v_1(\alpha_1,\alpha_2,\alpha_3)\,d\alpha_2$$

$$+\frac{1}{h_3(\alpha_3)}\frac{\partial}{\partial \alpha_3}\left[\int_0^{\varepsilon} h_1(\alpha_3)v_3(\alpha_1,\alpha_2,\alpha_3)\,d\alpha_2\right] = 0. \tag{45}$$

Moreover,

$$\Gamma(\alpha_2,p,C_1,A) \equiv \int_0^{\alpha_2}\sqrt{\Xi(\alpha_2,p,C_1,A)}\,d\alpha_2$$

$$-\frac{\alpha_2}{\varepsilon}\int_0^{\varepsilon}\sqrt{\Xi(\alpha_2,p,C_1,A)}\,d\alpha_2, \tag{46}$$

$$\Xi(\alpha_2,p,C_1,A) \equiv \left(\frac{A}{h_1}\frac{\partial p}{\partial \alpha_1}\alpha_2\right)^2 + \frac{2A}{h_1}\frac{\partial p}{\partial \alpha_1}\left(AC_1 + 2\eta_\infty - \eta_0\right)\alpha_2$$

$$+2AC_1\left(2\eta_\infty - \eta_0\right) + \left(AC_1\right)^2 + \eta_0^2, \tag{47}$$

$$C_1 = -\frac{2\eta_\infty}{\varepsilon}\omega h_1 + \frac{\eta_0}{A} - \frac{1}{2h_1}\frac{\partial p}{\partial \alpha_1}\varepsilon - \frac{1}{A\varepsilon}\int_0^\varepsilon \sqrt{\Xi\left(\alpha_2,p,C_1,A\right)}\,d\alpha_2 \tag{48}$$

$$\eta_{p1}(\Delta) \equiv \eta_{p1}(\alpha_2,p,C_1,A) = \frac{\dfrac{\eta_0}{\eta_\infty} + A\dfrac{\partial v_1}{\partial \alpha_2}}{1 + A\dfrac{\partial v_1}{\partial \alpha_2}} = \frac{2\eta_0 - \Delta(\alpha_2,p,C_1,A)}{2\eta_\infty - \Delta(\alpha_2,p,C_1,A)} \tag{49}$$

$$\Delta(\alpha_2,p,C_1,A) \equiv \frac{A}{2h_1}\frac{\partial p}{\partial \alpha_1}(\varepsilon - 2\alpha_2) - 2\eta_\infty A\frac{\omega h_1}{\varepsilon} + \sqrt{\Xi(\alpha_2,p,C_1,A)}$$

$$-\frac{1}{\varepsilon}\int_0^\varepsilon \sqrt{\Xi(\alpha_2,p,C_1,A)}\,d\alpha_2 \tag{50}$$

for $0 \le \alpha_1 \le 2\pi$ *in circumference direction,* $b_m \le \alpha_3 \le b_s$ *in width direction, and* $0 \le \alpha_2 \le \varepsilon$ *in the gap height direction.*

Proof. We integrate the equation (21) with respect to the variable α_2 and obtain:

$$\Omega_1 = \eta_p\frac{\partial v_1}{\partial \alpha_2} \qquad \text{for} \quad \Omega_1 \equiv \frac{1}{h_1}\frac{\partial p}{\partial \alpha_1}\alpha_2 + C_1 \tag{51}$$

where C_1 is the integral constant. We put the formulae (40) and (41) into the equation (51), and thus we obtain the following algebraic equation:

$$B\eta_\infty\Theta^3 + (A\eta_\infty - B\Omega_1)\Theta^2 + (\eta_o - A\Omega_1)\Theta - \Omega_1 = 0. \tag{52}$$

For $0 < \Theta^2 B \ll 1$ we can simplify equation (51) which obtains the following form:

$$A\eta_\infty\Theta^2 + (\eta_o - A\Omega_1)\Theta - \Omega_1 = 0. \tag{53}$$

In this case, the proper solution of the equation (57) has the form

$$\Theta = \frac{\partial v_1}{\partial \alpha_2} = \frac{A\Omega_1 - \eta_o + \sqrt{(\eta_o - A\Omega_1)^2 + 4A\Omega_1\eta_\infty}}{2A\eta_\infty} \tag{54}$$

For A equal zero, formula (54) has an indeterminacy point 0/0. We use the de l'Hospital rule to obtain the limit of the formula (54) as A tends to zero, i.e. when we consider the particular Newtonian case of the synovial fluid:

$$\lim_{A\to 0}\Theta = \frac{\Omega_1}{\eta_o}. \tag{55}$$

Let us find the solution for the small shear rates. We integrate equation (54) twice with respect to α_2. Hence, we obtain the circumference velocity component in the following form:

$$v_1(\alpha_1,\alpha_2,\alpha_3) = \frac{1}{2\eta_\infty}\left(\frac{1}{2h_1}\frac{\partial p}{\partial \alpha_1}\alpha_2^2 + C_1\alpha_2\right) - \frac{\eta_0}{2A\eta_\infty}\alpha_2$$
$$+\frac{1}{2A\eta_\infty}\int_0^{\alpha_2}\sqrt{\Xi(\alpha_2,C_1,A)}\,d\alpha_2 + C_2 \tag{56}$$

where function $\Xi(\alpha_2,p,C_1,A)$ has the form (47) and C_1, C_2 are the integration constants.

The boundary conditions (37), (38) for the velocity component (56) have the following form:

$$v_1(\alpha_2=0) = \omega h_1, \qquad v_1(\alpha_2=\varepsilon) = 0. \tag{57}$$

We impose condition (57) on the solution (56) and hence we obtain $C_2 = \omega h_1$ and $C_1(A)$ in the involved form (48). We introduce the dependence (48) for constant C_1 into solution (56), and so we shall get circumference fluid velocity component in the form (42), where function Γ determines formula (46) for $0 \le \alpha_2 \le \varepsilon$, $0 \le \alpha_1 < 2\pi$, $b_m \le \alpha_3 \le b_s$.

Now from equation (23) we determine the velocity component v_3. We neglect centrifugal acceleration term. Afterwards we integrate twice equation (23) with respect to the variable α_2. Hence we obtain:

$$v_3 = \frac{1}{\eta_\infty h_3}\frac{\partial p}{\partial \alpha_3}\int_0^{\alpha_2}\frac{\alpha_2}{\eta_{p1}(\alpha_2,p,C_1,A)}\,d\alpha_2 + C_3\int_0^{\alpha_2}\frac{1}{\eta_{p1}(\alpha_2,p,C_1,A)}\,d\alpha_2$$
$$+C_4 \tag{58}$$

where C_3, C_4 are integration constants. Dimensionless viscosity $\eta_{p1} = \eta_p/\eta_\infty$ has, by virtue of equations (39), (41), (58) the form (49) whereas the function $\Delta(\alpha_2,p,C_1,A)$ determines the formula (50). Now we impose the boundary conditions:

$$v_3(\alpha_2=0) = 0, \qquad v_3(\alpha_2=\varepsilon) = 0, \tag{59}$$

on the longitudinal velocity component (58). Thus we obtain constants:

$$C_4 = 0, \qquad C_3 = -\frac{1}{\eta_\infty h_3}\frac{\partial p}{\partial \alpha_3}\frac{\int_0^\varepsilon \alpha_2\dfrac{1}{\eta_{p1}(\Delta)}\,d\alpha_2}{\int_0^\varepsilon \dfrac{1}{\eta_{p1}(\Delta)}\,d\alpha_2}. \tag{60}$$

We substitute constants (60) into solution (58) hence we obtain the longitudinal velocity component of synovial fluid in the form (44). Now we integrate once the continuity equation (24) with respect to the variable α_2 and thus we obtain the radial velocity component of the synovial fluid in the general form:

$$v_2(\alpha_1,\alpha_2,\alpha_3) = -\frac{1}{h_1}\int_0^{\alpha_2}\frac{\partial v_1}{\partial \alpha_1}\,d\alpha_2 - \frac{1}{h_1 h_3}\int_0^{\alpha_2}\frac{\partial}{\partial \alpha_3}(h_1 v_3)\,d\alpha_2 + C_5, \quad (61)$$

where C_5 is the integration constant. We impose the boundary condition $v_2(\alpha_2 = 0) = 0$ on the solution (59), hence we obtain the integration constant $C_5 = 0$. Thus the radial velocity component has the form (43). Now we impose boundary condition $v_2(\alpha_2 = \varepsilon) = 0$ on the solution (61) and we take into account the following identities:

$$\frac{\partial}{\partial \alpha_1}\int_0^{\varepsilon} v_1(\alpha_1,\alpha_2,\alpha_3)\,d\alpha_2 = \int_0^{\varepsilon}\frac{\partial v_1(\alpha_1,\alpha_2,\alpha_3)}{\partial \alpha_1}\,d\alpha_2 \quad (62)$$

$$\frac{\partial}{\partial \alpha_3}\int_0^{\varepsilon} h_1(\alpha_3)v_3(\alpha_1,\alpha_2,\alpha_3)\,d\alpha_2 = \int_0^{\varepsilon}\frac{\partial[h_1(\alpha_3)v_3(\alpha_1,\alpha_2,\alpha_3)]}{\partial \alpha_3}\,d\alpha_2, \quad (63)$$

which are valid because $v_1(\alpha_1,\, \alpha_2 = \varepsilon,\, \alpha_3) \equiv 0$ and $v_3(\alpha_1,\, \alpha_2 = \varepsilon,\, \alpha_3) \equiv 0$. Hence, we obtain the modified Reynolds equation (45), which determines the unknown pressure function p. This result completes the proof of Lemma 7.1. ■

THEOREM 7.1. *Approximately unknown particular solutions of the non-linear partial differential system of second order (21), (22), (23), (24), for synovial non-Newtonian fluid velocity components and pressure in human joint gap resting on rotational bone surfaces with non-monotone generating line, can be shown to be composed of the following parts: the first part refers to the Newtonian properties of the synovial fluid, the second part is multiplied by the coefficient A and presents the corrections caused by the non-Newtonian fluid properties, while the subsequent parts are estimated by terms of the order of A^2:*

$$v_1 = -\frac{1}{2\eta_o}\frac{1}{h_1}\frac{\partial p}{\partial \alpha_1}s\varepsilon^2(1-s) + \omega h_1(1-s)$$

$$-\frac{A\kappa_1}{8\eta_\infty}s\varepsilon(1-s)\frac{1}{h_1}\frac{\partial p^{(0)}}{\partial \alpha_1}\left[2\omega h_1 + \frac{\varepsilon^2}{3\eta_0}\left(\frac{1}{h_1}\frac{\partial p^{(0)}}{\partial \alpha_1}\right)(1-2s)\right]$$

$$+O(A^2), \quad (64)$$

$$v_2 = \frac{1}{6}s^2(1-s)\frac{\varepsilon^3}{\eta_o}\left[\frac{1}{h_1^2}\frac{\partial^2 p}{\partial \alpha_1^2} + \frac{1}{h_1 h_3}\frac{\partial}{\partial \alpha_3}\left(\frac{h_1}{h_3}\frac{\partial p}{\partial \alpha_3}\right)\right] + A\frac{\omega\kappa_1\varepsilon}{6\eta_\infty}s^2(1-s)$$

$$\times\left\{\frac{1}{2}\frac{1}{h_1}\frac{\partial}{\partial \alpha_1}\left(\varepsilon\frac{\partial p^{(0)}}{\partial \alpha_1}\right) + \frac{1}{4}\frac{1}{h_1 h_3}\frac{\partial}{\partial \alpha_3}\left(\frac{\varepsilon h_1^2}{h_3}\frac{\partial p^{(0)}}{\partial \alpha_3}\right)\right.$$

$$\left.-\frac{1}{h_1}\frac{\partial p^{(0)}}{\partial \alpha_1}\frac{\partial \varepsilon}{\partial \alpha_1} - \frac{1}{2h_3}\frac{\partial p^{(0)}}{\partial \alpha_3}\frac{h_1}{h_3}\frac{\partial \varepsilon}{\partial \alpha_3}\right\} - A\frac{\kappa_1 s^2(1-s)}{48\eta_0\eta_\infty}$$

$$\times\left\{(1-s)\varepsilon\left[\frac{1}{h_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\varepsilon^{3/2}}{h_1}\frac{\partial p^{(0)}}{\partial\alpha_1}\right)^2+\frac{1}{h_1h_3}\frac{\partial}{\partial\alpha_3}\left(\frac{\varepsilon^3}{h_3}\frac{\partial p^{(0)}}{\partial\alpha_1}\frac{\partial p^{(0)}}{\partial\alpha_3}\right)\right]\right.$$

$$\left.-(1-3s)\varepsilon^3\left[\frac{1}{h_1}\frac{\partial\varepsilon}{\partial\alpha_1}\left(\frac{1}{h_1}\frac{\partial p^{(0)}}{\partial\alpha_1}\right)^2+\frac{1}{h_3}\frac{\partial\varepsilon}{\partial\alpha_3}\frac{1}{h_3}\frac{\partial p^{(0)}}{\partial\alpha_3}\frac{1}{h_1}\frac{\partial p^{(0)}}{\partial\alpha_1}\right]\right\}$$

$$+O(A^2), \tag{65}$$

$$v_3=-\frac{1}{2\eta_o}\frac{1}{h_3}\frac{\partial p}{\partial\alpha_3}s\cdot\varepsilon^2(1-s)-\frac{A\kappa_1}{8\eta_\infty}s\cdot\varepsilon(1-s)\frac{1}{h_3}\frac{\partial p^{(0)}}{\partial\alpha_3}$$

$$\times\left[\omega h_1+\frac{\varepsilon^2}{3\eta_0}\left(\frac{1}{h_1}\frac{\partial p^{(0)}}{\partial\alpha_1}\right)(1-2s)\right]+O(A^2), \tag{66}$$

$$p\equiv p^{(0)}+Ap^{(1)}+O(A^2) \tag{67}$$

where functions $p^{(0)}$, $p^{(1)}$ satisfy the following modified Reynolds partial equations:

$$\frac{1}{h_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\varepsilon^3}{\eta_0}\frac{\partial p^{(0)}}{\partial\alpha_1}\right)+\frac{1}{h_3}\frac{\partial}{\partial\alpha_3}\left(\frac{h_1\varepsilon^3}{h_3\eta_0}\frac{\partial p^{(0)}}{\partial\alpha_3}\right)=6\omega h_1\frac{\partial\varepsilon}{\partial\alpha_1}, \tag{68}$$

$$\frac{1}{h_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\varepsilon^3}{\eta_0}\frac{\partial p^{(1)}}{\partial\alpha_1}\right)+\frac{1}{h_3}\frac{\partial}{\partial\alpha_3}\left(\frac{h_1\varepsilon^3}{h_3\eta_0}\frac{\partial p^{(1)}}{\partial\alpha_3}\right)$$

$$=-\frac{1}{2}\omega\frac{\partial}{\partial\alpha_1}\left(\frac{\varepsilon^2\kappa_1}{\eta_\infty}\frac{\partial p^{(0)}}{\partial\alpha_1}\right)-\frac{1}{4}\omega\frac{1}{h_3}\frac{\partial}{\partial\alpha_3}\left(\frac{h_1^2\varepsilon^2\kappa_1}{\eta_\infty h_3}\frac{\partial p^{(0)}}{\partial\alpha_3}\right), \tag{69}$$

for $s\equiv\alpha_2/\varepsilon$, $0\le\alpha_2\le\varepsilon$, $b_m\le\alpha_3\le b_s$, $0\le\alpha_1<2\pi$, $-1/50\le\kappa_1\equiv 4\left(\eta_\infty^2-\eta_o\eta_\infty\right)/\eta_o^2\le-1/25$.

Proof. Formula (48) presents the involved equation with respect to the unknown constant $C_1(A)$. To obtain constant C_1 in the analytical form we expand the right hand side of equation (48) in two terms of Taylor series in the neighborhood of the point $A=0$. We obtain:

$$C_1=-\frac{2\eta_\infty}{\varepsilon}\omega h_1-\frac{\varepsilon}{2h_1}\frac{\partial p}{\partial\alpha_1}$$

$$+\lim_{A\to0}f(A,C_1)+\left[\lim_{A\to0}\frac{\partial f(A,C_1)}{\partial A}\right]\frac{A}{1!}+O(A^2), \tag{70}$$

where

$$f(A,C_1)\equiv\frac{\eta_o}{A}-\frac{1}{A\varepsilon}\int^{\varepsilon}\sqrt{\Xi(\alpha_2,p,C_1,A)}\,d\alpha_2 \tag{70*}$$

Since an indeterminate form of type 0/0 is obtained as A tends to zero, we use the de l'Hospital's rule to obtain the limits in the above formulae (70), (70*). In this way from formula (70) we obtain the square algebraic equation with respect to the constant $C_1(A)$. The proper real root of this square algebraic equation has the following form:

$$C_1(A) = \frac{\eta_o}{2A\kappa_1}\left[1 - \frac{\varepsilon\kappa_1 A}{\eta_o}\cdot\frac{1}{h_1}\frac{\partial p}{\partial\alpha_1} - \Pi(A)\right] \tag{71}$$

where:

$$\Pi(A) \equiv \sqrt{1 - \frac{1}{3}\left(\frac{\varepsilon\kappa_1 A}{\eta_o}\cdot\frac{1}{h_1}\frac{\partial p}{\partial\alpha_1}\right)^2 + 4\kappa_1 A\frac{\omega h_1}{\varepsilon}}. \tag{72}$$

As A tends to zero, we obtain the indeterminate form of type 0/0 in $C_1(A)$. Hence, by applying the de l'Hospital's rule, we find:

$$\lim_{A\to 0} C_1(A) = -\frac{1}{2}\varepsilon\frac{1}{h_1}\frac{\partial p}{\partial\alpha_1} - \frac{\omega h_1}{\varepsilon}\eta_o. \tag{73}$$

Now we remove the constant C_1 in the circumference fluid velocity component (42), i.e. we remove constant C_1 from function $\Xi(C_1)$, see equation (47), and from function $\Gamma[\Xi(C_1)]$, see equation (46). After this elimination we obtain:

$$\Xi(\alpha_2, p, A) = \left\{\frac{A}{h_1}\frac{\partial p}{\partial\alpha_1}\left(\alpha_2 - \frac{\varepsilon}{2}\right) + \frac{\eta_o}{2\kappa_1}\left[1 - \Pi(A)\right] + 2\eta_\infty - \eta_o\right\}^2 - \kappa_1\eta_o^2 \tag{74}$$

We use de l'Hospital's rule to obtain the limit of the last term on the right hand side of equation (46) as A tends to zero. In these calculations we take into account equation (47). We obtain finally:

$$\lim_{A\to 0}\frac{1}{2A\eta_\infty}\Gamma(\alpha_2, p, A) = -\frac{2\eta_\infty - \eta_o}{4\eta_o\eta_\infty}\cdot\frac{1}{h_1}\frac{\partial p^{(o)}}{\partial\alpha_1}\alpha_2\varepsilon\left(1 - \frac{\alpha_2}{\varepsilon}\right). \tag{75}$$

Hence, as A tends to zero, the circumference velocity component assumes the following classical form:

$$\lim_{A\to 0} v_1 = -\frac{1}{2\eta_o}\frac{1}{h_1}\frac{\partial p^{(o)}}{\partial\alpha_1}\alpha_2\varepsilon\left(1 - \frac{\alpha_2}{\varepsilon}\right) + \omega h_1\left(1 - \frac{\alpha_2}{\varepsilon}\right),$$

$$\text{where } p^{(o)} \equiv p(A=0) \tag{76}$$

The circumference velocity component (42) (after elimination of C_1 in functions ΞC_1 and $\Gamma[\Xi C_1]$) is expanded in two terms of Taylor series in the neighborhood of the point $A = 0$ in the form:

$$v_1 = \lim_{A\to 0} v_1 + \left[\lim_{A\to 0}\frac{\partial v_1}{\partial A}\right]\frac{A}{1!} + O(A^2). \tag{77}$$

Now we calculate the first derivative of the function (42) with respect to A:

$$\frac{\partial v_1}{\partial A} = \frac{\partial}{\partial A}\Bigg[-\frac{1}{4\eta_\infty h_1}\frac{\partial p}{\partial \alpha_1}\varepsilon\alpha_2\left(1-\frac{\alpha_2}{\varepsilon}\right)+\omega h_1\left(1-\frac{\alpha_2}{\varepsilon}\right)$$

$$+\frac{1}{2A\eta_\infty}\Gamma(\alpha_2,p,A)\Bigg] = \frac{A(\partial\Gamma/\partial A)-\Gamma}{2A^2\eta_\infty} - \frac{1}{4\eta_\infty h_1}\frac{\partial p^{(1)}}{\partial \alpha_1}\alpha_2\varepsilon\left(1-\frac{\alpha_2}{\varepsilon}\right), \quad (78)$$

whereas

$$\frac{\partial \Gamma}{\partial A} = \int_0^{\alpha_2}\frac{\partial\Xi/\partial A}{2\sqrt{\Xi}}\,d\alpha_2 - \frac{\alpha_2}{\varepsilon}\int_0^{\varepsilon}\frac{\partial\Xi/\partial A}{2\sqrt{\Xi}}\,d\alpha_2. \quad (79)$$

From equations (47), (46) it follows that functions Ξ, Γ and their first derivatives with respect to A have in point $A=0$ the following values:

$$\Xi(A=0)=\eta_o^2, \qquad \Gamma(A=0)=0, \quad (80)$$

$$\left(\frac{\partial\Xi}{\partial A}\right)_{A=0} = 2\left(2\eta_\infty-\eta_0\right)\left[\frac{1}{h_1}\frac{\partial p^{(o)}}{\partial \alpha_1}\left(\alpha_2-\frac{\varepsilon}{2}\right)-\eta_o\frac{\omega h_1}{\varepsilon}\right]$$

$$\left(\frac{\partial\Gamma}{\partial A}\right)_{A=0} = \frac{1}{2}\left(\frac{\eta_\infty}{\eta_o}-1\right)\frac{1}{h_1}\frac{\partial p^{(o)}}{\partial \alpha_1}\alpha_2\left(\alpha_2-\varepsilon\right). \quad (81)$$

$$\left(\frac{\partial^2\Xi}{\partial A^2}\right)_{A=0} = 2\left(\frac{1}{h_1}\frac{\partial p}{\partial \alpha_1}\right)^2\left[\alpha_2^2-\alpha_2\varepsilon+\frac{1}{4}\varepsilon^2-\frac{1}{6}\varepsilon^2\kappa_1\frac{2\eta_\infty-\eta_0}{\eta_0}\right]$$

$$-4\frac{\omega h_1}{\varepsilon}\eta_0\frac{1}{h_1}\frac{\partial p}{\partial \alpha_1}\left(\alpha_2-\frac{\varepsilon}{2}\right)+\left(\frac{\omega h_1}{\varepsilon}\right)^2\left[2\eta_0^2+4\alpha_2\kappa_1\eta_0\left(2\eta_\infty-\eta_0\right)\right]. \quad (82)$$

The first derivative of function v_1 with respect to A, see equation (78), has indeterminate form of type 0/0 as A tends to zero. Therefore we use de l'Hospital's rule to find the limit of function (78) in the following form:

$$\lim_{A\to 0}\frac{\partial v_1}{\partial A} = \frac{1}{4\eta_\infty}\lim_{A\to 0}\frac{\partial^2\Gamma}{\partial A^2} - \frac{1}{4\eta_\infty h_1}\frac{\partial p^{(1)}}{\partial \alpha_1}\alpha_2\varepsilon\left(1-\frac{\alpha_2}{\varepsilon}\right). \quad (83)$$

We use the first derivative (80) and we calculate the second derivative in the following form:

$$\frac{\partial^2\Gamma}{\partial A^2} = \frac{1}{2}\int_0^{\alpha_2}\left[-\frac{1}{2}\frac{1}{\Xi\sqrt{\Xi}}\left(\frac{\partial\Xi}{\partial A}\right)^2+\frac{1}{\sqrt{\Xi}}\frac{\partial^2\Xi}{\partial A^2}\right]d\alpha_2$$

$$-\frac{1}{2}\frac{\alpha_2}{\varepsilon}\frac{1}{2}\int_0^{\varepsilon}\left[-\frac{1}{2}\frac{1}{\Xi\sqrt{\Xi}}\left(\frac{\partial\Xi}{\partial A}\right)^2+\frac{1}{\sqrt{\Xi}}\frac{\partial^2\Xi}{\partial A^2}\right]d\alpha_2. \quad (84)$$

Into equation (77) we introduce equations (76) and (83). Afterwards in formula (83) we substitute the limits of function (84) as A tends to zero. To obtain these limits we must use the values (81), (80) and the value of the second derivative of function Ξ with respect to A in point $A = 0$. After calculations we obtain finally the expression (64), which determines the circumference velocity component.

The longitudinal velocity component (44) is expanded in two terms of Taylor series in the neighborhood of the point $A = 0$ in the following form:

$$v_3(\alpha_1, \alpha_2, \alpha_3) = \lim_{A\to 0} v_3(\alpha_1, \alpha_2, \alpha_3) + \left[\lim_{A\to 0} \frac{\partial v_3}{\partial A}\right] \frac{A}{1!} + O(A^2). \tag{85}$$

It is easy to see by virtue of equation (47) that function (50) $\Delta \equiv \Delta(\alpha_1, \alpha_2, \alpha_3) \equiv \Delta[\alpha_2, p, C_1, A]$ tends to zero if A tends to zero. Hence the longitudinal fluid velocity component (44), as A tends to zero, approaches the following classical form:

$$\lim_{A\to 0} v_3(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{2\eta_0 h_3} \frac{\partial p}{\partial \alpha_3} \left(\alpha_2^2 - \varepsilon\alpha_2\right). \tag{86}$$

Now we calculate the first derivative of the function (44) with respect to A:

$$\frac{\partial v_3}{\partial A} = \frac{1}{\eta_\infty} \frac{\partial p}{h_3 \partial \alpha_3} \left\{ \int_0^{\alpha_2} \alpha_2 \frac{\partial}{\partial A}[Y] d\alpha_2 - \int_0^{\alpha_2} \frac{\partial}{\partial A}[Y]\, d\alpha_2 \frac{\int_0^\varepsilon \alpha_2 Y\, d\alpha_2}{\int_0^\varepsilon Y\, d\alpha_2} \right\}$$

$$-\frac{1}{\eta_\infty h_3} \frac{\partial p}{\partial \alpha_3} \int_0^{\alpha_2} Y\, d\alpha_2 \frac{1}{\left[\int_0^\varepsilon Y\, d\alpha_2\right]^2} \left\{ \int_0^\varepsilon \alpha_2 \frac{\partial}{\partial A}[Y]\, d\alpha_2 \int_0^\varepsilon Y\, d\alpha_2 \right.$$

$$\left. - \int_0^\varepsilon \alpha_2 Y\, d\alpha_2 \int_0^\varepsilon \frac{\partial}{\partial A}[Y]\, d\alpha_2 \right\}, \tag{87}$$

where $Y = \frac{1}{\eta_{p1}(\Delta)}$. From equation (49) we obtain the first derivative of the reciprocal of viscosity function η_{p1} with respect to A in the following form:

$$\frac{\partial}{\partial A}[Y] = \frac{\partial}{\partial A}\left[\frac{2\eta_\infty - \Delta}{2\eta_0 - \Delta}\right] = \frac{2(\eta_\infty - \eta_0)\frac{\partial \Delta}{\partial A}}{(2\eta_0 - \Delta)^2} \equiv W(A). \tag{88}$$

We substitute expression (88) in formula (87). Hence, the first derivative of function v_3 with respect to A has following indeterminate form of type 0/0 as A tends to zero:

$$\lim_{A\to 0} \frac{\partial v_3}{\partial A} = \frac{1}{\eta_\infty h_3}$$

$$\times \lim_{A\to 0} \frac{\partial p}{\partial \alpha_3} \left\{ \int_0^{\alpha_2} \alpha_2 W(A)\, d\alpha_2 - \int_0^{\alpha_2} W(A)\, d\alpha_2 \cdot \frac{\int_0^\varepsilon \alpha_2 Y\, d\alpha_2}{\int_0^\varepsilon Y d\alpha_2} \right\}$$

$$-\frac{1}{\eta_\infty h_3}\lim_{A\to 0}\frac{\partial p}{\partial \alpha_3}\int_0^{\alpha_2} Y\, d\alpha_2$$
$$\times\left\{\frac{\int_0^\varepsilon \alpha_2 W(A)\, d\alpha_2 \int_0^\varepsilon Y\, d\alpha_2 - \int_0^\varepsilon Y\, d\alpha_2 \int_0^\varepsilon W(A)\, d\alpha_2}{\left[\int_0^\varepsilon Y\right]^2}\right\}. \tag{89}$$

To obtain the limit of formula (89), we must calculate the limits of expressions W, see (88) and the limit of the first derivative of Δ, see (50), with respect to A as A tends to zero:

$$\lim_{A\to 0}\frac{\partial \Delta}{\partial A} = -\frac{(\eta_\infty-\eta_0)}{\eta_0}\frac{1}{h_1}\frac{\partial p}{\partial \alpha_1}(\varepsilon-2\alpha_2) - 2\eta_\infty\frac{\omega h_1}{\varepsilon}, \tag{90}$$

$$\lim_{A\to 0} W(A) = -\frac{(\eta_\infty-\eta_0)}{\eta_0^2}\left[\frac{(\eta_\infty-\eta_0)}{2\eta_0}\frac{1}{h_1}\frac{\partial p}{\partial \alpha_1}(\varepsilon-2\alpha_2) + \eta_\infty\frac{\omega h_1}{\varepsilon}\right]. \tag{91}$$

Now, into equation (85) we introduce limits (86), (89) obtained by virtue of expressions (90), (91). After calculations we obtain finally the longitudinal velocity component in the form (66). We insert the longitudinal and circumference fluid velocity components (66), (64) in equation (45). After calculations we equate the coefficients of the same power k of small parameter A^k to zero. For $k = 0$ and $k = 1$ we obtain the classical (68) and modified Reynolds equations (69).

Equation (68) determines the pressure function $p^{(0)}$ and equation (69) determines the pressure corrections $p^{(1)}$ which are caused by the non-Newtonian fluid properties.

We put the circumference and longitudinal velocity components (64), (66) in the formula (43) and we take into account equations (68), (69), and thus we obtain finally the form (65) of the radial velocity component.

This result completes the proof of Theorem 7.1. ■

8. Example illustrating the capability of solutions

By virtue of the presented theory we determine the analytical solutions of a particular case of pressure distribution for the axial symmetrical flow between two rotational hyperbolic bone surfaces and variable gap height. In this case synovial fluid flow will be given as axial symmetrical in thin gap, thus the hyperbolic coordinate system will be taken in the form (see Fig. 7):

$$\alpha_1 = \alpha_{11}, \qquad \alpha_2 = \varepsilon_0\alpha_{21}, \qquad \alpha_3 = \Lambda^{-1}\alpha_{31}. \tag{92}$$

The Lamé coefficients are as follows:

$$h_1 = a\cos^{-2}(\alpha_3\Lambda), \qquad h_2 = 1,$$
$$h_3 = \cos^{-2}(\alpha_3\Lambda)\sqrt{1+4(a\Lambda)^2\tan^2(\alpha_3\Lambda)}. \tag{93}$$

whereas

$$\Lambda \equiv b^{-1}\sqrt{wa^{-1}}, \qquad 0 \le \alpha_1 \le 2\pi, \qquad 0 \le \alpha_2 \le \varepsilon,$$

$$|\alpha_3| \le \Lambda^{-1} \arccos \sqrt{a\,(a+w)^{-1}}. \tag{94}$$

We use the following notations: a—the smallest radius of the bone cross section, $a_1 = a + w$—the largest radius of the bone cross section, $2b$—the joint length, $\varepsilon_1(\alpha_{31}) \equiv \varepsilon(\alpha_3)/\varepsilon_o$—dimensionless gap height, ε_o—dimensional average value of gap height.

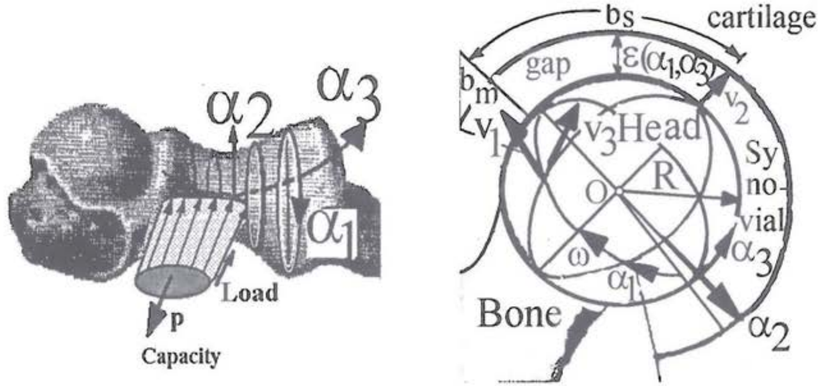

Figure 7. Radial elbow joint in hyperbolic co-ordinates and hip joint.

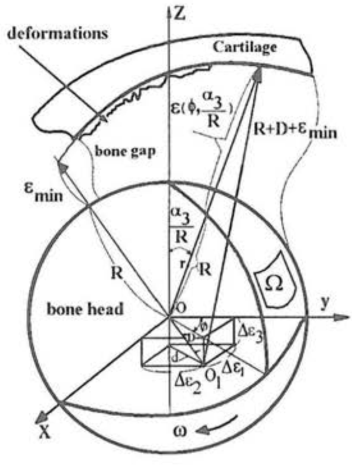

Figure 8. Human gap height in spherical coordinates for deformation ability cartilage.

Pressure distribution depends on variable α_3 only. In this case we obtain the function of pressure by virtue of equations (68), (69) for coefficients (92) in the following form:

$$p(\alpha_{31}) = p_w + (p_z - p_w)\left(\int_{\alpha_{31}}^{b_{s1}} \frac{\eta_{p1}\Omega_h}{\varepsilon_1^3(\alpha_{31})}\,d\alpha_{31}\right)\left(\int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1}\Omega_h}{\varepsilon_1^3(\alpha_{31})}\,d\alpha_{31}\right)^{-1}$$

$$+\frac{3}{20}\rho\omega^2 a^2\left\{\left[\sec^4(b_{s1}) - \sec^4(b_{m1})\right]\left(\int_{\alpha_{31}}^{b_{s1}} \frac{\eta_{p1}\Omega_h}{\varepsilon_1^3(\alpha_{31})}\,d\alpha_{31}\right)\right.$$

$$\left.\times\left(\int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1}\Omega_h}{\varepsilon_1^3(\alpha_{31})}\,d\alpha_{31}\right)^{-1} - \left[\sec^4(b_{s1}) - \sec^4(\alpha_{31})\right]\right\}, \tag{95}$$

where: $b_{m1} \equiv b_m\Lambda \le \alpha_{31} \le b_{bs}\Lambda \equiv b_{s1}$, $0 \le \alpha_{11} \le \alpha_e \le 2\pi$, $0 \le \alpha_{21} \equiv \alpha_2/\varepsilon_o \le \varepsilon_1(\alpha_{31})$. Symbols b_{m1} and b_{s1} denote the dimensionless upper and lower limits of the lubrication region, respectively. Moreover, we introduce following dimensionless notation:

$$\Omega_h \equiv \sqrt{1 + 4a\,(a_1 - a)\,b^{-2}\tan^2(\alpha_{31})} \tag{96}$$

Now for axial unsymmetrical synovial fluid flow for the gap resting on spherical bone surface in human hip joint (see Fig. 8) we have the following Lamé coefficients:

$$h_1 = R\sin(\vartheta/R), \qquad h_2 = 1, \qquad h_3 = 1, \tag{97}$$

where R is radius of sphere. We denote: $\alpha_1 \equiv \varphi$ circumference direction, $\alpha_2 \equiv r$ gap height direction, $\alpha_3 \equiv \vartheta$ (meridian) direction. The Reynolds equation (68) has following form:

$$\frac{\partial}{\partial\varphi}\left(\frac{\varepsilon^3}{\eta_o}\frac{\partial p^{(o)}}{\partial\varphi}\right) + R^2\sin\left(\frac{\vartheta}{R}\right)\frac{\partial}{\partial\vartheta}\left[\frac{\varepsilon^3}{\eta_o}\frac{\partial p^{(o)}}{\partial\vartheta}\sin\left(\frac{\vartheta}{R}\right)\right]$$

$$= 6\omega\,R^2\frac{\partial\varepsilon}{\partial\varphi}\sin^2\left(\frac{\vartheta}{R}\right) \tag{98}$$

in Ω region: $0 \le \varphi \le \pi$, $\pi R/8 \le \vartheta \le \pi R/2$. Gap height has the following form:

$$\varepsilon(\varphi,\vartheta/R) = \Delta\varepsilon_1\cos\varphi\sin\vartheta/R + \Delta\varepsilon_2\sin\varphi\sin\vartheta/R - \Delta\varepsilon_3\cos\vartheta/R - R$$

$$+[(\Delta\varepsilon_1\cos\varphi\sin\vartheta/R + \Delta\varepsilon_2\sin\varphi\sin\vartheta/R - \Delta\varepsilon_3\cos\vartheta/R)^2$$

$$+(R+\varepsilon_{\min})(R+2D+\varepsilon_{\min})]^{0.5}. \tag{99}$$

The center point of the bone head (see Fig. 8) can be written down in the following form: $O_1(x-\Delta\varepsilon_1,\ y-\Delta\varepsilon_2,\ z+\Delta\varepsilon)$, while D is the distance between the centre of bone head and the acetabulum (sleeve) centre.

9. Deformation ability gap height

The minimum of gap height, see Dowson (1998), for spherical hip joint is obtained from the formula:

$$\frac{\varepsilon_{\min}}{R} \equiv \sqrt[5]{2\pi} S_1^{0.4} \left(\frac{\omega R^2 \eta}{C}\right)^{0.6}, \qquad S_1 \equiv \frac{C}{ER^2},$$

$$\frac{1}{E} = \frac{1}{2}\left(\frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2}\right) \tag{100}$$

where E_1, E_2,v_1, v_2 are the elastic modules and Poisson ratios for bone head and cartilage, C—load, and quantities: η, ω, R are as defined previously. Dependence (39) for $\Theta \approx \omega R/\varepsilon_{\min}$ can be written in the following form:

$$\frac{\omega R^2 \eta}{C} \equiv \frac{S_2}{S_1}\left(\frac{\eta_\infty}{\eta_o} + \frac{\frac{\eta_o - \eta_\infty}{\eta_o}}{1 + S_3 \frac{R}{\varepsilon_{\min}}}\right), \qquad S_2 \equiv \frac{\omega R \eta_o}{ER}, \qquad S_3 \equiv A\omega. \tag{101}$$

By combining equations (101), (100) we obtain the system of two equations for determination of two unknown quantities, namely the dynamic viscosity η of synovial fluid and the minimal value $\varepsilon_{\min}$ of gap height, where elastic deformations of cartilage are taken into account. If we assume the following data: $R = 2.6 \cdot 10^{-2}$ m, $E = 2 \cdot 10^5$ Pa, $\omega R = 3 \cdot 10^{-1}$ m/s, $\eta_\infty = 0.10$ Pas, $2\pi R/C = 3 \cdot 10^{-4}$ m/N, $\eta_o/\eta_\infty \cong 1000$, $A = 1.88$ s, $C = 544.26$ N, then from eqs. (101), (100) we obtain: $\varepsilon_{\min} = 0.0000208\mu$ m $= 20.88\mu$ m and $\eta = 0.1036$ Pas. If we take in the computations the following quantities: $A = 1.88$ s, $\eta_o = 100.00$ Pas, $\eta_\infty = 0.10$ Pas, $R = 0.020$ m, $C = 544$ N, $0.50\,\mathrm{s}^{-1} \le \omega \le 10.00\,\mathrm{s}^{-1}$, $2 \cdot 10^5\mathrm{Pa} \le E \le 2 \cdot 10^7$ Pa, then we obtain the minimal value of the gap height in the interval: 0.29μ m $\le \varepsilon_{\min} \le 19.90\mu$ m.

10. Numerical example

We solve equation (98) for the region $\Omega(\varphi, \vartheta)$ resting on bone head and indicated in Fig. 5. We assume atmospheric pressure on the boundary of the region $\Omega(\alpha_1, \alpha_3)$. For this region we calculate also the capacity values. In numerical calculations we assume the following values for the joint gap: $\Delta\varepsilon_1 = 5\mu$m, $\Delta\varepsilon_2 = 5\mu$m, $\Delta\varepsilon_3 = 5\mu$m, radius of bone head $R = 0.026575$ m.

Now we calculate the normal hip joint. For angular velocity of bone head $\omega = 3\,\mathrm{s}^{-1}$ and average value of synovial fluid dynamic viscosity $\eta_o = 1.00$ Pas, we obtain the smallest gap height $\varepsilon_{\min} = 10.0$ m for normal joint and hence hydrodynamic pressure $p^{(o)}$ has maximal value equal $35.30 \cdot 10^5$ N/m^2 ≈ 35.30 at and capacity $C_{\mathrm{tot}} = 2133$ N, see Fig. 9. Lubrication surface has value $\pi R^2 \cos(\pi/8) \approx 20.50$ cm^2.

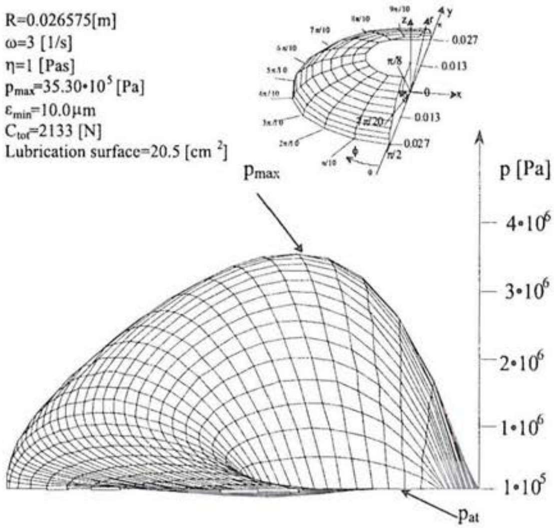

Figure 9. Pressure distribution in normal spherical hip joint gap for hydrodynamic lubrication caused by rotation.

Numerical calculations were performed with Mathcad 2000 Professional Program, with the help of the Method of Finite Differences. This method satisfies the requirement of stability of numerical solutions to the partial differential equations (98).

11. Final comments

The present paper shows the method of determination of approximate solutions to partial non-linear differential equations of non-Newtonian, asymmetrical synovial fluid flow in the thin gap occurring in human joint in curvilinear, orthogonal co-ordinates.

The method presented enables to obtain solutions in the form of Taylor series with increasing powers of the small parameter A obtained in experimental way for synovial fluid. In the particular case of the symmetrical flow we can, by virtue of theory presented, find analytical solutions in a simple form. The percentage corrections of velocity $\nu_i^{(1)}$ and of pressure $p^{(1)}$ caused by the non-Newtonian properties of the synovial fluid, see equations (64)–(67), are examined numerically through following ratio form:

$$100\frac{Ap^{(1)}+O(A^2)}{p^{(o)}} \text{ in percent.} \tag{102}$$

For large shear rates: $100\,\mathrm{s}^{-1} \le \Theta \le 1000\,\mathrm{s}^{-1}$,the viscosity of synovial fluid is small and has values 10^{-1} Pas $\le \eta \le 1$ Pas, see Fig. 6. In this case we obtain from eq. (102) small pressure changes from 2% to 4%. For small shear rates: $10^{-1}\,s^{-1} \leq \Theta \leq 10\,s^{-1}$, when viscosity is large, i.e. $10\,\text{Pas} \leq \eta \leq 100\,\text{Pas}$ we obtain from eq. (102) pressure changes from 7% to 15%.

Acknowledgement

Author thanks for the financial support from the Polish Grant KBN 8-T 11E-021-17.

References

BATHELOR, A.W., STACHOWIAK, G.W. (1996) Arthritis and the interacting mechanisms of synovial joint lubrication, *Journal of Orthopaedic Rheumatology* **9**, 3–10.

DOWSON, D. (1990) *Biomechanics of Diarthrodial Joints*, chapter 29: Bio-Tribology of Natural and Replacement Synovial Joints, N.Y., Berlin, Springer Verlag.

DOWSON, D. (1998) *Advances in Medical Tribology*, Mech. Eng. Publ. Limited, Bookcraft UK Limited, Bath.

GRUCA, A. (1993) *Chirurgia ortopedyczna*, Warszawa, PZWL (in Polish).

HAYES, A., HARRIS, B., DIEPPE, P.A.and CLIFT, S.E. (1993) Wear of articular cartilage:the effect of cristals, *Proc. Inst. Mech. Eng.* Part H, Engineering in Medicine, 207, 41–58.

MAQUET, P.G.J. (1984) *Biomechanics of the Knee* Springer Verlag, Berlin, Heidelberg, New York.

MAUREL,W., WU, Y., THALMANN, D. (1998) *Biomechanical Models for Soft Tissue Simulation*, Springer Verlag, Berlin/Heidelberg.

MOW, V.C. (1969) The role of lubrication in biomechanical joints. *J. Lub. Tech. Trans. ASME*, **91**, 320–329.

MOW, V.C.,HOLMES,M.H.,LAI, W.M. (1984) Fluid transport and mechanical properties of articular cartilage. *J. Biomech.*, **17**, 337–394.

MOW, V.C., RATCLIFFE,A., WOO S. (1990) *Biomechanics of Diarthrodial Joints*, Springer Verlag, Berlin, Heidelberg, New York.

MOW, V.C., SOSLOWSKY, L.J. (1991) Friction, lubrication and wear of diarthrodial joints. In: Mow, V.C., Hayes, W.C., eds. *Basic orthopedic biomechanics*, New York, Raven Press, 254–291.

MOW, V.C., GUILAK, F. (1994) *Cell Mechanics and Cellular Engineering*, Springer Verlag, Berlin, Heidelberg, New York.

MOW, V.C., WILSON C., HAYES W.C. (1998) *Biomechanics*, Lippincott-Wiliams, Wilkins Publishers, Philadelphia, Baltimore, New York, London.

PYTKO, S. and WIERZCHOLSKI, K. (1995) Analytical Biobearing Calculation for Experimental Dependencies Between Shear Rate and Synovial Fluid Viscosity *Proceedings of International Tribology Conference in Yokohama*, **3**, 1975-1980.

TRUESDELL, C.(1972) *A first Course in Rational Continuum Mechanics*, Maryland, John Hopkins University, Baltimore.

UNGETHM, M., WINKLER-GNIEWEK, W.(1990) Tribologie in Medizin. *Tribologie—Schmierungstechnik*, 5 , 268-277.

WIERZCHOLSKI, K., PYTKO, S. (1993) Metoda wyznaczania parametrów biołożyska smarowanego cieczą nienewtonowską. *Tribologia* **1**, 9–12 (in Polish).

WIERZCHOLSKI, K., NOWOWIEJSKI, R. and MISZCZAK, A. (1995) Numerical Analysis of Synovial Fluid Flow in Biobearing Gap. *Proceedings of International Conference System Modelling Control*, Conference, **8**, 2, 382–387.

WIERZCHOLSKI, K. (1993) *Mathematical Methods in Hydrodynamic Theory of Lubrication*, Szczecin Technical University Press, ISSN 0209-1151.

WIERZCHOLSKI, K., NOWOWIEJSKI, R. and PYTKO, S. (1994) Investigations of dynamic viscosity of synovial fluid. *Proc.Mechanics in Medicine*, **2**, 73–80.

WINTERS, J. M., WOO, S.L. (1990) *Multiple Muscle Systems*, Springer Verlag, Berlin, Heidelberg, New York.

WISLICKI, B. (1980) On the similarity of tribological processes in technical frictional pairs and human joints. *Exploitation Problems of Machines*, **3**, 15, 309–325.

YAO, J.Q., UNSWORTH, A. (1993) Asperity lubrication in human joints, *Proc. Inst. Mech. Eng.* Part H: Engineering in Medicine, **207**, 245–254.

ZIMMERMANN GORSKA, I. (1995) *Atlas płynu stawowego*, Warszawa, PZWL (in Polish).