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A large displacement framework for buckling and formation studies of elastic plates[1]

by

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Abstract: We study the problem of changing the geometric configuration of an elastic plate by means of attached and embedded actuators. For this purpose we use the so-called "full" von Karman plate equations, incorporating geometric nonlinearities, and we develop a model for internal actuation based on the same principles and assumptions. We show that the von Karman model predicts azimuthal buckling for a thin, centrally supported disk-shaped plate with uniform transverse boundary loading and we indicate that behavior of this type poses a significant problem in attempting reformation of the elastic plate into a rotationally symmetric, bowl-shaped shell, a problem of some importance in projected applications. We study two different systems of actuator deployment and indicate why one of them appears to deal with this problem more effectively than the other.

Keywords: formation, elasticity, elastic plate, von Karman equations.

1. Introduction and geometric setting

The first goal of the present work is to provide an alternate - we feel more elementary - derivation of the so-called "full" von Karman plate system (see, e.g. Lagnese, 1989, Lagnese and Lions, 1988, Benabdallah and Lasiecka, 2000) in the static configuration. The derivation, as presented here, relies on a rigorous *order assumption* relating the magnitudes of admitted in-plane displacements

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to the squares of the magnitudes of the admitted transverse displacements, replacing the somewhat arcane assumptions of finite elasticity and allowing us to provide better explanations for the terms occurring in these equations than are ordinarily available. In many engineering studies the model analysed here is referred to as the *large deflection model*; for brevity we will refer to it as the *LD (plate) model*.

We do not specialize to the system usually identified as "the" von Karman plate because that provides no special advantage in what we do here and because we do wish to admit the possibility of clamped boundary conditions. Additionally, we at all times retain the potential energy, or variational form, of the system, whose minimization yields the equilibrium state, rather than proceeding to the partial differential equations constituting necessary conditions for a minimum. This has the advantage of allowing us to work with lower order derivatives than those occurring in the partial differential equations. Restriction to the variational framework also allows us to dispense with a listing, or intricate parametrization of all of the possibilities for natural boundary conditions. Since the commonly used finite element approximation techniques almost always take the energy form as their point of departure, we feel the gains in this approach outweigh the losses.

Our objective is to study a number of developments relative to the LD model. One of these is *finite amplitude buckling*, a necessarily nonlinear phenomenon. This is done in the context of a plate of annular, or disk-shaped, cross section; we consider azimuthally sinusoidal, or near sinusoidal, buckling resulting from constant transverse forces applied to the outer boundary.

Many buckling phenomena associated with the LD model can be explained by the fact that, as the thickness of the plate tends to zero, the equations increasingly model an elastic membrane, resistant to in-plane stresses but with vanishing resistance to bending. In fact it is the presence of the membrane energy terms in the potential energy expression which distinguishes the LD model from the classical Kirchhoff plate model (Lagnese, 1989). It is mathematically natural, therefore, and very pertinent in the light of the developing advanced materials technology, to consider variations of the LD model explicitly allowing for the possible presence of a finite number of thin membranes, structures resistant to selected in-plane stresses only, distinct from the substrate plate but embedded in it or forming one or both of its transverse boundary surfaces. In general such an *augmented LD model* is not isotropic but it includes a subclass of isotropic models with embedded membranes resistant to in-plane dilatation and shear.

Some of the membranes described in the preceding paragraph incorporate, in the formation studies presented later in the paper, actuators by means of which formative stresses can be introduced into the plate. Our specific objective is to study the re-formation of an annular, or disk-shaped, plate into a bowl-shaped shell, a process important in the development of many "smart" structures, in-

We will see that the study of this formation question in the LD plate context, which we feel to be the minimally adequate context short of a full-blown elastic shell study, which we do not attempt here, inevitably indicates the buckling studies indicated in the preceding paragraph.

To develop the geometric setting of the model, we consider an elastic plate of uniform thickness occupying a region $\mathcal{R} \equiv \{(x, y, z) \in \Omega\} \times [-h, h]$, where Ω is a bounded domain in $\mathbf{R}^2$ with smooth boundary Γ. In equilibrium, the neutral plane of the plate coincides with the set $\mathcal{R}_0 \equiv \Omega \times \{0\}$ in the plane $z = 0$. In a general admitted deformation the set $\mathcal{R}_0$ undergoes in-plane as well as transverse displacements; these are described in terms of the vector function $\mathbf{F}_0 : \mathbf{R}^2 \to \mathbf{R}^3$ given for $(x, y) \in \Omega$ by

$$\mathbf{F}_0(x, y) \equiv \begin{bmatrix} x + \xi(x, y) \\ y + \eta(x, y) \\ \zeta(x, y) \end{bmatrix}. \tag{1}$$

In the present article we assume that ξ, η and ζ have whatever smoothness is required to admit the partial derivatives introduced. Existence and regularity results for the von Karman system, which require more rigorous specification of the state space of admitted displacements, may be found in, e.g., Ciarlet and Rabier (1982).

Throughout our discussion we use the subscript notation, e.g., $\mathbf{F}_x$, and the equivalent partial derivative symbol, e.g., $\frac{\partial \mathbf{F}}{\partial \mathbf{X}}$ interchangeably; the first saves space while the second is clearer in certain situations, particularly when applied to variables which are already subscripted. Thus we have

$$\frac{\partial}{\partial x}\mathbf{F}_0 = \begin{bmatrix} 1 + \xi_x \\ \eta_x \\ \zeta_x \end{bmatrix}. \tag{2}$$

and

$$\frac{\partial}{\partial y}\mathbf{F}_0 = \begin{bmatrix} \xi_y \\ 1 + \eta_y \\ \zeta_y \end{bmatrix}. \tag{3}$$

With the *Kirchhoff assumption*, to the effect that lines perpendicular to the neutral surface $z = 0$ in unforced equilibrium remain lines perpendicular to the displaced neutral surface under the admitted displacements, there is no shearing deformation between displaced two dimensional layers of the plate parallel to the neutral surface. This is one of the features distinguishing the LD model under present consideration from the Mindlin - Timoshenko, Mindlin (1951), for example.

The underlying analytical assumption operative in our model derivation is

Order Assumption: *The in-plane (horizontal) displacements* ξ and η *and their partial derivatives are of the same order of magnitude as the squares of the transverse displacement* ζ *and its partial derivatives.*

Remark: In the case of totally free boundary conditions the order assumption requires modification to a statement about displacements modulo uniform translations and rotations.

Proceeding now to general points in $\mathcal{R}$, not necessarily on the neutral surface, the displacement of the point $(x, y, 0) \in \mathcal{R}_0$ indicated by (1) results in the point $X = (x, y, z) \in \mathcal{R}$ being transferred to the displaced point whose coordinates, within the degree of accuracy mandated by the Order Assumption, are given by

$$\mathbf{F}(x,y,z) = X + \Xi(x,y,z) = \begin{pmatrix} x + \xi(x,y) - z\,\zeta_x(x,y) \\ y + \eta(x,y) - z\,\zeta_y(x,y) \\ z + \zeta(x,y) - \frac{z}{2}\left({\zeta_x}^2 + {\zeta_y}^2\right) \end{pmatrix}. \tag{4}$$

For the partial derivatives of $\mathbf{F}$ with respect to x and y we then have

$$\frac{\partial}{\partial x}\mathbf{F} = \begin{pmatrix} 1 + \xi_x - z\zeta_{xx} \\ \eta_x - z\,\zeta_{yx} \\ \zeta_x - \frac{z}{2}\frac{\partial}{\partial x}\left({\zeta_x}^2 + {\zeta_y}^2\right) \end{pmatrix};\ \frac{\partial}{\partial y}\mathbf{F} = \begin{pmatrix} \xi_y - z\,\zeta_{xy} \\ 1 + \eta_y - z\,\zeta_{yy} \\ \zeta_y - \frac{z}{2}\frac{\partial}{\partial y}\left({\zeta_x}^2 + {\zeta_y}^2\right) \end{pmatrix}. \tag{5}$$

The Order Assumption is motivated, of course, by the realization that the change, due to an admitted displacement, in the length of a short material line segment parallel to the neutral surface, is affected in a first order manner by the partial derivatives $\xi_x,\ \xi_y,\ \eta_x$ and η_y whereas the effect arising from the transverse displacement $\zeta(x,y)$ is proportional to $1 - \cos v$, where v is the angle of inclination of the surface $z = \zeta(x,y)$, and thus to the square of the norm of the gradient of ζ, provided that gradient remains small. To explore the details of this length change, and for later use in formulae related to inclusion of layers of monotropic actuators, we consider two material points with coordinates (x, y, z) and $(x + \Delta x, y + \Delta y, z)$ in the nominal equilibrium configuration. We assume Δx and Δy are small relative to distances over which partial derivatives of the displacement components change appreciably. Under a general displacement, as described previously, these two points are carried, to first order in Δx, Δy and $\|\nabla\zeta\|$, into the image points

$$\left(x + \xi(x,y) - z\,\frac{\partial\zeta}{\partial x}(x,y),\ y + \eta(x,y) - z\,\frac{\partial\zeta}{\partial y}(x,y),\ z + \zeta(x,y)\right)$$

and

$$\left(x + \xi(x,y) + \frac{\partial\xi}{\partial x}(x,y)\,\Delta x + \frac{\partial\xi}{\partial y}(x,y)\,\Delta y - z\,\frac{\partial\zeta}{\partial x}(x + \Delta x, y + \Delta y)\right.,$$

$$z + \zeta(x,y) + \frac{\partial \zeta}{\partial x}(x,y)\,\Delta x + \frac{\partial \zeta}{\partial y}(x,y)\,\Delta y\Big).$$

We let $\Delta \ell = \sqrt{\Delta x^2 + \Delta y^2}$ denote the original distance between the two points in question. Suppressing (x, y) as an argument for notational brevity, the distance between the displaced points is, again to first order in Δx, Δy and z, the norm of the vector

$$\begin{pmatrix} \Delta x + \frac{\partial \xi}{\partial x}\,\Delta x + \frac{\partial \xi}{\partial y}\,\Delta y - \frac{\partial^2 \zeta}{\partial x^2}\,\Delta x\, z - \frac{\partial^2 \zeta}{\partial x \partial y}\,\Delta y\, z, \\ \Delta y + \frac{\partial \eta}{\partial x}\,\Delta x + \frac{\partial \eta}{\partial y}\,\Delta y - \frac{\partial^2 \zeta}{\partial x \partial y}\,\Delta x\, z - \frac{\partial^2 \zeta}{\partial y^2}\,\Delta y\, z, \\ \frac{\partial \zeta}{\partial x}\,\Delta x + \frac{\partial \zeta}{\partial y}\,\Delta y \end{pmatrix} \tag{6}$$

A complicated, but straightforward computation shows that if we discard products of derivatives of ξ with derivatives of η and products of derivatives of either of these with derivatives of ζ, as prescribed by the Order Assumption, and if we use the standard approximation $\sqrt{1+a} \approx 1 + \frac{a}{2}$ valid for small a, we obtain for the norm of (6) the approximate expression

$$\Delta \ell + \Delta \ell\, \Phi^* \mathbf{M}\, \Phi$$

where $\mathbf{M}$ is the matrix

$$\mathbf{M} = \begin{pmatrix} \xi_x - \zeta_{xx}\, z + \frac{1}{2}\,\zeta_x^2 & \xi_y - \zeta_{xy}\, z + \frac{1}{2}\,\zeta_x \zeta_y \\ \eta_x - \zeta_{xy}\, z + \frac{1}{2}\,\zeta_x \zeta_y & \eta_y - \zeta_{yy}\, z + \frac{1}{2}\,\zeta_y^2 \end{pmatrix}. \tag{7}$$

If we amend the Order Assumption by further supposing that ζ and its partial derivatives should be of the same order as the thickness, $2h$, of the plate, then, since $-h \le z \le h$, we see that all terms in the entries of the matrix $\mathbf{M}$ are of the same order. This provides an additional rationale for the Order Assumption and an indication of the magnitude of displacements for which the LD model, based on the this assumption, is likely to be satisfactory.

2. The energy expression for the matrix plate

As we have indicated in Section 1, we envision the membrane components of the composite as being embedded in a matrix, or substrate, plate structure. The first step in model development is to obtain a potential energy expression arising from displacements of this basic structure.

Our starting point is to note that the potential energy for two dimensional elasticity, involving only displacements $\xi(x,y)$, $\eta(x,y)$ in the (x,y) plane, may be expressed in terms of the integral

$$\frac{1}{2}\int_\Omega \Big((\lambda+\nu)\big(\xi_x + \eta_y\big)^2 + \nu\,\big(\xi_y + \eta_x\big)^2 + \nu\,\big(\xi_x - \eta_y\big)^2 \Big)\, dx\, dy, \tag{8}$$

where λ and ν are the Lamé constants. The first term involves the square of the measures of shear corresponding to the two independent modes of shear deformation arising in the two dimensional context. Referring to the area increment as dA and to the two shear angles as α and β, within the accuracy mandated by the Order Assumption (8) may be replaced by the equivalent expression

$$\frac{1}{2}\int_{\Omega}\Big((\lambda+\nu)\,dA^2+\nu\,\alpha^2+\nu\,\beta^2\Big)\,dx\,dy. \tag{9}$$

Fixing attention on the displaced surface

$$\mathbf{S}_z=\left\{\,\mathbf{F}(x,y,z)\,\middle|\,(x,y)\,\in\,\Omega\,\right\}$$

with z fixed, which is the image of

$$\mathcal{R}_z\equiv\left\{\,(x,y,z)\,\middle|\,(x,y)\,\in\,\Omega\,\right\},$$

we approximate the potential energy of a thin lamina of thickness dz centered on this surface by (see (9))

$$\frac{dz}{2}\int_{\Omega}\Big((\lambda+\nu)\,dA^2+\nu\,\alpha^2+\nu\,\alpha^2\Big)\,dx\,dy,$$

but we now measure dA, α and β with reference to the surface $\mathbf{S}_z$ and the deformation function $\mathbf{F}(x,y,z)$.

We begin with the energy term corresponding to the area increment. Suppose that an elemental rectangular region $R_0\,\in\,\Omega_z$ of area $\mathcal{A}(R_0)=dx\,dy$ is transformed via $\mathbf{F}$ into a two-dimensional surface element R in $\mathbf{S}_z$. The area of this surface element may be approximated to first order (O'Neil, 1995) by

$$\mathcal{A}(R)\approx\|\mathbf{F}_x\,\times\,\mathbf{F}_y\|\;dx\,dy.$$

Using (5) we readily compute

$$\mathbf{F}_x\,\times\,\mathbf{F}_y=$$

$$\mathbf{i}\Big(\big(\eta_x-z\,\zeta_{yx}\big)\big(\zeta_y-z\,\zeta_x\,\zeta_{xy}-\zeta_y\,\zeta_{yy}\big)-$$

$$\big(1+\eta_y-z\,\zeta_{yy}\big)\big(\zeta_x-z\,\zeta_x\,\zeta_{xx}-z\,\zeta_y\,\zeta_{yx}\big)\Big)$$

$$-\mathbf{j}\Big(\big(1+\xi_x-z\,\zeta_{xx}\big)\big(\zeta_y-z\,\zeta_x\,\zeta_{xy}-\zeta_y\,\zeta_{yy}\big)-$$

$$\big(\xi_y-z\,\zeta_{xy}\big)\big(\zeta_x-z\,\zeta_x\,\zeta_{xx}-z\,\zeta_y\,\zeta_{yx}\big)\Big)$$

Applying the Order Assumption we may discard terms to obtain the simplified approximation

$$\mathbf{F}_x \times \mathbf{F}_y \approx -\mathbf{i}\,\zeta_x - \mathbf{j}\,\zeta_y + \mathbf{k}\Big(1 + \xi_x - z\,\zeta_{xx} + \eta_y - z\,\zeta_{yy}\Big).$$

Then we have

$$\|\mathbf{F}_x \times \mathbf{F}_y\|^2 \approx {\zeta_x}^2 + {\zeta_y}^2 + 1 + 2\,\xi_x + 2\,\eta_y - 2z\,(\zeta_{xx} + \zeta_{yy}).$$

and then

$$\|\mathbf{F}_x \times \mathbf{F}_y\| - 1 \approx \xi_x + \eta_y + \frac{1}{2}({\zeta_x}^2 + {\zeta_y}^2) - z\,(\zeta_{xx} + \zeta_{yy}).$$

Thus we have

$$\big(\mathcal{A}(R) - \mathcal{A}(R_0)\big)^2 \approx \Big(\xi_x + \eta_y + \frac{1}{2}({\zeta_x}^2 + {\zeta_y}^2) - z\,(\zeta_{xx} + \zeta_{yy})\Big)^2.$$

Integrating this expression over $\Omega \times [-h, h]$ and recognizing that odd powers of z integrated over $[-h, h]$ yield zero, we obtain the component of potential energy due to local change of area in the form

$$V_A = \frac{(\lambda + \nu)}{2} \int_\Omega \int_{-h}^{h} \Big(\big(\xi_x + \eta_y + \frac{1}{2}({\zeta_x}^2 + {\zeta_y}^2)\big)^2 + z^2\big(\zeta_{xx} + \zeta_{yy}\big)^2\Big)\, dz\, dx\, dy$$

$$= (\lambda + \nu) \int_\Omega \left(h\left(\xi_x + \eta_y + \frac{1}{2}({\zeta_x}^2 + {\zeta_y}^2)\right)^2 + \frac{h^3}{3}\left(\zeta_{xx} + \zeta_{yy}\right)^2 \right) dx\, dy. \qquad (10)$$

Next we consider potential energy arising from the first shear mode shown in (8). Measured relative to the deformed surface $S_z = \mathbf{F}(\Omega_z)$ the angle of shear, α, is such that

$$\sin\alpha = \frac{\mathbf{F}_x \cdot \mathbf{F}_y}{\|\mathbf{F}_x\|\;\|\mathbf{F}_y\|}.$$

The numerator here is

$$\mathbf{N}(x, y, z) = \big(1 + \xi_x - z\,\zeta_{xx}\big)\big(\xi_y - z\,\zeta_{xy}\big) + \big(\eta_x - z\,\zeta_{yx}\big)\big(1 + \eta_y - z\,\zeta_{yy}\big)$$

$$+ \big(\zeta_x - z\,\zeta_x\zeta_{xx}\big)\big(\zeta_y - z\,\zeta_x\zeta_x y - z\,\zeta_y\zeta_{yy}\big),$$

from which, following our rules, we retain only the approximation

$$\mathbf{N}(x, y, z) \approx \xi_y + \eta_x - 2z\,\zeta_{xy} + \zeta_x\,\zeta_y.$$

The denominator is

$$\times \left(\left(\xi_y - z\,\zeta_{xy}\right)^2 + \left(1+\eta_y - z\,\zeta_{yy}\right)^2 + \left(\zeta_y - z\,\zeta_x\zeta_{xy} - z\,\zeta_y\zeta_{yy}\right)^2 \right)^{\frac{1}{2}};$$

again, following our rules, we retain only

$$\mathbf{D}(x,y,z) \approx \left(1 + 2\,\xi_x - 2z\,\zeta_{xx} + {\zeta_x}^2\right)^{\frac{1}{2}} \times \left(1 + 2\,\eta_y - 2z\,\zeta_{yy} + {\zeta_y}^2\right)^{\frac{1}{2}}$$

$$\approx \left(1 + \xi_x - z\,\zeta_{xx} + \frac{1}{2}\,{\zeta_x}^2\right) \times \left(1 + \eta_y - z\,\zeta_{yy} + \frac{1}{2}\,{\zeta_y}^2\right)$$

$$\approx 1 + \xi_x - z\left(\zeta_{xx} + \zeta_{yy}\right) + \frac{1}{2}\left({\zeta_x}^2 + {\zeta_y}^2\right),$$

and from this we have the approximation

$$\frac{1}{\mathbf{D}(x,y,z)} \approx 1 - \xi_x + z\left(\zeta_{xx} + \zeta_{yy}\right) - \frac{1}{2}\left({\zeta_x}^2 + {\zeta_y}^2\right).$$

When we multiply this reciprocal by our approximation for $\mathbf{N}(x,y,z)$, we see that all terms except the "1" in the expression for $\mathbf{D}(x,y,z)$ are included in non-retained terms of the product, and so we have, assuming the angle α to be small,

$$\alpha \approx \sin\alpha \approx \mathbf{N}(x,y,z) \approx \xi_y + \eta_x - 2z\,\zeta_{xy} + \zeta_x\,\zeta_y.$$

Again recognizing that odd powers of z integrate to zero over $[-h,h]$, the potential energy due to this shear mode can be approximately expressed as

$$V_\alpha = \frac{\nu}{2}\int_\Omega \int_{-h}^{h} \left(\left(\xi_y + \eta_x + \zeta_x\,\zeta_y\right)^2 + 4z^2\,{\zeta_{xy}}^2 \right) dz\,dx\,dy$$

$$= \int_\Omega \left(\nu h\left(\xi_y + \eta_x + \zeta_x\,\zeta_y\right)^2 + \frac{4\nu h^3}{3}\,{\zeta_{xy}}^2 \right) dx\,dy. \tag{11}$$

The angle β corresponds to shearing relative to the directions making angles of $\frac{\pi}{4}$ with the x and y axes; otherwise the two shear modes are identical. (It can be shown that shearing relative to any two orthogonal axes can be expressed as a linear combination of these two.) Thus, in order to obtain the corresponding potential energy term V_β it is only necessary to rotate the x,y plane by this angle in the expression (11). This results in

$$V_\beta = \int_\Omega \left(\nu h\left(\xi_x - \eta_y + \frac{1}{2}({\zeta_x}^2 - {\zeta_y}^2)\right)^2 + \frac{\nu h^3}{3}\left(\zeta_{xx} - \zeta_{yy}\right)^2 \right) dx\,dy. \tag{12}$$

Adding the three potential energy components together, we define

$$= (\lambda+\nu)\int_\Omega \left(h\left(\xi_x+\eta_y+\frac{1}{2}({\zeta_x}^2+{\zeta_y}^2)\right)^2+\frac{h^3}{3}\left(\zeta_{xx}+\zeta_{yy}\right)^2\right) dx\,dy$$

$$+\int_\Omega \left(\nu h\left(\xi_y+\eta_x+\zeta_x\,\zeta_y\right)^2+\frac{4\nu h^3}{3}\,{\zeta_{xy}}^2\right) dx\,dy$$

$$+\int_\Omega \left(\nu h\left(\xi_x-\eta_y+\frac{1}{2}({\zeta_x}^2-{\zeta_y}^2)\right)^2+\frac{\nu h^3}{3}\left(\zeta_{xx}-\zeta_{yy}\right)^2\right) dx\,dy. \qquad (13)$$

This potential energy form may be viewed as defining the unforced, unstressed von Karman system in the static configuration; what we call the LD model here. We add stress terms in later sections of this paper. This model is developed in the dynamic, time varying context in Lagnese (1989) along with the partial differential equations of motion obtained by application of Hamilton's (or Lagrange's, or Green's) principle. We do not develop the static counterpart of these equations here for reasons cited in Section 1.

3. Cylindrical coordinates and buckling problems

In this section we will convert the energy expression (13) into the form corresponding to the use of polar coordinates r, θ in place of the independent variables x, y, and cylindrical coordinates ζ, ρ, ψ in place of the dependent variables ζ, ξ, η. This will enable us to demonstrate the rotational symmetry of the model and will give us energy expressions facilitating the study of rotationally symmetric deflection states and angularly dependent bifurcations about those states.

Introducing the polar coordinates

$$r=\sqrt{x^2+y^2},\ \ \theta=\tan^{-1}\frac{y}{x},$$

we find that the Cartesian and cylindrical planar displacement components are related by

$$\xi(r\,\cos\theta, r\,\sin\theta)=\cos\theta\,\rho(r,\theta)-\sin\theta\,\psi(r,\theta),$$

$$\eta(r\,\cos\theta, r\,\sin\theta)=\sin\theta\,\rho(r,\theta)+\cos\theta\,\cos\theta\,\psi(r,\theta).$$

From this we obtain the planar divergence expression

$$\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}=\frac{\partial\rho}{\partial r}+\frac{1}{r}\left(\rho+\frac{\partial\psi}{\partial\theta}\right)$$

and we also compute that

$$\left(\frac{\partial\zeta}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial y}\right)^2=\left(\frac{\partial\zeta}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\zeta}{\partial\theta}\right)^2$$

resulting in an overall dilatation term

$$\left(\frac{\partial\rho}{\partial r}+\frac{1}{r}\left(\rho+\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{2}\left(\left(\frac{\partial\zeta}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\zeta}{\partial\theta}\right)^2\right)^2\right). \tag{14}$$

The combined shear energy terms

$$\left(\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial x}+\frac{\partial\zeta}{\partial x}\frac{\partial\zeta}{\partial y}\right)^2$$

$$+\left(\frac{\partial\xi}{\partial x}-\frac{\partial\eta}{\partial y}+\frac{1}{2}\left(\left(\frac{\partial\zeta}{\partial x}\right)^2-\left(\frac{\partial\zeta}{\partial y}\right)^2\right)\right)^2$$

are best treated together; their combined polar/cylindrical form is

$$\left(\frac{\partial\rho}{\partial r}-\frac{1}{r}\left(\rho+\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{2}\left(\left(\frac{\partial\zeta}{\partial r}\right)^2-\frac{1}{r^2}\left(\frac{\partial\zeta}{\partial\theta}\right)^2\right)\right)^2$$

$$+\left(\frac{\partial\psi}{\partial r}-\frac{1}{r}\left(\psi-\frac{\partial\rho}{\partial\theta}\right)+\frac{1}{r}\frac{\partial\zeta}{\partial r}\frac{\partial\zeta}{\partial\theta}\right)^2. \tag{15}$$

The square of the Laplacian of ζ in polar coordinates is

$$\left(\frac{\partial^2\zeta}{\partial r^2}+\frac{1}{r}\frac{\partial\zeta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\zeta}{\partial\theta^2}\right)^2$$

and we also verify that

$$\left(\frac{\partial^2\zeta}{\partial x^2}-\frac{\partial^2\zeta}{\partial y^2}\right)^2+4\left(\frac{\partial^2\zeta}{\partial x\partial y}\right)^2$$

$$=\left(\frac{\partial^2\zeta}{\partial r^2}-\frac{1}{r}\frac{\partial\zeta}{\partial r}-\frac{1}{r^2}\frac{\partial^2\zeta}{\partial\theta^2}\right)^2+4\left(\frac{1}{r}\frac{\partial^2\zeta}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial\zeta}{\partial\theta}\right)^2.$$

Combining all of these we now obtain the potential energy in the form

$$V=(\lambda+\nu)\int_\Omega\left(h\left(\frac{\partial\rho}{\partial r}+\frac{1}{r}\left(\rho+\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{2}\left(\left(\frac{\partial\zeta}{\partial r}\right)^2\right.\right.\right.$$

$$\left.\left.\left.+\frac{1}{r^2}\left(\frac{\partial\zeta}{\partial\theta}\right)^2\right)\right)^2+\frac{h^3}{3}\left(\frac{\partial^2\zeta}{\partial r^2}+\frac{1}{r}\frac{\partial\zeta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\zeta}{\partial\theta^2}\right)^2\right)r\,dr\,d\theta$$

$$\int\left(\;\left(\partial\psi\quad 1\left(\quad\partial\rho\right)\quad 1\,\partial\zeta\,\partial\zeta\right)^2\right.$$

$$+\frac{4\nu h^3}{3}\left(\frac{1}{r}\frac{\partial^2\zeta}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial\zeta}{\partial\theta}\right)^2\Bigg)\,r\,dr\,d\theta+\int_\Omega\left(\nu h\left(\frac{\partial\rho}{\partial r}-\frac{1}{r}\left(\rho+\frac{\partial\psi}{\partial\theta}\right)\right.\right.$$

$$\left.+\frac{1}{2}\left(\left(\frac{\partial\zeta}{\partial r}\right)^2-\frac{1}{r^2}\left(\frac{\partial\zeta}{\partial\theta}\right)^2\right)\right)^2+\frac{\nu h^3}{3}\left(\frac{\partial^2\zeta}{\partial r^2}-\frac{1}{r}\frac{\partial\zeta}{\partial r}-\frac{1}{r^2}\frac{\partial^2\zeta}{\partial\theta^2}\right)^2\Bigg)\,r\,dr\,d\theta. \quad (16)$$

THEOREM 3.1 *Let $\zeta_0(r)$, $\rho_0(r)$ ($\psi(r)\equiv 0$) be twice continuously differentiable and continuously differentiable functions, respectively, on the interval $r_0\le r\le r_1$, $r_0\ge 0$ with*

$$\zeta_0(r_0)=\frac{\partial\zeta_0}{\partial r}(0)=0;\ \rho(r_0)=0$$

minimizing, relative to other states independent of θ, the Hamiltonian

$$V_f=V-\int_0^{2\pi} f\,\zeta(r_1,\theta)\,r\,dr,$$

arising through augmentation of the potential energy (16) with the indicated term, reflecting a constant transverse negative force $-f$ acting on the outer boundary $r=r_0$ of the plate. Assuming $\frac{\partial\rho_0}{\partial r}+\frac{1}{r}\rho_0+\frac{1}{2}\left(\frac{\partial\zeta_0}{\partial r}\right)^2<0$ and $\rho_0(r)<0$, $r_0<r\le r_1$, this state cannot be a stable equilibrium for the nonlinear plate system if the positive thickness parameter h is sufficiently small.

Remark This result corresponds to the experimentally familiar fact that a very thin disc-shaped plate supported at the center and subject to a uniform transverse force at the outer perimeter will buckle in the azimuthal, or "angular" direction rather than undergo deformation into a rotationally symmetric roughly paraboloid configuration as would be predicted by the corresponding linear (Kirchhoff) plate model.

Proof We consider perturbations

$$\zeta(r,\theta)=\zeta_0(r)+\zeta_1(r)\,\cos n\theta,\ \ \rho(r,\theta)=\rho_0(r)+\rho_1(r)\,\cos n\theta,$$

where n is a positive integer. We maintain $\psi(r,\theta)\equiv 0$. Because the plate bending terms in (16), (3.1) are all multiplied by h^3 while the "membrane" terms are multiplied by h, it is only necessary to demonstrate that we can design the indicated perturbation in such a way that the portion of the potential energy corresponding to the membrane effects is decreased from the value it assumes with the state $\zeta_0(r)$, $\rho_0(r)$.

First of all, the perturbed membrane dilatation term is

$$(\lambda+\nu)h\int_0^{2\pi}\int_{r_0}^{r_1}\left(\frac{\partial\rho_0}{\partial r}+\frac{\partial\rho_1}{\partial r}\,\cos n\theta+\frac{1}{r}(\rho_0+\rho_1\,\cos n\theta)+\right.$$

$$\left.\frac{1}{2}\left(\frac{\partial\zeta_0}{\partial r}+\frac{\partial\zeta_1}{\partial r}\cos n\theta\right)^2+\frac{1}{2r^2}\left(\zeta_1\sin n\theta\right)^2\right)^2 r\,dr\,d\theta \quad (17)$$

Because any odd power of $\cos n\theta$ or $\sin n\theta$ will integrate to 0 over the interval $[0, 2\pi]$ we will ignore any such terms. Also, we regard ζ_1 and ρ_1 as being very small and do not retain powers of these beyond the second (note that since these perturbation terms are entirely independent of ζ_0 and ρ_0 this does not violate the original *ansatz* upon which our model was based; if we had prefaced ζ_1 and ρ_1 with "ϵ" parameters it would simply correspond to neglecting terms of order greater than or equal to 3 in ϵ). With these conventions (17) reduces to

$$(\lambda+\nu)h \int_0^{2\pi}\int_{r_0}^{r_1} \left(\frac{\partial\rho_0}{\partial r} + \frac{1}{r}\rho_0 + \frac{1}{2}\Big(\frac{\partial\zeta_0}{\partial r}\Big)^2\right)^2 + 2\Big(\frac{\partial\rho_1}{\partial r} + \frac{\rho_1}{r}\Big)\frac{\partial\zeta_0}{\partial r}\frac{\partial\zeta_1}{\partial r}\cos^2 n\theta$$

$$+\left(\frac{\partial\rho_0}{\partial r} + \frac{1}{r}\rho_0 + \frac{1}{2}\Big(\frac{\partial\zeta_0}{\partial r}\Big)^2\right)\left(\Big(\frac{\partial\zeta_1}{\partial r}\Big)^2\cos^2 n\theta + \frac{n^2}{r}\,\zeta_1^2\sin^2 n\theta\right) r\,dr\,d\theta. \qquad (18)$$

Next we consider the membrane shear terms. With the same conventions as indicated above the first of these, as shown in (16), reduces to

$$\nu h \int_0^{2\pi}\int_{r_0}^{r_1} \left(-\frac{n}{r}\,\rho_1\sin n\theta + \frac{1}{r}\Big(\frac{\partial\zeta_0}{\partial r} + \frac{\partial\zeta_1}{\partial r}\cos n\theta\Big)\Big(-n\,\zeta_1\sin n\theta\Big)\right)^2 r\,dr\,d\theta.$$

$$= \nu h \int_0^{2\pi}\int_{r_0}^{r_1} \left(-\frac{n}{r}\Big(\rho_1 + \frac{\partial\zeta_0}{\partial r}\,\zeta_1\Big)\sin n\theta - \frac{n}{r}\frac{\partial\zeta_1}{\partial r}\,\zeta_1\cos n\theta\,\sin n\theta\right)^2 r\,dr\,d\theta. \qquad (19)$$

We now stipulate that

$$\rho_1(r) \equiv -\frac{\partial\zeta_0}{\partial r}(r)\,\zeta_1(r),\ \ r_0 < r < r_1.$$

Then (19) becomes

$$\nu h \int_0^{2\pi}\int_{r_0}^{r_1} \frac{n^2}{r}\Big(\frac{\partial\zeta_1}{\partial r}\,\zeta_1\Big)^2\cos^2 n\theta\,\sin^2 n\theta\,r\,dr\,d\theta. \qquad (20)$$

Since this term is of fourth order in ζ_1 it is not retained.

Finally, the second shear term is

$$(\lambda+\nu)h \int_0^{2\pi}\int_{r_0}^{r_1} \left(\frac{\partial\rho_0}{\partial r} - \frac{1}{r}\rho_0 + \frac{1}{2}\Big(\frac{\partial\zeta_0}{\partial r}\Big)^2\right)^2 + 2\Big(\frac{\partial\rho_1}{\partial r} - \frac{\rho_1}{r}\Big)\frac{\partial\zeta_0}{\partial r}\frac{\partial\zeta_1}{\partial r}\cos^2 n\theta$$

$$+\left(\frac{\partial\rho_0}{\partial r} - \frac{1}{r}\rho_0 + \frac{1}{2}\Big(\frac{\partial\zeta_0}{\partial r}\Big)^2\right)\left(\Big(\frac{\partial\zeta_1}{\partial r}\Big)^2\cos^2 n\theta - \frac{n^2}{r}\,\zeta_1^2\sin^2 n\theta\right) r\,dr\,d\theta. \qquad (21)$$

The terms with subscript 0 correspond to the unperturbed membrane energy. Omitting these, we combine terms in (18) and (21) and carry out some straightforward algebraic manipulation to obtain a sum of two terms, which we will list separately. The first of these is

$$\int_0^{2\pi}\int_{r_0}^{r_1}\Big(\frac{\partial\rho_0}{\partial r}\ \ \frac{1}{}\ \ \frac{1}{}\Big(\frac{\partial\zeta_0}{}\Big)^2\Big)^2\ \ \Big(\frac{\partial\rho_1}{}\ \ \frac{\rho_1}{}\Big)\frac{\partial\zeta_0}{}\frac{\partial\zeta_1}{}$$

$$+\left(\frac{\partial\rho_0}{\partial r}+\frac{1}{r}\rho_0+\frac{1}{2}\Big(\frac{\partial\zeta_0}{\partial r}\Big)^2\right)\left(\Big(\frac{\partial\zeta_1}{\partial r}\Big)^2\cos^2 n\theta+\frac{n^2}{r}\,\zeta_1^2\,\sin^2 n\theta\right) r\,dr\,d\theta.$$

The second term is

$$\nu h\int_0^{2\pi}\int_{r_0}^{r_1}\left(\frac{\partial\rho_0}{\partial r}+\frac{1}{2}\Big(\frac{\partial\zeta_0}{\partial r}\Big)^2+\frac{1}{r}\,\rho_0\right)\left(\Big(\frac{\partial\zeta_1}{\partial r}\Big)^2\cos^2 n\theta+\frac{n^2}{r}\,\zeta_1^2\,\sin^2 n\theta\right)$$

$$+\left(\frac{\partial\rho_0}{\partial r}+\frac{1}{2}\Big(\frac{\partial\zeta_0}{\partial r}\Big)^2-\frac{1}{r}\,\rho_0\right)\left(\Big(\frac{\partial\zeta_1}{\partial r}\Big)^2\cos^2 n\theta-\frac{n^2}{r}\,\zeta_1^2\,\sin^2 n\theta\right)$$

$$=\nu h\int_0^{2\pi}\int_{r_0}^{r_1}\left(4\,\frac{\partial\rho_1}{\partial r}\,\frac{\partial\zeta_0}{\partial r}\,\frac{\partial\zeta_1}{\partial r}\,\cos^2 n\theta+2\left(\frac{\partial\rho_0}{\partial r}+\frac{1}{2}\Big(\frac{\partial\zeta_0}{\partial r}\Big)^2\right)\right.$$

$$\left.\Big(\frac{\partial\zeta_1}{\partial r}\Big)^2\cos^2 n\theta+\frac{2}{r}\,\rho_0\,\frac{n^2}{r}\,\zeta_1^2\,\sin^2 n\theta\right) r\,dr\,d\theta$$

$$\nu h\int_0^{2\pi}\int_{r_0}^{r_1}\left(-4\,\frac{\partial}{\partial r}\Big(\frac{\partial\zeta_0}{\partial r}\,\zeta_1\Big)\,\frac{\partial\zeta_0}{\partial r}\,\frac{\partial\zeta_1}{\partial r}\,\cos^2 n\theta+2\left(\frac{\partial\rho_0}{\partial r}+\frac{1}{2}\Big(\frac{\partial\zeta_0}{\partial r}\Big)^2\right)\right.$$
$$\left.\Big(\frac{\partial\zeta_1}{\partial r}\Big)^2\cos^2 n\theta+\frac{2}{r}\,\rho_0\,\frac{n^2}{r}\,\zeta_1^2\,\sin^2 n\theta\right) r\,dr\,d\theta. \tag{22}$$

The squared trigonometric terms integrate, of course, to π. Since we have assumed that both $\frac{\partial\rho_0}{\partial r}+\frac{1}{r}\rho_0+\frac{1}{2}\left(\frac{\partial\zeta_0}{\partial r}\right)^2$ and ρ_0 are negative, the terms involving n^2 in (3.) and (22) are negative for a perturbation $\zeta_1(r)>0,\ r_0<r\le r_1$. Keeping ζ_1 and ρ_1, related to ζ_1 by our earlier assumption, both fixed, it follows that the integrals (3.) and (22) are both negative if n is sufficiently large. If we then go back to (16) and replace ζ_1 and ρ_1 by $\epsilon\,\zeta_1$ and $\epsilon\,\rho_1$ we see that the the sum of (3.) and (22) correspond to the second derivative of the membrane potential energy with respect to ϵ based on the state $\zeta_0,\ \rho_0$; the first derivative involves only with $\cos n\theta$ and $\sin n\theta$ to the first power. These integrate to zero, showing that the first derivative of the membrane potential energy with respect to ϵ is zero. Then, reducing h until the change in the membrane potential energy dominates any change in the terms multiplied by h^3 in (16) we conclude that the potential energy is decreased in the direction of a perturbation such as we have described, relative to its value at $\epsilon=0$, i.e., relative to the state $\zeta_0,\ \rho_0$, for small positive values of ϵ. Since the applied transverse force f is constant, the second term appearing in the Lagrangian (3.1) does not vary with ϵ; thus the decrease in the Lagrangian is the same as that in the potential energy and the proof of the theorem is complete. ■

It seems likely that we cannot escape the requirement that n should be sufficiently large because no decrease in the potential energy/Lagrangian is to of n simply amounts to tilting the plate, with some increase in bending near the origin $r = 0$. It is interesting, however, to ask if we always obtain the energy decrease we have just established in the case $n = 2$ for sufficiently small h.

We consider, for the purpose of providing a computational example, the deformation of a nonlinear annular plate with inner boundary at $r_0 = 0.1$ and outer boundary at $r_1 = 1.1$. A constant in-plane force g is applied around the outer boundary. Clamped boundary conditions are enforced around the inner boundary while the outer boundary is free. The potential energy functional given in (16) is used with $\psi = 0$, $\rho = \rho(r,\theta)$, and $\zeta = \zeta(r,\theta)$. For ease the Lamé constants are taken to be $\lambda = nu = 10$ and the plate is assumed to be of uniform thickness $h = 0.2$. Designating the form obtained from (16) by $\mathcal{V} = \mathcal{V}(\rho,\zeta)$, we seek to minimize the Lagrangian

$$L(\rho,\zeta) = \mathcal{V}(\rho,\zeta) - (f,\rho)$$

over the class of deformations of the form

$$\rho(r,\theta) = \rho_0(r) + \rho_1(r) cos\theta$$

and

$$\zeta(r,\theta) = \zeta_0(r) + \zeta_1(r) cos\theta$$

with the clamped boundary condition enforced by means of penalization. The interval $(0.1, 1.1)$ is subdivided into four subintervals of equal length. Cubic b-splines spanned by seven basis functions $b_i(r)$ for $i = 1,...7$ are used as radial basis functions. Basis functions for the two dimensional annular domain are then obtained as a tensor product of the cubic b-spline radial basis functions and the theta-dependent functions defined on $(0, 2\pi)$ consisting of $d_1(\theta) = 1$ and $d_2(\theta) = cos(\theta)$. By representing $\rho(r,\theta)$ and $\zeta(r,\theta)$ in the form of arbitrary linear combinations of these two dimensional basis functions and appropriately substituting these forms into the Lagrangian $L(\rho,\zeta)$, one obtains an objective function for a finite dimensional minimization problem, i.e., to find the coefficients associated with $\rho(r,\theta)$ and $\zeta(r,\theta)$ for which the potential energy is minimized. In our work we used a Levenburg $-$ Marquardt minimization method to set a minimizing direction for each iteration step. The result of the computation is depicted in Fig. 1 for the case in which $g = -0.25$. It should be noted that we have plotted the independent variables r and θ on a rectangular grid here.

4. Inclusion of membrane structures

For the purposes of this paper a membrane is a two dimensional elastic structure, subject to the standard linear stress-strain relations of two dimensional linear elasticity, but embedded in three dimensional space. In using the term *two dimensional* we mean, of course, that the physical structure being modelled is

Two dimensional buckling of an annular plate

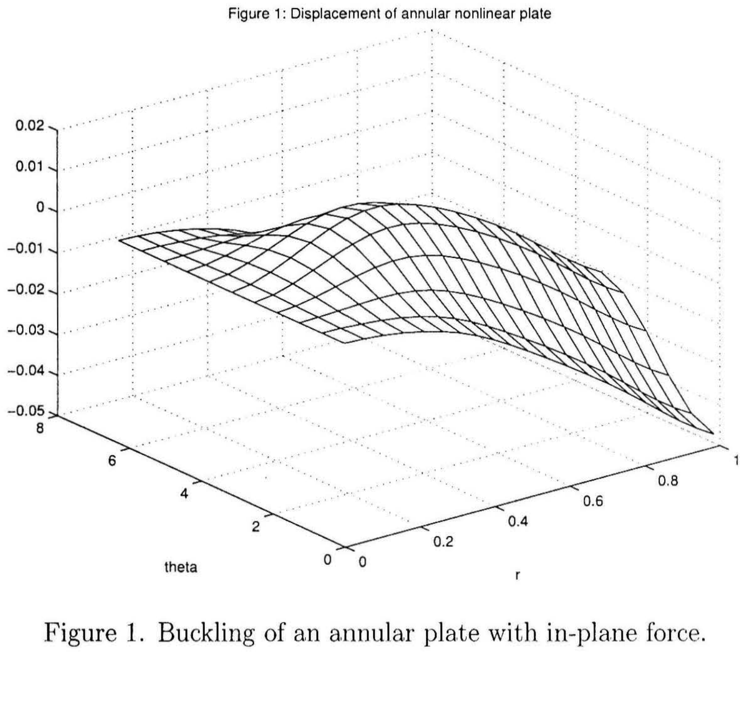

Figure 1. Buckling of an annular plate with in-plane force.

is negligible. The membranes under discussion are coextensive with the plate domain Ω introduced in Section 1, forming a layer of the plate located, when the plate is in its unforced equilibrium configuration, parallel to the x, y plane at $z = \tilde{z}$ or some other specified constant value of z. The embedding is assumed "perfect" in that there is no "slipping" between the membrane and the plate substrate. In mathematical terms this means that membrane deformations are described using the same displacement functions $\xi(x,y)$, $\eta(x,y)$, $\zeta(x,y)$ as we have hitherto used for the plate itself, via (4) with $z = \tilde{z}$.

For the first instance we consider an isotropic membrane with Lamé constants $\tilde{\lambda}$ and $\tilde{\nu}$. To avoid the minor complication of having to include first powers of z in the potential energy expression we will consider only situations symmetric with respect to the elastic axis here. Thus we assume we have a single membrane layer of double strength (i.e., with λ and ν replaced by 2λ and 2ν, respectively, located at $z = \tilde{z} = 0$ or "twinned" layers consisting of identical single strength membranes located at $z = \pm\tilde{z} \neq 0$). We will maintain this convention in all subsequent cases considered in this paper as well. Using the fixed value $z = \tilde{z}$ and omitting the integration with respect to z, the same steps as led to (13) as a sum of (10), (11) and (12) lead us to assign to the "twinned" membranes under discussion the potential energy

$$\tilde{V} = \left(\tilde{\lambda} + \tilde{\nu}\right) \int_\Omega \left(\left(\xi_x + \eta_y + \frac{1}{2}(\zeta_x^2 + \zeta_y^2)\right)^2 + \tilde{z}^2 \left(\zeta_{xx} + \zeta_{yy}\right)^2 \right) dx\, dy$$

$$+\tilde{\nu} \int_\Omega \left(\left(\xi_y + \eta_x + \zeta_x \zeta_y\right)^2 + 4\,\tilde{z}^2 \zeta_{xy}^2 \right) dx\, dy$$

$$+\tilde{\nu} \int_\Omega \left(\left(\xi_x - \eta_y + \frac{1}{2}(\zeta_x^2 - \zeta_y^2)\right)^2 + \tilde{z}^2 \left(\zeta_{xx} - \zeta_{yy}\right)^2 \right) dx\, dy. \tag{23}$$

For a plate with membranes of this type, embedded as indicated earlier, the net potential energy form is then $V + \tilde{V}$, with V as in (13) and $\tilde{V}$ as here in (23). Multiple embedded membranes will result in energy forms $V + \tilde{V}_1 + \tilde{V}_2 + \cdots$, etc.

Next we envision a membrane consisting of a film with negligible elastic properties in which are embedded a large number of elastic filaments, or strips. In general we will suppose the location coordinates and orientation angles to be random variables with particular distributions. For our first case study we will suppose that the filaments have uniform spatial density but the orientation angle θ is a random variable in the interval $[0, \pi)$ with probability density $d(\theta)$ defined on that interval (the angle θ is equivalent to $\theta + \pi$ for this purpose because the latter orientation just involves end for end reversal of the filament). We will suppose the filaments in question have equilibrium length $\Delta\ell$ which is short with respect to distances over which the partial derivatives of the displacement components vary appreciably and that they have modulus of elasticity ϵ and displacement functions $\xi(x,y)$, $\eta(x,y)$, $\zeta(x,y)$ the potential energy may be seen to be

$$\frac{\epsilon\delta a}{2}\frac{\delta\ell^2}{\Delta\ell} = \frac{\epsilon a}{2}\left(\Phi^* \mathrm{M}\,\Phi\right)^2, \tag{24}$$

where

$$\mathrm{M} = \begin{pmatrix} m & p \\ q & n \end{pmatrix}, \;\; \Phi = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{25}$$

are the matrix in (7) and the unit orientation vector for the filament, respectively. Clearly, then, we have

$$\Phi^* \mathrm{M}\,\Phi = m\cos^2\theta + (p+q)\sin\theta\cos\theta + n\sin^2\theta.$$

If we assume the filaments are strips whose thickness in the transverse direction of the membrane is fixed, then the energy (24) can be rewritten as

$$\frac{\epsilon\alpha}{2}\delta\sigma\left(m^2\cos^4\theta + (p+q)^2\sin^2\theta\cos^2\theta + n^2\sin^4\theta\right.$$

$$\left.+2m(p+q)\cos^3\theta\sin\theta + 2n(p+q)\cos\theta\sin^3\theta + 2mn\cos^2\theta\sin^2\theta\right),$$

where $\delta\sigma = \frac{\delta a}{\alpha}\Delta\ell$ is the two-dimensional surface area of the strip filament within the membrane and α is an appropriate constant of proportionality. The expected value of this energy for a given angular density $d(\theta)$ is then

$$\frac{\epsilon\alpha}{2}\,\delta\sigma\int_0^\pi\left(m^2\cos^4\theta + (p+q)^2\sin^2\theta\cos^2\theta + n^2\sin^4\theta\right.$$

$$\left.+2m(p+q)\cos^3\theta\sin\theta + 2n(p+q)\cos\theta\sin^3\theta + 2mn\cos^2\theta\sin^2\theta\right)d(\theta)\,d\theta. \tag{26}$$

If we suppose that the mean fractional membrane area occupied by strip elements is β, then as $\delta\sigma \to 0$ and the number of strip filaments tends to infinity in inverse proportion, the potential energy of the membrane composed of all of these strip filaments tends to

$$\frac{\epsilon\alpha\beta}{2}\int_\Omega(\cdots)\,dx\,dy \tag{27}$$

where $(\cdots)$ is an abbreviation for the integral in (26). This integral cannot, in general, be further simplified without more information about $d(\theta)$ but some special cases merit more detailed treatment. In the case of the constant angular density $d(\theta) = \frac{1}{\pi}$ the terms involving $\cos^3\theta\sin\theta$ and $\cos\theta\sin^3\theta$ are readily seen to be odd functions with respect to $\theta = \frac{\pi}{2}$ and those terms integrate to zero. Then, (27) becomes

$$\epsilon\alpha\beta \int \int^{\pi} \ldots$$

$$+2mn\cos^2\theta\sin^2\theta\big)\; d(\theta)\,d\theta\,dx\,dy \tag{28}$$

$$= \frac{\epsilon\alpha\beta}{2}\int_\Omega \left(\frac{3}{8}(m^2+n^2)+\frac{1}{8}((p+q)^2+2mn)\right)\,dx\,dy$$

$$= \frac{1}{2}\int_\Omega \frac{\epsilon\alpha\beta}{4}(m+n)^2+\frac{\epsilon\alpha\beta}{8}(m-n)^2+\frac{\epsilon\alpha\beta}{8}(p+q)^2\,dx\,dy.$$

Defining $\hat{\nu} = \frac{\epsilon\alpha\beta}{2}$ and invoking our "twinning convention" the membrane potential energy becomes

$$\hat{V} = \frac{1}{2}\int_\Omega 2\hat{\nu}\,(m+n)^2+\hat{\nu}\,(m-n)^2+\hat{\nu}\,(p+q)^2\,dx\,dy. \tag{29}$$

Noting the forms of m, n, p and q, a short computation shows that the same process as led to (10), (11) and (12) again yields $\hat{V}$ in the form corresponding to (23) but with $\tilde{\lambda}$ and $\tilde{\nu}$ both replaced by $\hat{\nu}$. The membrane is isotropic and is of the type consistent with what is called the *rariconstant* theory, i.e., the original Navier theory, in the literature on the history of the strength of materials, Timoshenko (1983).

Filament distributions other than the uniform one $d(\theta) \equiv \frac{1}{\pi}$ lead, in general, but not always, to anisotropic membranes. A very special case is $d = \delta_{\theta_0}$, the Dirac distribution with point support θ_0. Replacing $d(\theta)$ by δ_{θ_0} in (28) we obtain the membrane energy form

$$V_{\theta_0} = \epsilon\alpha\beta\int_\Omega \big(m^2\cos^4\theta_0 + (p+q)^2\sin^2\theta_0\cos^2\theta_0 + n^2\sin^4\theta_0 +$$

$$+\,2mn\cos^2\theta_0\sin^2\theta_0 + 2m(p+q)\cos^3\theta_0\sin\theta_0 + 2n(p+q)\sin^3\theta_0\cos\theta_0\big)\;dx\,dy$$

$$= \epsilon\alpha\beta\int_\Omega \big(m\cos^2\theta_0 + n\sin^2\theta_0 + (p+q)\cos\theta_0\sin\theta_0\big)^2\;dx\,dy. \tag{30}$$

Application of the twinning convention requires in addition that in this formula the integrand be replaced by the average of its values for $z = \pm z_0$, where $|z_0|$ is the distance between the membrane and the neutral surface of the plate. We do not carry out this process in detail here but we will do so in two special cases to be discussed in the next section.

Different specifications of (constant) θ_0 lead, of course, to different formulae (30) and its averaged counterpart, as described in the preceding paragraph. For example, with $\theta_0 = 0$ we have $\cos 0 = 1,\ \sin 0 = 0$ and we obtain

$$V_0 = \frac{\epsilon\alpha\beta}{2}\int_\Omega \left(\left(\zeta_x - z_0\,\zeta_{xx} + \frac{1}{2}(\zeta_x)^2\right) + \left(\zeta_x + z_0\,\zeta_{xx} + \frac{1}{2}(\zeta_x)^2\right)\right)\,dx\,dy$$

In general, the orientation angle θ_0 and $\mu \equiv \epsilon\alpha\beta$ can be allowed to vary with x, y:

$$\theta_0 = \theta_0(x,y), \;\; \mu = \mu(x,y).$$

We will refer to a membrane of this type, with potential energy (30), as a *monotropic* membrane, uniform if θ_0 and μ are both constant, nonuniform otherwise. As discussed more fully in Russell (1995) and Russell (1997) a membrane of this type can be fully specified by giving z_0 and the "filament field"

$$F_0(x,y) = \mu(x,y) \begin{pmatrix} \cos\theta_0(x,y) \\ \sin\theta_0(x,y) \end{pmatrix}.$$

We can also consider "screen type" membranes corresponding to the superposition of two or more filament fields corresponding to fields $F_{0,1}(x,y)$, $F_{0,2}(x,y)$, $F_{0,3}(x,y), \ldots$, etc. If the number of fields is two, $\mu_1(x,y) \equiv \mu_2(x,y)$ and $\theta_2(x,y) = \theta_1(x,y) \pm \frac{\pi}{2}$, we have what we might call an "orthotropic" membrane (it is not isotropic). In the case of three or more filament fields with equal $\mu_k(x,y)$ and orientation angles $\theta_{0,k}$ uniformly spaced in the periodic interval obtained from $[0,\pi]$ by identifying π with 0 it can be shown that the resulting membrane is isotropic and has a potential energy expression equivalent to $\hat{V}$ discussed following (29).

5. A formation problem for an annular/disk plate

Let us think of the filaments introduced in Section 4 as microactuators of monotropic type, generating stresses in particular directions in response to external control signals. For a given field of such actuators with density factor $\mu = \mu(x,y)$ and orientation angle $\theta_0 = \theta_0(x,y)$, as introduced in Section 4, we will suppose that the external control $u = u(x,y)$ has a linear effect on the equilibrium length of the actuator filament. Then, if u is appropriately normalized, the effect of u is to change the equilibrium length of the filament from $\Delta\ell$ to $(1+u)\Delta\ell$. Assuming the elastic properties of the actuator and substrate materials are not changed by application of the control, the potential energy of a double strength actuator field may be seen to be (with a qualification to be described in the paragraph to follow)

$$V_u = \int_\Omega \mu \left(m\cos^2\theta_0 + n\sin^2\theta_0 + (p+q)\cos\theta_0\sin\theta_0 - u\right)^2 dx\,dy, \quad (32)$$

where m, n, p, q are as in (25) with $\mathbf{M}$ as in (7) and it is understood that μ and θ_0 may depend on x, y as indicated previously.

Actuators designed to induce exclusively in-plane stresses will normally be located at the neutral surface $z = 0$; in this case the formula (32) requires no modification. Actuators designed to induce bending stresses, or other stresses to occur as twinned, or paired, membranes, as described in Section 4, off the neutral surface at $z = \pm z_0,\ z_0 \in (0, h]$. Often, in such paired situations, we also assume that the values of the control u, for given coordinates (x, y), take different signs; i.e., we suppose $u(x, y)$ is associated with z_0 and $-u_b(x, y)$ is associated with $-z_0$. In these circumstances the integrand of (32) is replaced by the average of its values for $z = z_0$ and $z = -z_0$. Using the forms of m, n, p, q from (25) and (7) we can see that the modified form of (32) is then

$$V_u =$$

$$\int_\Omega \mu \left[\Big(\big(\xi_x + \frac{1}{2}(\zeta_x)^2\big) \cos^2\theta_0 + \big(\eta_y + \frac{1}{2}(\zeta_y)^2\big) \sin^2\theta_0 \right.$$

$$+ \big(\xi_y + \eta_x + \zeta_x\zeta_y\big) \sin\theta_0 \cos\theta_0\Big)^2$$

$$\left. \Big(z_0\big(\zeta_{xx}\cos^2\theta_0 + \zeta_{yy}\sin^2\theta_0 + 2\zeta_{xy}\cos\theta_0\sin\theta_0\big) + u\Big)^2 \right] dx\, dy. \tag{33}$$

Our specific interest for this article lies with a generally annular plate corresponding to the domain

$$\Omega = \left\{ (x, y) \,\middle|\, r_0 \le r \le r_1 \right\}$$

with $0 \le r_0 < r_1$; if $r_0 = 0$ we have a disk, of course. Our goal is to describe actuator arrays, or "screens", suitable to the purpose of reforming the plate into a bowl-shaped, rotationally symmetric, shell. Objectives of this type may be expected to have some importance in connection with special purpose, "non-articulated" valves, especially in medical applications such as artificial hearts where articulated structures run the risk of causing damage to certain blood components. We will describe two types of actuator arrays that appear to be suitable for formation objectives of this sort.

For the first actuator array to be considered we suppose that we have radially oriented bending actuators located at $z = \pm h$ together with azimuthally (or "circumferentially") oriented, actuators located at $z = 0$. We will suppose that the density parameter μ takes on equal constant values for the two actuator families; thus the actuators are uniformly and equally distributed over the domain Ω. The corresponding control variables will be designated by u and v, respectively. In the first instance we have $\theta_0(x, y) = \theta_0(r, \theta) \equiv \theta$ while in the second case we have $\theta_0(x, y) = \theta_0(r, \theta) \equiv \theta + \frac{\pi}{2}$. Since $\cos(\theta + \frac{\pi}{2}) = -\sin\theta$ and $\sin(\theta + \frac{\pi}{2}) = \cos\theta$, the two actuator potential energy expressions become

$$V_u = \mu \int_\Omega \left[\Big(\big(\xi_x + \frac{1}{2}(\zeta_x)^2\big) \cos^2\theta + \big(\eta_y + \frac{1}{2}(\zeta_y)^2\big) \sin^2\theta \right.$$

$$\left(z_0\left(\zeta_{xx}\cos^2\theta+\zeta_{yy}\sin^2\theta+2\zeta_{xy}\cos\theta\sin\theta\right)+u\right)^2\Big]\,dx\,dy. \tag{34}$$

and

$$V_v=\mu\int_\Omega\Big(\big(\xi_x+\frac{1}{2}(\zeta_x)^2\big)\sin^2\theta+\big(\eta_y+\frac{1}{2}(\zeta_y)^2\big)\cos^2\theta$$

$$-\big(\xi_y+\eta_x+\zeta_x\zeta_y\big)\cos\theta\sin\theta-v\Big)^2\,dx\,dy. \tag{35}$$

In order to model the plate with actuation of the type we have been discussing, it is necessary to add the energy forms (34) and (35) to the energy form (16) developed earlier. However, our initial interest lies in the strictly radial situation, wherein $\psi\equiv 0$ and ζ,ρ,u and v depend only on the radial coordinate r. If we make the changes so indicated in (16), (34) and (35), omit the factor 2π resulting from integration with respect to θ and drop the u^2 and v^2 terms (since these are assumed constant and have no effect on the location or value of the minimum potential energy) we obtain the following potential energy expression for the plate under actuation via u and v:

$$V_{rad,u,v}=(\lambda+\nu)\int_{r_0}^{r_1}\left(h\left(\frac{\partial\rho}{\partial r}+\frac{\rho}{r}+\frac{1}{2}\left(\frac{\partial\zeta}{\partial r}\right)^2\right)^2+\frac{h^3}{3}\left(\frac{\partial^2\zeta}{\partial r^2}+\frac{1}{r}\frac{\partial\zeta}{\partial r}\right)^2\right)r\,dr$$

$$+\nu\int_{r_0}^{r_1}\left(h\left(\frac{\partial\rho}{\partial r}-\frac{\rho}{r}+\frac{1}{2}\left(\frac{\partial\zeta}{\partial r}\right)^2\right)^2+\frac{h^3}{3}\left(\frac{\partial^2\zeta}{\partial r^2}-\frac{1}{r}\frac{\partial\zeta}{\partial r}\right)^2\right)r\,dr \tag{36}$$

$$+\mu\int_{r_0}^{r_1}\left(\left(\frac{\partial\rho}{\partial r}+\frac{1}{2}\left(\frac{\partial\zeta}{\partial r}\right)^2\right)^2+h^2\left(\frac{\partial^2\zeta}{\partial r^2}\right)^2+\left(\frac{\rho}{r}\right)^2\right)r\,dr \tag{37}$$

$$+2\mu\int_{r_0}^{r_1}\left(h\frac{\partial^2\zeta}{\partial r^2}u-\frac{\rho}{r}v\right)r\,dr. \tag{38}$$

Here the two lines ending with the label (36) constitute the elastic potential energy for the substrate plate, the line labeled (37) is the elastic potential energy for the actuator membrane structure, and the line labeled (38) is what we refer to as the *control potential.* The presence of (37) corresponds to what we have elsewhere (Russell, 1995, 1997) referred to as the *actuator stiffening effect.*

Why are two actuator families required? In principle the radial, bending actuators may be shown to be sufficient to produced the desired re-formation of the plate into the desired shell configuration. However, if this type of actuator is used alone, the arguments of Section 3 can be repeated to show, in comparable circumstances, that the plate will eventually buckle in the azimuthal direction as described in that section. (Under these circumstances, of course, the radial form (36) - (38) would not be adequate to describe all that is going on; for this we

One would need to introduce active control strategies, with controls of the form $u(r, \theta, t)$, to stabilize the system against such an eventuality. This would greatly complicate the control process and, in all likelihood, decrease reliability.

The azimuthal control v is sufficient to hold the plate in the desired shell configuration but inadequate to initiate the re-formation process. In fact, such controls cannot determine whether the re-formed plate/shell will be concave upward or concave downward; the bending control u is needed at the outset in order to determine the development in this respect.

Now we consider a different actuator configuration. We suppose that only actuators oriented in the azimuthal direction are used but they are deployed in two layers at $z = z_0 = h$ and $z = -z_0 = -h$. The control values for the two layers are different but not of opposite sign, in general; we will denote the control used at $z = h$ by $v + w$ and the control used at $z = -h$ by $v - w$. In both cases we have $\theta_0(x, y) = \theta_0(r, \theta) \equiv \theta + \frac{\pi}{2}$. The relevant actuator potential energy expression then becomes

$$\mu \int_\Omega \Big[\Big(\Big(\xi_x + \frac{1}{2}(\zeta_x)^2 \Big) \sin^2\theta + \Big(\eta_y + \frac{1}{2}(\zeta_y)^2 \Big) \cos^2\theta$$

$$+ \big(\xi_y + \eta_x + \zeta_x \zeta_y \big) \sin\theta \cos\theta - v \Big)^2$$

$$\Big(h \big(\zeta_{xx} \sin^2\theta + \zeta_{yy} \cos^2\theta + 2\zeta_{xy} \cos\theta \sin\theta \big) + w \Big)^2 \Big] \, dx \, dy. \tag{39}$$

Then, going over to the rotationally symmetric case again with polar coordinates and omitting the v^2, and w^2 terms for the same reasons as cited earlier in connection with u and v, we find that we have

$$V_{rad,v,w} = (36) + \mu \int_{r_0}^{r_1} \left(\frac{h^2}{r^2} \Big(\frac{\partial \zeta}{\partial r} \Big)^2 + \Big(\frac{\rho}{r} \Big)^2 \right) r \, dr \tag{40}$$

$$+ 2\mu \int_{r_0}^{r_1} \left(\frac{h}{r} \frac{\partial \zeta}{\partial r} w - \frac{\rho}{r} v \right) r \, dr. \tag{41}$$

The second term in (40) is the elastic potential energy for the actuator structure in this configuration and (41) is the control potential. One of the advantages associated with this actuator configuration is that there is less actuator stiffening effect in the radial direction than there is with the earlier configuration involving the bending actuators. This indicates the likelihood that this second configuration may be superior to the first for formation objectives of the type we have described.

6. Computational experience

We have carried out extensive numerical computations connected with the for-section we will present some of the results of these computations in graphic form.

All of our computations have been carried out within the variational framework described in this paper. Except as otherwise indicated we have studied strictly radial situations and, within that context, we have used quadratic spline finite element representations of the lateral deflection ζ and linear spline finite element representations of the in-plane displacement ρ. Our procedure has been to solve the linear (Kirchhoff) system first and then use a Newton-type procedure to correct for the nonlinear terms present in the LD (i.e., the full von Karman) system. The graphics presented here are based on the use of ten finite elements in ten equal subintervals along the radial axis from $r_0 = 0$ to $r_1 = 1$ with three additional points added in the interior of each of these subintervals for plotting purposes. These interior points are also used for approximate evaluation, via Simpson's rule, of some integrals involving products of finite elements or their derivatives in the computations leading to these plots.

The most common type of buckling associated with a disk-shaped LD plate is strictly radial buckling resulting from application of an in-plane force in an inward (i.e., negative) direction along the outer boundary. In Figs. 2 and 3 we show the finite amplitude character of this type of buckling phenomenon as it arises in connection with the LD model. For the computations which resulted in Fig. 2 the plate thickness h is set at .1. A very small negative value of f, corresponding to the lateral force, is used to start the buckling in the desired direction. Then, different values of the boundary force, g, are used, retaining the same value of f. Corresponding to $g = -0.03, -0.06, -0.09$ we have ζ deflections labelled $z1$, $z2$, $z3$, respectively, with corresponding in-plane displacements ρ labelled $r1$, $r2$, $r3$. In each case the label lies directly above the curve in question on the graph.

In the next set of figures we display the results of some formation studies using controls w, v as in the Lagrangian/Hamiltonian expression (41) and described prior to that in Section 5. In each case a small positive value of w was used to ensure that bending would take place in the negative ζ direction. Then, successively larger values of v were used to obtain the ζ deflection curves labelled $z1$, $z2$, $z3$. The corresponding in-plane displacement functions ρ are labelled $r1$, $r2$, $r3$; in each case the label lies directly above the curve to which it applies. It should be noted that the ζ curves are not plotted over the interval $[r_0, r_1](= [0,1]$ in all these cases), but rather over the interval $[0, 1 + \rho(1)]$ ($\rho(1)$ is negative in all these cases) resulting from the in-plane displacement primarily induced by the in-plane control v. Fig. 4 shows the results obtained in the limiting case where both v and w have support confined to the outer boundary $r = r_1 = 1$. Fig. 5 shows corresponding results but with both v and w uniformly distributed over the interval $[0, 1]$; the integrals in this case correspond to the point densities of the corresponding controls used to obtain the plots of Fig. 4. Finally, the plots of Fig. 6 were obtained with v and w densities of the form

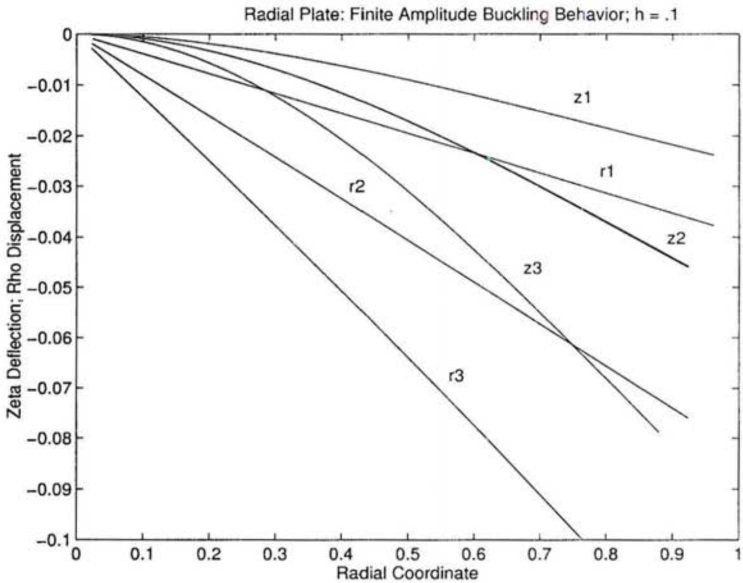

Figure 2. Buckling for $h = 0.1$

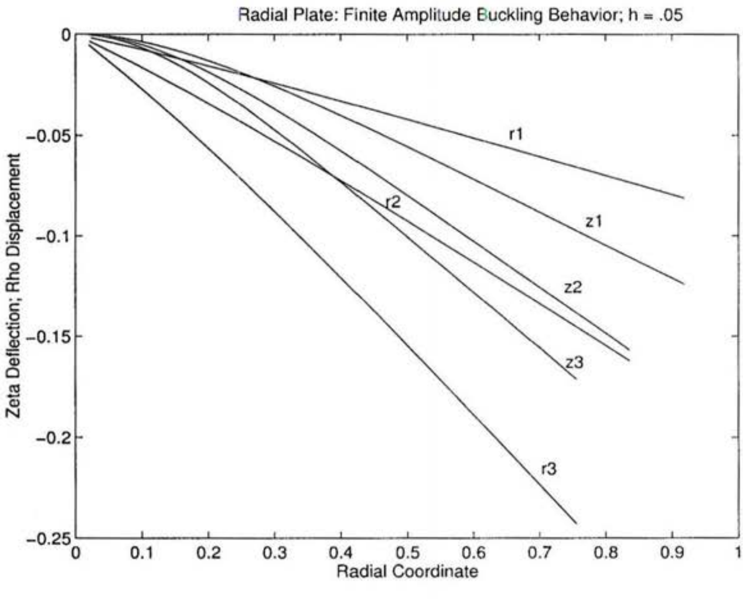

Figure 3. Buckling for $h = 0.05$

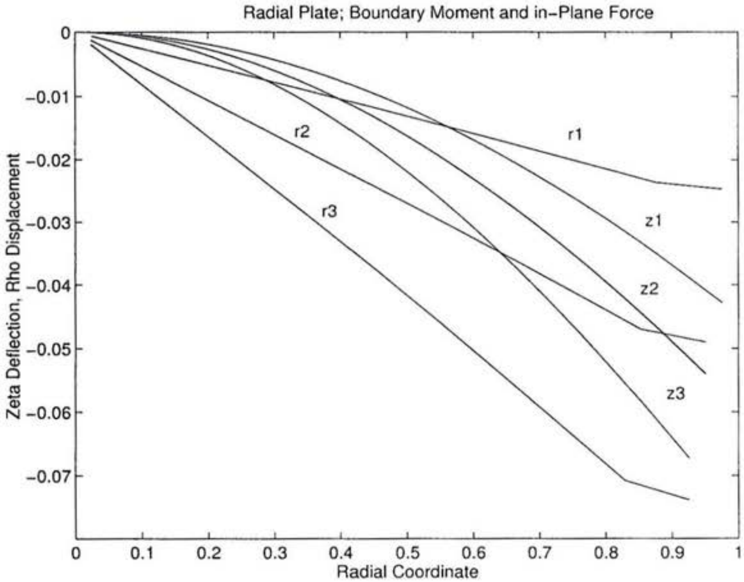

Figure 4. ζ deflections and ρ displacements for boundary controls.

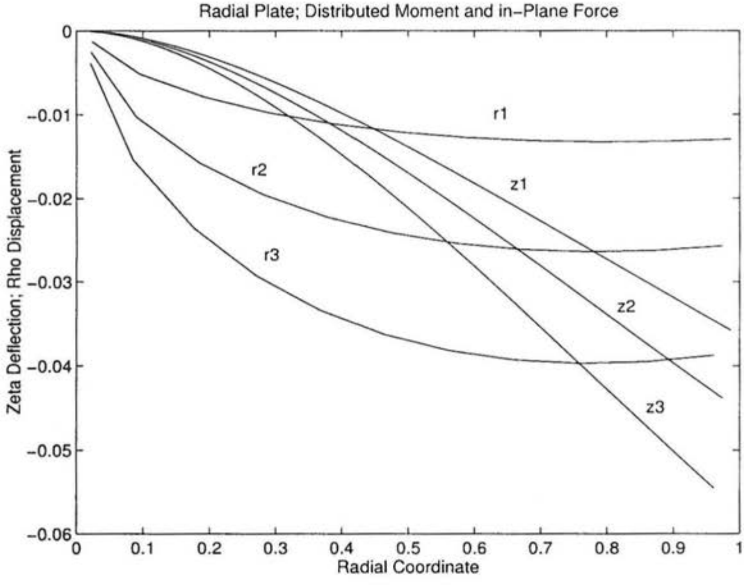

Figure 5. ζ deflections and ρ displacements for uniformly distributed controls

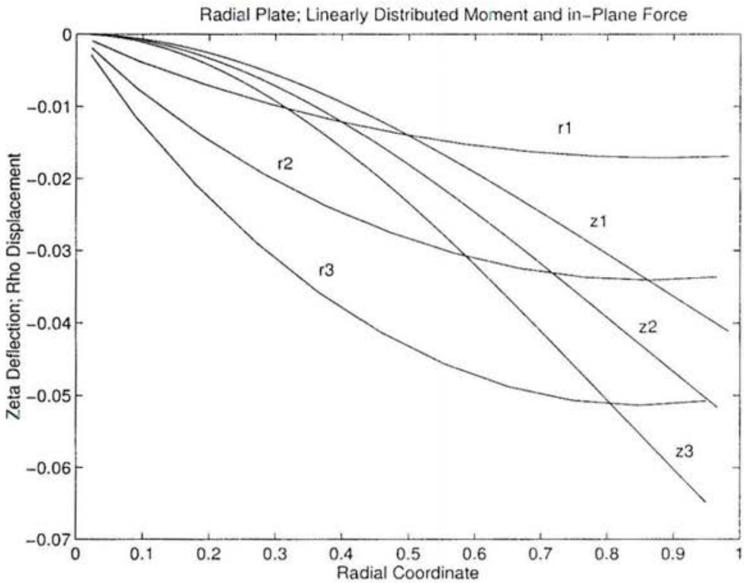

Figure 6. ζ deflections and ρ displacements for linearly distributed controls.

In future work we hope to carry out optimization studies, posing a desired target deflection profile and computing optimal controls relative to appropriate "fit-to-data" criteria.

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