

Inverse/observability estimates for Schrödinger equations  
with variable coefficients<sup>1</sup>

by

Roberto Triggiani\* and Peng-Fei Yao\*\*

\* Mathematics, University of Virginia, Charlottesville, VA 22903

\*\* Institute of Systems Science, Academia Sinica,  
Beijing 100080, China

**Abstract:** We consider a general Schrödinger equation defined on an open bounded domain  $\Omega \subset \mathbf{R}^n$  with variable coefficients in both the elliptic principal part and in the first-order terms as well. At first, no boundary conditions (B.C.) are imposed. Our main result (Theorem 3.5) is a reconstruction, or inverse, estimate for solutions  $w$ : under checkable conditions on the coefficients of the principal part, the  $H^1(\Omega)$ -energy at time  $t = T$ , or at time  $t = 0$ , is dominated by the  $L_2(\Sigma)$ -norms of the boundary traces  $\frac{\partial w}{\partial \nu_A}$  and  $w_t$ , modulo an interior lower-order term. Once homogeneous B.C. are imposed, our results yield – under a uniqueness theorem, needed to absorb the lower order term – continuous observability estimates for both the Dirichlet and Neumann case, with an arbitrarily short observability time; hence, by duality, exact controllability results. Moreover, no artificial geometrical conditions are imposed on the controlled part of the boundary in the Neumann case. In contrast to existing literature, the first step of our method employs a Riemann geometry approach to reduce the original variable coefficient principal part problem in  $\Omega \subset \mathbf{R}^n$  to a problem on an appropriate Riemannian manifold (determined by the coefficients of the principal part), where the principal part is the Laplacian. In our second step, we employ explicit Carleman estimates at the differential level to take care of the variable first-order (energy level) terms. In our third step, we employ micro-local analysis yielding a sharp trace estimate to remove artificial geometrical conditions on the controlled part of the boundary in the Neumann case.

**Keywords:** Schrödinger equation, inverse/observability estimates, exact controllability, Riemannian manifold, Carleman estimates.

---

<sup>1</sup>R. Triggiani's research was partially supported by the Army Research Office under Grant DAAH04-96-1-0059, and by the National Science Foundation under Grant DMS-9504822. P.-F. Yao's research was partially supported by the National Science Foundation and The

## 1. Introduction. The dual problem: continuous observability inequalities. Literature

**Standing assumptions.** (H.1): Let  $\Omega \subset \mathbf{R}^n$  be a bounded, open domain with boundary  $\Gamma = \partial\Omega$  of class  $C^2$ . Let  $\Gamma_0$  and  $\Gamma_1$  be open disjoint subsets of  $\Gamma$  with  $\Gamma = \Gamma_0 \cup \Gamma_1$ . Let

$$\mathcal{A}w \equiv - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial w}{\partial x_j} \right), \quad x = [x_1, \dots, x_n] \quad (1)$$

be a second-order differential operator, with real coefficients  $a_{ij} = a_{ji}$  of class  $C^1$ , satisfying the uniform ellipticity condition:

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq a \sum_{i=1}^n \xi_i^2, \quad x \in \Omega, \quad (2)$$

for some positive constant  $a > 0$ . Assume further that

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j > 0, \quad \forall x \in \mathbf{R}^n, \quad \xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbf{R}^n, \quad \xi \neq 0. \quad (3)$$

(H.2): Let  $F(w)$  be a linear, first-order differential operator in all variables  $\{t, x_1, \dots, x_n\}$  on  $w$  with  $L_\infty(Q)$ -coefficients, thus satisfying the following pointwise estimate: there exists a constant  $C_T > 0$  such that

$$|F(w)|^2 \leq C_T [|\nabla w|^2 + w^2], \quad \forall t, x \in Q, \quad (4)$$

where  $Q = (0, T] \times \Omega$  and  $w(t, x) \in C^1(Q)$ . Let  $(0, T] \times \Gamma_i \equiv \Sigma_i$ ,  $i = 0, 1$ ;  $(0, T] \times \Gamma \equiv \Sigma$ .

**Dirichlet control.** We consider the Dirichlet mixed problem for the Schrödinger equation in the unknown  $w(t, x)$  and its dual homogeneous problem in  $\psi(t, x)$ :

$$\left\{ \begin{array}{ll} iw_t + \mathcal{A}w = F_1(w) & \text{in } Q; \\ w(0, \cdot) = w_0; & \text{in } \Omega; \\ w|_{\Sigma_0} \equiv 0 & \text{in } \Sigma_0; \\ w|_{\Sigma_1} = u & \text{in } \Sigma_1; \end{array} \right. \quad \left\{ \begin{array}{ll} i\psi_t + \mathcal{A}\psi = F(\psi) & \text{in } Q; \\ \psi(T, \cdot) = \psi_0, & \text{in } \Omega; \\ \psi|_{\Sigma} \equiv 0 & \text{in } \Sigma, \end{array} \right. \quad (5)$$

with control function  $u \in L_2(0, T; L_2(\Gamma_1))$  in the Dirichlet B.C., where  $F_1(\psi)$  is a suitable first-order differential operator, depending on the original operator  $F$ , and satisfying the same pointwise bound such as (4) for  $F$ .

**Continuous observability inequality in the Dirichlet case.** As our

following *a-priori* inequality for the homogeneous Dirichlet  $\psi$ -problem (5): for all  $T > 0$ , there is a constant  $c_T > 0$  for which

$$\int_0^T \int_{\Gamma_1} \left| \frac{\partial \psi}{\partial \nu_A} \right|^2 d\Sigma_1 \geq c_T \|\psi_0\|_{H_0^1(\Omega)}^2. \quad (6)$$

In (6),  $\frac{\partial w}{\partial \nu_A} = \sum_{i,j=1}^n a_{ij} \frac{\partial w}{\partial x_j} \nu_i$  is the co-normal derivative, where  $\nu = [\nu_1, \dots, \nu_n]$  is the unit outward normal on  $\Gamma$ . Eqn. (6) is the *continuous observability inequality* for the  $\psi$ -problem (5) in the established terminology of Dolecki and Russell (1977). As is well-known, e.g., Lasiecka and Triggiani (1991), Triggiani (1996), inequality (6) for the  $\psi$ -problem (5) is, by duality or transposition, *equivalent* to the exact controllability property of the non-homogeneous  $w$ -problem (5) at the arbitrary time  $T$ , on the space  $Y = H^{-1}(\Omega)$ , within the class of  $L_2(0, T; L_2(\Gamma_1))$ -controls; in other words, such exact controllability is the property that the map  $L_T$ :

$$\begin{cases} \{u, w_0 = 0\} \rightarrow L_T u \equiv w(T, \cdot) \text{ is surjective} \\ \text{from } L_2(0, T; L_2(\Gamma_1)) \text{ onto } H^{-1}(\Omega), \end{cases} \quad (7)$$

with  $w(T, \cdot)$  solution of the  $w$ -problem (5) at  $t = T$ ; while inequality (6) is a restatement, Lasiecka and Triggiani (1991), Triggiani (1996), of the following standard, Taylor and Lay (1980), p. 235, inequality from below of the corresponding adjoint:

$$\|L_T^* z\|_{L_2(0, T; L_2(\Gamma_1))} \geq c_T \|z\|_{H^{-1}(\Omega)}, \quad (8)$$

which is well known to be equivalent to the surjectivity property (7).

**Remark 1.1.** The converse (trace regularity) of inequality (6) always holds true, for any  $T > 0$ , Lasiecka and Triggiani (1991), Theorem 1.1.

**Neumann control.** Here we let  $\Gamma_0 \neq \emptyset$ ,  $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ , and consider the Neumann mixed Schrödinger problem in the unknown  $w(t, x)$  and its dual homogeneous version in  $\psi(t, x)$ :

$$\begin{cases} iw_t + \mathcal{A}w = F_1(w); \\ w(0, \cdot) = w_0; \\ w|_{\Sigma_0} \equiv 0; \\ \left. \frac{\partial w}{\partial \nu_A} \right|_{\Sigma_1} = u; \end{cases} \quad \begin{cases} i\psi_t + \mathcal{A}\psi = F(\psi) & \text{in } Q; \\ \psi(T, \cdot) = \psi_0, & \text{in } \Omega; \\ \psi|_{\Sigma_0} \equiv 0 & \text{in } \Sigma_0; \\ \left[ \frac{\partial \psi}{\partial \nu_A} + \beta\psi \right]_{\Sigma_1} \equiv 0 & \text{in } \Sigma_1, \end{cases} \quad (9)$$

with control function  $u \in L_2(0, T; L_2(\Gamma_1)) \equiv L_2(\Sigma_1)$  in the Neuman B.C., where  $F_1$  is a suitable first-order differential operator depending on  $F$ , and satisfying the same pointwise estimate such as (4) for  $F$ , and  $\beta$  is a suitable function.

**Continuous observability inequality in the Neumann case.** As our *second goal* we seek to establish – under a suitable additional assumption – the following *a-priori* inequality for the homogeneous Neumann  $\psi$ -problem (9): for all  $T > 0$ , there is a constant  $c_T > 0$  for which

$$\int_0^T \int_{\Gamma_1} |\psi_t|^2 d\Sigma_1 \geq c_T \|\psi_0\|_{H_{\Gamma_0}^1(\Omega)}^2, \quad (10)$$

where  $H_{\Gamma_0}^1(\Omega) = \{f \in H^1(\Omega) : f|_{\Gamma_0} = 0\}$ , whenever the left-hand side is finite. This is the *continuous observability inequality* for the  $\psi$ -problem (9) Dolecki and Russell (1977). Again, by duality or transposition, inequality (10) is *equivalent* (see e.g., Triggiani, 1996) to the exact controllability property of the non-homogeneous  $w$ -problem (9) at time  $T$ , on the space  $H_{\Gamma_0}^1(\Omega)$ , within the class of  $L_2(0, T; L_2(\Gamma_1))$ -controls; in other words, such exact controllability is the property that the map  $L_T$ :

$$\left\{ \begin{array}{l} \{u, w_0 = 0\} \rightarrow L_T u \equiv w(T, \cdot) \text{ is surjective} \\ \text{from } L_2(0, T; L_2(\Gamma_1)) \text{ onto } H_{\Gamma_0}^1, \end{array} \right. \quad (11)$$

with  $w(T, \cdot)$  being a solution of the  $w$ -problem (9) at  $t = T$ , while inequality (10) is a restatement, Triggiani (1996), of the following standard, Taylor and Lay (1980), p. 235, inequality from below of the corresponding adjoint:

$$\|L_T^* z\|_{L_2(0, T; L_2(\Gamma_1))} \geq c_T \|z\|_{H_{\Gamma_0}^1}, \quad (12)$$

which is well known to be equivalent to the surjectivity property (11), Lasiecka and Triggiani (1991), Triggiani (1996).

**Literature.** Our results are more general than just continuous observability estimates, or – by duality – exact controllability statements. The latter are generally obtained in the literature through the former, Dolecki and Russell (1977), on the basis of the standard Functional Analysis result, Taylor and Lay (1980), p. 235, quoted before. One exception is the approach pursued by W. Littman, who seeks exact controllability results directly, without passing through continuous observability inequalities, Littman (1987, 1992), Littman and Taylor (1992), Horn and Littman (1996a, b).

A detailed analysis of the various methods used in the literature to establish continuous observability inequalities, particularly with reference to second-order hyperbolic equations, along with a description of their virtues and shortcomings was already given in our previous works Lasiecka, Triggiani, and Yao (1997, 1998, 1999). Here we shall focus on the counterpart of these considerations, as they apply to the Schrödinger equation (9):  $i w_t + \mathcal{A} w = F_1(w)$  in  $Q$  with  $F_1$  a first-order differential operator in  $x_1, \dots, x_n$  satisfying (4). The energy (multiplier) method, based on the principal multiplier  $h(x) \cdot \nabla \bar{w}(x)$ ,  $h(x)$  being a suitable coercive vector field over  $\bar{\Omega}$ , permits to establish a number of key

(i) the “regularity inequality” in the Dirichlet homogeneous case  $w|_{\Sigma} \equiv 0$  (the  $L_2(\Sigma_T)$ -norm of  $\frac{\partial w}{\partial \nu}$  is bounded above by  $E_w(0)$ , for all  $T$ ), Lasiecka and Triggiani (1991), Theorem 1.1, indeed, even in the case of a (symmetric) principal part with variable coefficients;

(ii) the reverse “continuous observability inequalities,” such as (6) and (10), when coupled with the second multiplier  $\bar{w} \operatorname{div} h$ , however, *only when*  $F_1$  is actually a zero-order operator, Lasiecka and Triggiani (1991), Machtyngier (1990). If  $F_1$  is a bonafide first-order operator, the method fails. To obtain “continuous observability” reverse inequalities, more sophisticated methods were subsequently introduced:

- (a) Methods of microlocal analysis, after a rescaling of time, depending on the frequency, Lebeau (1992): the final statement, which assumes analytic boundary and delivers a control acting on a pair  $(\bar{\Gamma}, T)$ , which geometrically controls  $\Omega$ , refers, however, to the *pure*  $w$ -Schrödinger equation (9) with  $\mathcal{A} = -\Delta$  and  $F_1 \equiv 0$ . However, it is not an easy matter to verify in applications and examples the (sharp) sufficient conditions that all the rays of geometric optics hit the effective controlled part  $\Sigma_1 = (0, T] \times \Gamma_1$  of the lateral boundary  $\Sigma$  of the cylinder  $Q$  at a non-diffractive point. This condition was first obtained in Littman (1987) for hyperbolic systems and then re-obtained and refined in Bardos, Lebeau and Rauch (1992) for second-order hyperbolic equations. Moreover, the method uses  $C^\infty$  data and  $\Gamma$ , at least at present. Extension to other models such as general plate-like equations, seems a serious issue.
- (b) Pseudo-differential methods derived from pseudo-convex functions to extend Carleman estimates – which were available in the literature, Hormander (1985), for solutions with compact support and, generally, isotropic operators – to the case of domains with boundary and to anisotropic operators, as carried out in the general and unifying work of Tataru (1992, 1994, 1995). However, they require the existence of a pseudo-convex function, a property which essentially could be verified mostly if not exclusively in the case of constant coefficients  $a_{ij}$  of the principal part  $\mathcal{A}$ . Moreover, at least in Tataru (1992, 1994, 1995), the control is taken to be active in the entire boundary  $\Gamma$ .
- (c) An altogether different approach is proposed and pursued in Littman (1987, 1992), Horn and Littman (1996a, b), Littman and Taylor (1992), which aims at obtaining steering controls directly through the principle of *local smoothing + reversibility + uniqueness*  $\rightarrow$  *exact controllability*. This method allows for variable  $C^\infty$ -coefficients of the (strongly elliptic and self-adjoint) principal part, but delivers only controls which belong to  $C^\infty(\partial\Omega)$  for  $t > 0$ . For many purposes, we would, instead, need a precise relationship, in terms of Sobolev spaces, between the space  $L_2(0, T; L_2(\Gamma))$  of controls on  $[0, T]$ , and the space  $Y$  of exact controllability at  $t = T$ ; i.e.,  $Y = H^{-1}(\Omega)$  (Dirichlet case), and  $Y = H^1(\Omega)$  (Neumann case).

(much more flexible than the classical differential multipliers in (a), tuned to either the second-order hyperbolic equations, Fursikov and Imanuvilov (1996), Imanuvilov (1990), Lasiecka and Triggiani (1994), or else to Schrödinger equations Triggiani (1996) with  $\mathcal{A} = -\Delta$ . In these cases, the drawback of the existence of a pseudo-convex function remains, of course, for general  $\mathcal{A}$ , while now a more detailed analysis – this time at the differential rather than pseudo-differential level – allows the control to act on a suitable part of the boundary. These differential Carleman multipliers can be viewed as a non-trivial generalization of the original multipliers  $h \cdot \nabla \bar{w}$ ,  $\bar{w} \operatorname{div} h$  in (a), over which they possess an added flexibility via the parameter  $\tau$  below, which allows to handle also those first-order terms  $F_1$  as in (4) that the original multipliers could not deal with. See also Remark 4.2.1 further on.

- (e) Differential geometric methods, originally introduced in Yao (1996) in the case of second-order hyperbolic equations, which could handle at first only the case of variable coefficient principal part, but no genuine first-order energy level terms. They were the generalization of (a) from the Euclidean to a suitable Riemannian metric. Subsequently, in Lasiecka, Triggiani and Yao (1997, 1998, 1999), these Riemannian geometric methods have been extended to the counterpart of (c), thereby handling both variable coefficient principal part and first-order energy level terms.

**Contribution of the present paper.** The present paper generalizes Triggiani (1996) from the constant coefficient  $\mathcal{A} = -\Delta$  and general first-order energy level terms to general  $\mathcal{A}$ , in the case of Schrödinger equations, by using a Riemann metric, Yao (1996), in the same way as Lasiecka, Triggiani and Yao (1997, 1998, 1999) generalized the case of second-order hyperbolic equations from constant coefficients to variable coefficients in  $\mathcal{A}$ , and first-order energy level terms. More precisely, in this paper we present a successful combination of three key ingredients which allow to establish the validity of the continuous observability inequalities (6) and (10) in the case of (a) variable coefficients  $a_{ij}(x)$  of the principal part  $\mathcal{A}$ , subject to verifiable conditions, and (b) genuine first-order, energy level terms  $F$ , and (c) with no artificial geometric conditions in the Neumann case. These three ingredients are: (1) the Riemann geometric approach of Yao (1996) for variable  $a_{ij}(x)$  as improved in Lasiecka, Triggiani and Yao (1997, 1998, 1999) for the addition of genuine first-order energy level terms; (2) the Carleman differential multipliers used in Triggiani (1996), which now replace the original classical differential multipliers of Lasiecka and Triggiani (1991), though in the Riemann metric; (3) the pseudo-differential approach in Lasiecka and Triggiani (1994), Triggiani (1996), which led to an  $L_2$ -estimate of the tangential derivative (gradient) of the solution  $w$  in terms of  $L_2$ -boundary estimates of  $w_t$  and  $\frac{\partial w}{\partial \nu_{\mathcal{A}}}$ , modulo lower-order terms; see Lemma 6.2 further on.

It is ingredients (1) and (2) that permit to consider variable coefficients  $a_{ij}$

ometrical conditions present in the literature in the Neumann case, Machtyngier (1990), on the controlled part of the boundary.

The present approach provides an arbitrarily small time for the validity of the continuous observability inequalities (6) and (10), as is the case with pseudo-convex functions.

Our new main differential multipliers are (see statements (82) and (91)):

$$e^{\tau\phi(x,t)} \langle \nabla_g \phi, \nabla_g \bar{w} \rangle_g \quad \text{and} \quad \bar{w} \operatorname{div}_0(e^{\tau\phi} \nabla_g \phi) \quad (13)$$

in the Riemann metric  $(\mathbf{R}^n, g)$ , where  $\phi$  is the pseudo-convex function defined in (50).

## 2. Riemannian metric generated by the principal part $\mathcal{A}$

Recalling the coefficients  $a_{ij} = a_{ji}$  of  $\mathcal{A}$ , let  $A(x)$  and  $G(x)$  be, respectively, the coefficient matrix and its inverse

$$A(x) = (a_{ij}(x)); \quad G(x) = [A(x)]^{-1} = (g_{ij}(x)), \quad i, j = 1, \dots, n; \quad x \in \mathbf{R}^n. \quad (14)$$

Both  $A(x)$  and  $G(x)$  are  $n \times n$  matrices.  $A(x)$  is positive definite for any  $x \in \mathbf{R}^n$  by assumption (3).

**Riemannian metric.** Let  $\mathbf{R}^n$  have the usual topology and  $x = [x_1, x_2, \dots, x_n]$  be the natural coordinate system. For each  $x \in \mathbf{R}^n$ , define the inner product and the norm on the tangent space  $\mathbf{R}_x^n = \mathbf{R}^n$  by

$$g(X, Y) = \langle X, Y \rangle_g = \sum_{ij=1}^n g_{ij}(x) \alpha_i \beta_j, \quad (15)$$

$$|X|_g = \langle X, X \rangle_g^{\frac{1}{2}}, \quad \forall X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i} \in \mathbf{R}_x^n. \quad (16)$$

It is easily checked from (3) that  $(\mathbf{R}^n, g)$  is a Riemannian manifold with the Riemannian metric  $g$ . We shall denote  $g = \sum_{ij=1}^n g_{ij} dx_i dx_j$ . (If  $A(x) \equiv I$ , i.e.,  $\mathcal{A} = -\Delta$ , then  $G(x) \equiv I$ , and  $g$  is the Euclidean  $\mathbf{R}^n$ -metric.)

**Euclidean metric.** For each  $x \in \mathbf{R}^n$ , denote by

$$X \cdot Y = \sum_{i=1}^n \alpha_i \beta_i, \quad |X|_0 = (X \cdot Y)^{\frac{1}{2}}, \quad \forall X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i} \in \mathbf{R}_x^n, \quad (17)$$

the Euclidean metric on  $\mathbf{R}^n$ . For  $x \in \mathbf{R}^n$ , and with reference to (14), set

$$A(x)X = \sum_{i=1}^n \left( \sum_{j=1}^n a_{ij}(x) \alpha_j \right) \frac{\partial}{\partial x_i}, \quad \forall X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i} \in \mathbf{R}_x^n. \quad (18)$$

Thus, recalling the co-normal derivative defined in (6), we have

$$\frac{\partial w}{\partial \nu_A} = \sum_{i=1}^n \left( \sum_{j=1}^n a_{ij}(x) \frac{\partial w}{\partial x_j} \right) \nu_i = (A(x)\nabla_0 w) \cdot \nu. \tag{19}$$

In (17), and hereafter, we denote by a sub “0” entities in the Euclidean metric. Thus, for  $f \in C^1(\Omega)$  and  $X = \sum_{i=1}^n \alpha_i(x) \frac{\partial}{\partial x_i}$  a vector field on  $\mathbf{R}^n$ ,

$$\nabla_0 f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}, \text{ and } \operatorname{div}_0(X) = \sum_{i=1}^n \frac{\partial \alpha_i(x)}{\partial x_i} \tag{20}$$

denote gradient of  $f$  and divergence of  $X$  in the Euclidean metric.

**Further relationships.** If  $f \in C^1(\bar{\Omega})$ , we define the gradient  $\nabla_g f$  of  $f$  in the Riemannian metric  $g$ , via the Riesz representation theorem, by

$$X(f) = \langle \nabla_g f, X \rangle_g, \tag{21}$$

where  $X$  is any vector field on the manifold  $(\mathbf{R}^n, g)$ . The following lemma provides further relationships, Yao (1996), Lemma 2.1.

**Lemma 2.1.** *Let  $x = [x_1, x_2, \dots, x_n]$  be the natural coordinate system in  $\mathbf{R}^n$ . Let  $f, h \in C^1(\bar{\Omega})$ . Finally, let  $H, X$  be vector fields. Then, with reference to the above notation, we have*

(a)

$$\langle H(x), A(x)X(x) \rangle_g = H(x) \cdot X(x), \quad x \in \mathbf{R}^n; \tag{22}$$

(b)

$$\nabla_g f(x) = \sum_{i=1}^n \left( \sum_{j=1}^n a_{ij}(x) \frac{\partial f}{\partial x_j} \right) \frac{\partial}{\partial x_i} = A(x)\nabla_0 f, \quad x \in \mathbf{R}^n; \tag{23}$$

(c) If  $X = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i}$ , then by (21) and (23),

$$X(f) = \langle \nabla_g f, X \rangle_g = \langle A\nabla_0 f, X \rangle_g = \nabla_0 f \cdot X = \sum_{i=1}^n \xi_i \frac{\partial f}{\partial x_i}; \tag{24}$$

(d) By (19) and (23),

$$\frac{\partial w}{\partial \nu_A} = (A(x)\nabla_0 w) \cdot \nu = \nabla_g w \cdot \nu; \tag{25}$$

(e) by (21), (23), (22),

$$\langle \nabla_g f, \nabla_g h \rangle_g = \nabla_g f(h) = \langle A(x)\nabla_0 f, \nabla_g h \rangle_g = \nabla_0 f \cdot \nabla_g h$$



(f) If  $H$  is a vector field in  $(\mathbf{R}^n, g)$  (see, e.g., (29) below),

$$\begin{aligned} \langle \nabla_g f, \nabla_g(H(f)) \rangle_g &= DH(\nabla_g f, \nabla_g f) + \frac{1}{2} \operatorname{div}_0(|\nabla_g f|_g^2 H)(x) \\ &\quad - \frac{1}{2} |\nabla_g f|_g^2(x) \operatorname{div}_0(H)(x), \quad x \in \mathbf{R}^n, \end{aligned} \tag{27}$$

where  $DH$  is the covariant differential discussed below;

(g) by (1), (20), (23),

$$\begin{aligned} Aw &= - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \sum_{j=1}^n a_{ij}(x) \frac{\partial w}{\partial x_j} \right) = -\operatorname{div}_0(A(x)\nabla_0 w) = -\operatorname{div}_0(\nabla_g w), \\ w &\in C^2(\Omega). \end{aligned} \tag{28}$$

**Covariant differential.** Denote the Levi-Civita connection in the Riemannian metric  $g$  by  $D$ . Let

$$H = \sum_{k=1}^n h_k \frac{\partial}{\partial x_k}; \quad X = \sum_{k=1}^n \xi_k \frac{\partial}{\partial x_k}, \tag{29}$$

be vector fields on  $(\mathbf{R}^n, g)$ . The covariant differential  $DH$  of  $H$  determines a bilinear form on  $\mathbf{R}_x^n \times \mathbf{R}_x^n$ , for each  $x \in \mathbf{R}^n$ , defined by

$$DH(Y, X) = \langle D_X H, Y \rangle_g, \quad \forall X, Y \in \mathbf{R}_x^n, \tag{30}$$

where  $D_X H$  is the covariant derivative of  $H$  with respect to  $X$ . This is computed as follows, in the notation of (29), (24), by using the axioms of a connection,

$$\begin{aligned} D_X H &= \sum_{k=1}^n D_X \left( h_k \frac{\partial}{\partial x_k} \right) = \sum_{k=1}^n X(h_k) \frac{\partial}{\partial x_k} + \sum_{k=1}^n h_k D_X \left( \frac{\partial}{\partial x_k} \right) \\ &= \sum_{k=1}^n X(h_k) \frac{\partial}{\partial x_k} + \sum_{k,i=1}^n h_k \xi_i D_{\partial/\partial x_i} \left( \frac{\partial}{\partial x_k} \right), \end{aligned} \tag{31}$$

where by definition, see (24),

$$\begin{aligned} X(h_k) &= \langle \nabla_g h_k, X \rangle_g = X \cdot \nabla_0 h_k = \sum_{i=1}^n \xi_i \frac{\partial h_k}{\partial x_i}; \\ D_{\partial/\partial x_i} \left( \frac{\partial}{\partial x_k} \right) &= \sum_{\ell=1}^n \Gamma_{i\ell}^k \frac{\partial}{\partial x_\ell} \end{aligned} \tag{32}$$

$\Gamma_{ik}^\ell$  being the connection coefficients (Christoffel symbols) of the connection  $D$ ,

$$\Gamma_{ik}^\ell = \frac{1}{2} \sum_{p=1}^n a_{\ell p} \left( \frac{\partial g_{kp}}{\partial x_i} + \frac{\partial g_{ip}}{\partial x_k} - \frac{\partial g_{ik}}{\partial x_p} \right), \quad (g_{ij}) = (a_{ij})^{-1}. \tag{33}$$

Inserting (33) into (32), and then (32) into (31) yields

$$\begin{aligned} D_X H &= \sum_{k=1}^n X(h_k) \frac{\partial}{\partial x_k} + \sum_{\ell=1}^n \left( \sum_{k,i=1}^n h_k \xi_i \Gamma_{ik}^\ell \right) \frac{\partial}{\partial x_\ell} = \\ &= \sum_{\ell=1}^n \left[ X(h_\ell) + \sum_{k,i=1}^n h_k \xi_i \Gamma_{ik}^\ell \right] \frac{\partial}{\partial x_\ell}. \end{aligned} \tag{34}$$

Finally, inserting (34) into (30), we obtain by (15), (29), and (32) for  $X(h_\ell)$ :

$$DH(X, X) = \langle D_X H, X \rangle_g = \sum_{\ell,j=1}^n \left[ X(h_\ell) + \sum_{k,i=1}^n h_k \xi_i \Gamma_{ik}^\ell \right] \xi_j g_{\ell j} \tag{35}$$

$$\text{(by (32))} \quad = \sum_{i,j=1}^n \left[ \sum_{\ell=1}^n \frac{\partial h_\ell}{\partial x_i} g_{\ell j} + \sum_{k,\ell=1}^n h_k g_{\ell j} \Gamma_{ik}^\ell \right] \xi_i \xi_j. \tag{36}$$

Thus, in  $\mathbf{R}_x^n \times \mathbf{R}_x^n$ ,  $DH(\cdot, \cdot)$  is equivalent to the  $n \times n$  matrix

$$\left( m_{ij} = \sum_{\ell=1}^n \frac{\partial h_\ell}{\partial x_i} g_{\ell j} + \sum_{k,\ell=1}^n h_k g_{\ell j} \Gamma_{ik}^\ell \right), \quad i, j = 1, \dots, n. \tag{37}$$

**Hessian in the Riemannian metric  $g$ .** Let  $f \in C^2(\mathbf{R}^n)$ . By definition, the Hessian of  $f$  with respect to the metric  $g$  is

$$D^2 f(X, X) \equiv \langle D_X(\nabla_g f), X \rangle_g \tag{38}$$

$$= \sum_{i,j=1}^n \xi_i \left( \sum_{\ell=1}^n \frac{\partial f_\ell}{\partial x_i} g_{\ell j} + \sum_{k,\ell=1}^n f_k g_{\ell j} \Gamma_{ik}^\ell \right) \xi_j, \tag{39}$$

where, by (23),  $f_\ell = (\nabla_g f)_\ell$  is the  $\ell$ -th coordinate of  $\nabla_g f$ :

$$(\nabla_g f)_\ell = f_\ell = \sum_{p=1}^n a_{\ell p} \frac{\partial f}{\partial x_p}, \quad \ell = 1, 2, \dots, n. \tag{40}$$

To prove (39), we recall (34) with  $H = \nabla_g f$ , hence with coordinates  $h_\ell = (\nabla_g f)_\ell = f_\ell$  as in (40), and obtain by (32):

$$D_X(\nabla_g f) = \sum_{\ell=1}^n \left[ \sum_{i=1}^n \xi_i \frac{\partial f_\ell}{\partial x_i} + \sum_{k,i=1}^n f_k \xi_i \Gamma_{ik}^\ell \right] \frac{\partial}{\partial x_\ell}. \tag{41}$$

Thus, (15), (29) for  $X$  and (41) yield

$$\langle D_X(\nabla_g f), X \rangle_g = \sum_{\ell, q=1}^n g_{\ell q} \left[ \sum_{i=1}^n \xi_i \frac{\partial f_\ell}{\partial x_i} + \sum_{k, i=1}^n f_k \xi_i \Gamma_{ik}^\ell \right] \xi_q \quad (42)$$

$$= \sum_{\ell, q, i=1}^n g_{\ell q} \xi_i \frac{\partial f_\ell}{\partial x_i} \xi_q + \sum_{\ell, q, k, i=1}^n g_{\ell q} f_k \xi_i \Gamma_{ik}^\ell \xi_q \quad (43)$$

$$= \sum_{i, q=1}^n \xi_i \left( \sum_{\ell=1}^n g_{\ell q} \frac{\partial f_\ell}{\partial x_i} \right) \xi_q + \sum_{i, q=1}^n \xi_i \left( \sum_{k=1}^n g_{\ell q} f_k \Gamma_{ik}^\ell \right) \xi_q, \quad (44)$$

and (44) proves (39), as desired with  $q = j$ .

Thus, by (39), we have that

$$\left\{ \begin{array}{l} D^2 f \text{ is positive on } \mathbf{R}_x^n \times \mathbf{R}_x^n \text{ if and only if the} \\ n \times n \text{ matrix } \left( m_{ij} = \sum_{\ell=1}^n \frac{\partial f_\ell}{\partial x_i} g_{\ell j} + \sum_{k, \ell=1}^n f_k g_{\ell j} \Gamma_{ik}^\ell \right), \\ i, j = 1, \dots, n, \text{ is positive, with } f_\ell \text{ given by (40).} \end{array} \right. \quad (45)$$

### 3. Main results. Preliminaries

Let the domain  $\Omega$  and the elliptic operator  $\mathcal{A}$  in (1) be given satisfying the standing assumption (H.1)=(2). The additional hypothesis which we shall need to establish the continuous observability inequalities (6) and (10) is the following:

**Main assumption (H.3).** We assume that there exists a function  $v : \bar{\Omega} \rightarrow \mathbf{R}$  of class  $C^2$  which is strictly convex on  $\bar{\Omega}$ , with respect to the Riemannian metric  $g$  defined in Section 2 modulo a translation, we may assume without loss of generality that  $v(x) \geq 0$ . This assumption means that the Hessian of  $v$  in the Riemannian metric  $g$  is positive on  $\bar{\Omega}$ , as defined by (38), (45):

$$D^2 v(X, X)(x) > 0, \quad \forall x \in \bar{\Omega}, X \in \mathbf{R}_x^n. \quad (46)$$

Since  $\bar{\Omega}$  is compact, it follows from (46) that there exists a positive constant  $\rho > 0$  such that

$$D^2 v(X, X) \geq \rho |X|_g^2, \quad \forall x \in \bar{\Omega}, X \in \mathbf{R}_x^n. \quad (47)$$

Under assumption (H.3), we then take the vector field

$$h(x) \equiv \nabla_g v(x) = \sum_{i=1}^n \left( \sum_{j=1}^n a_{ij}(x) \frac{\partial v}{\partial x_j} \right) \frac{\partial}{\partial x_i}, \quad (48)$$

defined as the gradient of  $v(x)$  with respect to the Riemannian metric  $g$ , see (23).

Section 9 below will provide some non-trivial illustrations where the standing assumption (H.1) as well as the main assumption (H.3) are guaranteed to hold.

**Main results. Continuous observability inequalities.** We are now in the position to state our main results concerning the validity of the continuous observability inequalities (6) and (10) for the Dirichlet and the Neumann case, respectively.

**Remark 3.1.** Both Theorems 3.1 and 3.2, which follow, require a uniqueness continuation result for the Schrödinger  $\psi$ -problem (5), respectively (9), with over-determined B.C.:

$$\frac{\partial \psi}{\partial \nu_{\mathcal{A}}} \Big|_{\Sigma_1} \equiv 0 \text{ for Theorem 3.1; } \psi|_{\Sigma} \equiv 0 \text{ for Theorem 3.2,} \quad (49)$$

which asserts that, then,  $\psi \equiv 0$  in  $Q$ , for  $T$  as given. This uniqueness continuation result is needed to absorb the lower order term from estimates (146), respectively (156), through a (by now standard) compactness/uniqueness argument. The known uniqueness continuation results include the following cases:

(a) Analyticity in time, or analyticity in space, subject to additional conditions Tataru (1995), Section 5.2, Hormander (1997). The sub-case of *time-independent* coefficients in  $\mathcal{A}$  and  $F$  (as in (1)–(4)) can be reduced to uniqueness results for the corresponding static problem, Hormander (1985), Theorem 17.2.6, p. 14.

(b) A result in Isakov (1998), Theorem 3.4.8, which has  $L_{\infty}(Q)$ -first order possibly time dependent terms  $F$ , and, for instance,  $\mathcal{A} = -\Delta$  (see also Lasiecka, Triggiani, Zhang, to appear).

(c) The case of (real) analytic data covered by Homgren-John's theorem, Isakov (1998), p. 52, Hormander (1985).

**Theorem 3.1.** (Dirichlet case) *Let  $\Omega, \mathcal{A}$ , and  $F$  satisfy the standing assumptions (H.1)=(2), (H.2)=(4). Let assumption (H.3)=(47) hold and define  $h(x)$  by (48). Let  $T > 0$  be arbitrary. Assume that  $h(x) \cdot \nu(x) \leq 0$  for  $x \in \Gamma_0$ , where we recall that  $\nu(x) = [\nu_1(x), \dots, \nu_n(x)]$  is the unit outward normal vector to  $\Gamma$ , and where  $h(x) \cdot \nu(x) = \sum_{i=1}^n h_i(x)\nu_i(x)$  is the dot product in  $\mathbf{R}^n$ . Assume the uniqueness continuation property of the over-determined  $\psi$ -problem (5) with*

*$\frac{\partial \psi}{\partial \nu_{\mathcal{A}}} \Big|_{\Sigma_1} \equiv 0$ , as described in Remark 3.1. Then, the observability inequality (6) for the Dirichlet  $\psi$ -problem (5) holds.*

*continuation property of the over-determined  $\psi$ -problem (9) with  $\psi|_{\Sigma} \equiv 0$ , as described in Remark 3.1. Then, the observability inequality (10) for the Neumann  $\psi$ -problem (9) holds.*

**Carleman estimates.** The results of Theorems 3.1 and 3.2 can be shown as a consequence of suitable Carleman estimates for Eqn. (5) with no boundary conditions imposed, which we now describe.

Let  $v : \bar{\Omega} \rightarrow \mathbf{R}^+$  be the strictly convex function, with respect to the Riemannian metric  $g$ , provided by assumption (H.3) = (47). Define the function  $\phi : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$  by

$$\phi(x, t) \equiv v(x) - c \left| t - \frac{T}{2} \right|^2, \quad T > 0. \quad (50)$$

For any  $T > 0$ , the constant  $c > 0$  can be taken sufficiently large so that such function  $\phi(x, t)$  has then the following properties:

(i)

$$\phi(x, 0) < -\delta \text{ and } \phi(x, T) < -\delta \text{ uniformly in } x \in \Omega, \quad (51)$$

for a suitable constant  $\delta > 0$ ;

(ii) there are  $t_0$  and  $t_1$  with  $0 < t_0 < \frac{T}{2} < t_1 < T$  such that

$$\min_{x \in \Omega, t \in [t_0, t_1]} \phi(x, t) \geq -\frac{\delta}{2}, \quad (52)$$

since  $\phi(x, \frac{T}{2}) = v(x) \geq 0$  for all  $x \in \Omega$ ; see statement above (46) (in fact, only the weaker property:  $\min \phi(x, t) \geq \sigma > -\delta$  is actually needed).

(iii) recalling (48),

$$\begin{aligned} \nabla_g \phi &= \nabla_g v = h; \quad \phi_t(x, t) \equiv -2c \left( t - \frac{T}{2} \right), \quad \phi_{tt} \equiv -2c; \quad \phi_t(x, 0) \equiv cT; \\ \phi(x, T) &\equiv -cT. \end{aligned} \quad (53)$$

**Remark 3.2.** (Optimal choice of  $T$ ) We have already noted that by choosing  $c$  large enough, we may obtain any  $T > 0$  small. Henceforth, in all results to follow,  $T > 0$  may be taken arbitrarily small, since the proofs put no further constraint on  $c$ .

The important property (51) will be invoked in the proof of (101) of Theorem 4.2.2 (same as Theorem 3.3). The important property (52) (in fact, only the weaker property:  $\min \phi(x, t) \geq \sigma > -\delta$  is actually needed) will be invoked in going from Eqn. (56) to Eqn. (57) in the statement of Theorem 3.3 (Carleman estimates, first version), but not before (56).

**Theorem 3.3.** (Carleman estimates, first version) *Assume (H.1) = (2), (H.2) = (4), and (H.3) = (47). Let  $f \in L_2(Q)$ . Let  $w$  be a solution of the Schrödinger equation*

$$iw_t + Aw = F(w) + f \quad \text{in } Q \tag{54}$$

[with no boundary conditions imposed], within the following class:

$$\begin{cases} w \in C([0, T]; H^1(\Omega)) \\ w_t, \frac{\partial w}{\partial \nu_A} \in L_2(0, T; L_2(\Gamma)). \end{cases} \tag{55}$$

Let  $\phi(x, t)$  be the function defined by (50),  $\rho > 0$  being the constant in (47),  $\delta > 0$  the constant in (51), and  $C_T$  a generic constant.

Then, for  $\tau > 0$ , the following one-parameter family of estimates holds true:

$$\begin{aligned} (BT_w)|_\Sigma + \frac{2}{\tau} \int_Q e^{\tau\phi} |f|^2 dQ + C_{T,\phi,\tau} \|w\|_{C([0,T];L_2(\Omega))}^2 \\ \geq \left(\rho - \frac{C_T}{\tau}\right) \int_Q e^{\tau\phi} |\nabla_g w|_g^2 dQ - \frac{e^{-\delta\tau}}{\tau} [E(T) + E(0)] \end{aligned} \tag{56}$$

$$\geq \left(\rho - \frac{C_T}{\tau}\right) e^{-\frac{\tau\delta}{2}} \int_{t_0}^{t_1} E(t) dt - \frac{e^{-\delta\tau}}{\tau} [E(T) + E(0)], \tag{57}$$

where the boundary terms  $(BT_w)|_\Sigma$  over  $\Sigma = [0, T] \times \Gamma$  are given by

$$\begin{aligned} (BT_w)|_\Sigma = \operatorname{Re} \left( \int_\Sigma e^{\tau\phi} \frac{\partial w}{\partial \nu_A} h(\bar{w}) d\Sigma \right) - \frac{1}{2} \int_\Sigma e^{\tau\phi} |\nabla_g w|_g^2 h \cdot \nu d\Sigma \\ + \frac{1}{2} \left| \int_\Sigma \frac{\partial w}{\partial \nu_A} \bar{w} \operatorname{div}_0(e^{\tau\phi} h) d\Sigma - i \int_\Sigma \bar{w} w_t e^{\tau\phi} h \cdot \nu d\Sigma \right|. \end{aligned} \tag{58}$$

Moreover, we have set for convenience

$$E(t) = E_w(t) = \int_\Omega |\nabla_g w(t, x)|_g^2 d\Omega, \tag{59}$$

and we recall that  $h(w) = \langle h, \nabla_g w \rangle_g = \langle \nabla_g v, \nabla_g w \rangle_g = \nabla_0 w \cdot h$  by (21), and (24), with  $h$  the vector field defined by (48).

**Remark 3.3.** By (26), (2), we have

$$\begin{aligned} a|\nabla_0 w(t, x)|^2 \leq |\nabla_g w(t, x)|_g^2 = \nabla_0 w(t, x) \cdot A(x) \nabla_0 w(t, x) \leq a_1 |\nabla_0 w(t, x)|^2, \\ x \in \Omega, \end{aligned} \tag{60}$$

where  $a > 0$  is the constant in (2). Thus, by (59) and (60) we have that

$$\dots \tag{61}$$

We shall henceforth use (61) freely, particularly for  $t = 0$  and  $t = T$ .

**Remark 3.4.** Property (51) is used to obtain (56). Property (52) is used to obtain (57).

**Remark 3.5.** The presence of the factor  $\frac{1}{\tau}$  in front of the integral term containing  $f$  in (56) is critical to extend Theorem 3.3 to a system of coupled Schrödinger equations as in Triggiani (1996), Theorem 1.1.

The proof of Theorem 3.3 is given in Section 4. The counterpart of Triggiani (1996), Theorem 2.1.2 is given next. To this end, we specialize the first-order operator  $F(w)$  to have *real* first-order coefficients; i.e., we assume that:

(H.4) the first-order term  $F(w)$  is of the form

$$\begin{cases} F(w) = R(w) + rw, & R(w) = R \cdot \nabla_0 w \text{ by (24) where } R \text{ is a real vector} \\ \text{field on } \mathbf{R}^n\text{-fields; and } r : \mathbf{R}^n \rightarrow \mathbf{C} \text{ is a function, which is } L_\infty \text{ on } \Omega. \end{cases} \quad (62)$$

**Theorem 3.4.** (Carleman estimates, second version) *Assume the hypotheses (H.1) = (2), (H.2) = (4), (H.3) = (47), and (H.4) = (62). Let  $f = 0$ . Then, for all  $\tau > 0$  sufficiently large, there exists a constant  $k_{\phi,\tau} > 0$  such that the following one-parameter family of estimates holds true:*

$$\begin{aligned} (BT_{1,w})|_\Sigma + C_{T,\phi,\tau} \|w\|_{C([0,T];L_2(\Omega))}^2 \\ \geq e^{-\frac{\tau t}{2}} \left\{ \left( \rho - \frac{C_T}{\tau} \right) \frac{e^{-kT}}{2} (t_1 - t_0) - \frac{e^{-\frac{\delta \tau}{2}}}{\tau} \right\} [E(T) + E(0)] \end{aligned} \quad (63)$$

$$\geq k_{\phi,\tau} [E(T) + E(0)], \quad (64)$$

$C_T$  a generic constant, where the boundary terms  $(BT_{1,w})|_\Sigma$  over  $\Sigma = (0, T] \times \Gamma$  are given by

$$(BT_{1,w})|_\Sigma = (BT_w)|_\Sigma + \text{const}_{\phi,\tau,\rho} \int_\Sigma \left| \frac{\partial w}{\partial \nu_{\mathcal{A}}} \right| [|w_t| + |W(w)| + |rw|] d\Sigma, \quad (65)$$

with  $(BT_w)|_\Sigma$  defined by (58), where  $W(w) = W \cdot \nabla_0 w$ , and  $W(x)$  is a vector field on the submanifold  $\Gamma$  such that  $W(x) \in \Gamma_x$  for  $x \in \Gamma$  (the tangent space to  $\Gamma$  at  $x$ ); see (108) below.

(b) Assume, further, that the solution  $w$  of (54) satisfies

$$w|_{\Sigma_0} \equiv 0, \quad \Sigma_0 = (0, T] \times \Gamma_0, \quad \text{and that } h(x) \cdot \nu(x) \leq 0, \quad x \in \Gamma_0. \quad (66)$$

with  $h = \nabla_g \phi = \nabla_g v$  by (53), and  $\nu(x)$  the unit outward normal vector at  $x \in \Gamma$ .

Then, estimate (64) holds true for  $\tau > 0$  sufficiently large, with the boundary terms  $(BT_{1,w})|_\Sigma$  replaced by  $(BT_{1,w})|_{\Sigma_1}$ , i.e., evaluated only on  $\Sigma_1 = (0, T] \times \Gamma_1$ , while the boundary terms  $(BT_{1,w})|_{\Sigma_0}$  evaluated on  $\Sigma_0 = (0, T] \times \Gamma_0$  are negative:

The proof of Theorem 3.4 is given in Section 5. Estimate (64) of Theorem 3.4 then readily yields Theorem 3.1 on the continuous observability inequality (6) in the Dirichlet case for  $\psi = w$  with  $f \equiv 0$ ,  $\psi|_{\Sigma} \equiv 0$  and  $h \cdot \nu \leq 0$  on  $\Gamma_0$ . This is done in Section 6. However, to prove Theorem 3.2 on the continuous observability inequality in the Neumann case for  $\psi = w$  with  $f \equiv 0$ ,  $\psi|_{\Sigma_0} \equiv 0$ ,  $\Gamma_0 \neq \emptyset$ , and  $h \cdot \nu \leq 0$  on  $\Gamma_0$ , an additional non-trivial step is needed. This is provided by a result of Triggiani (1996) which will be quoted in Lemma 7.2 below. Combined with Theorem 3.4, this result will permit us to obtain the following theorem, which may be viewed as the *main estimate (at the energy level)* of the present paper, the counterpart of Triggiani (1996), Theorem 2.1.5.

**Theorem 3.5.** *Assume (H.1), (H.2), (H.3), (H.4) and that  $f = 0$ . Let  $w$  be a solution of Eqn. (54) in the class (55).*

(a) *Then, the following estimate holds true. There exists a constant  $k_{\phi, \tau} > 0$  for  $\tau$  sufficiently large such that, for any  $\epsilon_0 > 0$ :*

$$\int_0^T \int_{\Gamma} \left[ \left| \frac{\partial w}{\partial \nu_{\mathcal{A}}} \right|^2 + |w_t|^2 \right] d\Sigma + C_{\phi, \epsilon_0} \|w\|_{L_2(0, T; H^{\frac{1}{2} + \epsilon_0}(\Omega))}^2 \geq k_{\phi, \tau} [E(T) + E(0)]. \tag{67}$$

(b) *Assume, further, that the solution  $w$  of (54) satisfies hypothesis (66).*

*Then, estimate (66) holds true with  $\int_{\Gamma}$  replaced by  $\int_{\Gamma_1}$ .*

Estimate (67) implies the continuous observability inequality (10) for  $\psi = w$ ,  $f = 0$ , under the required assumption (66):

$$\psi|_{\Sigma_0} \equiv 0, \quad \Gamma_0 \neq \emptyset, \quad h \cdot \nu \leq 0 \text{ on } \Gamma_0; \text{ and } \left. \frac{\partial \psi}{\partial \nu_{\mathcal{A}}} \right|_{\Sigma_1} = 0, \tag{68}$$

by dropping  $E(T)$  in (67) and by absorbing the lower-order interior term by compactness/uniqueness, see Section 8.

**Remark 3.6.** (Uniform Stabilization) Consider the well-posed (in the semi-group sense, Lasiecka and Triggiani, 1992) Neumann feedback problem with  $\Gamma_0 \neq \emptyset$ :

$$\begin{cases} iw_t + \mathcal{A}w = 0 & \text{in } Q; \\ w(0, \cdot) = w_0 & \text{in } \Omega; \\ w|_{\Sigma_0} \equiv 0; \left. \frac{\partial w}{\partial \nu_{\mathcal{A}}} \right|_{\Sigma_1} = -w_t & \text{in } \Sigma_i. \end{cases} \tag{69}$$

Then, inequality (67) permits to obtain a *uniform stabilization* (on  $H^1(\Omega)$ ) result for the Neumann feedback problem (69) in the case of *variable coefficients* in



there exist constants  $M \geq 1$ ,  $\mu > 0$ , such that the energy (see (59)) of problem (69) satisfies

$$E(t) \leq M e^{-\mu t} E(0), \quad t \geq 0. \quad (70)$$

The case of  $\mathcal{A} = -\Delta$  with geometrical conditions on  $\Gamma_1$  is given in Machtyngier (1990).

By contrast, the *uniform stabilization* (on  $H^{-1}(\Omega)$ ) of the Schrödinger equation under Dirichlet feedback is much more demanding. The case of  $\mathcal{A} = -\Delta$  is given in Lasiecka and Triggiani (1991). The general case  $\mathcal{A}$  will require the counterpart of the energy estimate for second-order hyperbolic equations obtained in Lasiecka, Triggiani and Yao (1998).

## 4. Proof of Theorem 3.3: Carleman estimate (first version)

### 4.1. Preliminaries

We collect here below a few formulas to be invoked in the sequel.

**Green's formula.** In the proof of Theorem 4.2.1, Eqn. (85), as well as Eqn. (91) and (103), we shall make use of the following Green's formula. Let  $z(x) \in C^1(\bar{\Omega})$ . Then, the following identity holds true:

$$\int_{\Omega} (\mathcal{A}w)z \, d\Omega = \int_{\Omega} \langle \nabla_g w, \nabla_g z \rangle_g \, d\Omega - \int_{\Gamma} z \frac{\partial w}{\partial \nu_{\mathcal{A}}} \, d\Gamma, \quad (71)$$

see also (26). In fact, to prove (71), we write by recalling (28) for  $\mathcal{A}w$ , and the usual divergence formula (Lasiecka and Triggiani, 1992, (A.1), or (88) below):

$$\int_{\Omega} (\mathcal{A}w)z \, d\Omega = - \int_{\Omega} z \operatorname{div}_0(\nabla_g w) \, d\Omega \quad (72)$$

$$= \int_{\Omega} \nabla_g w \cdot \nabla_0 z \, d\Omega - \int_{\Gamma} z \nabla_g w \cdot \nu \, d\Gamma. \quad (73)$$

Then, recalling identity (24), and (25) for  $\frac{\partial w}{\partial \nu_{\mathcal{A}}}$ , we see that (73) leads to (71), as desired.

**An identity.** Let  $\phi$  be the function in (50). Let  $H = e^{\tau\phi}h$ , with  $h = \nabla_g \phi$  by (53). Finally, let  $X = \nabla_g w$ . Then, with reference to (30), the following identity to be invoked in the proof of Theorem 4.2.1, (87), holds true:

$$DH(X, X) = \langle D_X H, X \rangle_g = \langle D_{\nabla_g w} (e^{\tau\phi}h), \nabla_g w \rangle_g \quad (74)$$

$$= \tau e^{\tau\phi} [h(w)]^2 + e^{\tau\phi} D^2\phi(\nabla_g w, \nabla_g w), \quad (75)$$

*Proof of (75).* We preliminarily compute, by using the axioms of the connection  $D$ ,

$$\begin{aligned} D_X H &= D_X(e^{\tau\phi}h) = X \cdot \nabla_0(e^{\tau\phi})h + e^{\tau\phi}D_X h \\ &= \tau e^{\tau\phi}X \cdot \nabla_0\phi h + e^{\tau\phi}D_X h. \end{aligned} \tag{76}$$

Thus, (76) yields by (24),

$$\langle D_X H, X \rangle_g = \tau e^{\tau\phi}X(\phi)\langle h, X \rangle_g + e^{\tau\phi}\langle D_X h, X \rangle. \tag{77}$$

As to the second term in (77), with  $h = \nabla_g\phi$  by (53), we have, recalling definition (38) of Hessian of  $\phi$ :

$$\langle D_X h, X \rangle_g = \langle D_X(\nabla_g\phi), X \rangle_g \equiv D^2\phi(X, X). \tag{78}$$

As to the first term in (77), we have with  $X = \nabla_g w$ ,  $h = \nabla_g\phi$ , recalling (21) or (24):

$$X(\phi) = \langle \nabla_g\phi, X \rangle_g = \langle h, X \rangle_g = \langle h, \nabla_g w \rangle_g = h(w). \tag{79}$$

Thus, (78) and (79), used on the R.H.S. of (77) yield for  $X = \nabla_g w$ ,  $h = \nabla_g\phi$ :

$$\langle D_X H, X \rangle_g = \tau e^{\tau\phi}[h(w)]^2 + e^{\tau\phi}D^2\phi(X, X), \tag{80}$$

which, in turn, proves (75).

#### 4.2. Energy methods in the Riemann metric: First Carleman estimate

We will complete the proof of Theorem 3.3 through several propositions. The strategy follows closely the proof of Triggiani (1996), Section 2, for constant coefficient principal part ( $\mathcal{A} = -\Delta$ ), except that it is carried out in the Riemann metric  $g$  defined by (15), rather than in the Euclidean metric as in Triggiani (1996). The close parallelism between the present treatment and that of Triggiani (1996) will be emphasized in the intermediate results as well. The counterpart of Triggiani (1996), Theorem 2.2.1, is

**Step 1. Theorem 4.2.1.** *Let  $w$  be a solution of Eqn. (54) within the class (55). Then the following one-parameter family of identities holds true for  $\tau > 0$ , where  $\Sigma = [0, T] \times \Gamma$ ;  $Q = [0, T] \times \Omega$ :*

$$\operatorname{Re} \left( \int_{\Sigma} e^{\tau\phi} \frac{\partial w}{\partial \nu_{\mathcal{A}}} h(\bar{w}) d\Sigma \right) - \frac{1}{2} \int_{\Sigma} e^{\tau\phi} |\nabla_g w|_g^2 h \cdot \nu d\Sigma$$

$$1 \int \frac{\partial w}{\partial \nu_{\mathcal{A}}} h(\bar{w}) d\Sigma - \frac{1}{2} \int e^{\tau\phi} |\nabla_g w|_g^2 h \cdot \nu d\Sigma$$

$$\begin{aligned}
 &= \int_Q e^{\tau\phi} D^2\phi(\nabla_g w, \nabla_g \bar{w})dQ + \tau \int_Q e^{\tau\phi} |h(w)|^2 dQ \\
 &- \operatorname{Re} \left( \int_Q [F(w) + f](e^{\tau\phi} h)(\bar{w})dQ \right) + \frac{1}{2} \int_Q \bar{w} \langle \nabla_g w, \nabla_g (\operatorname{div}_0(e^{\tau\phi} h)) \rangle_g dQ \\
 &- \frac{1}{2} \int_Q [F(w) + f] \bar{w} \operatorname{div}_0(e^{\tau\phi} h) dQ \\
 &+ \frac{i}{2} \int_Q \bar{w} \frac{d(e^{\tau\phi})}{dt} h(w) dQ - \frac{i}{2} \left[ \int_\Omega \bar{w} e^{\tau\phi} h(w) d\Omega \right]_0^T. \tag{81}
 \end{aligned}$$

In (81), we have  $h(x) = \nabla_g \phi = \nabla_g v(x)$ , see (48), (53), while  $D^2\phi(\cdot, \cdot)$  is the Hessian (as defined in (38)) of the function  $\phi$  in (50); finally,  $h(w) = \langle h, \nabla_g w \rangle_g = \langle \nabla_g v, \nabla_g w \rangle_g = \nabla_0 w \cdot h$  by (21), and (24), with the vector field  $h$  defined by (48).

*Proof.* We first set

$$a = \int_Q w_t e^{\tau\phi} h(\bar{w}) dQ. \tag{82}$$

(a) We multiply both sides of Eqn. (1) by the multiplier  $e^{\tau\phi} h(\bar{w})$ , see (13). We shall show that

(i)

$$\begin{aligned}
 ia &= \int_\Sigma e^{\tau\phi} h(\bar{w}) \frac{\partial w}{\partial \nu_A} d\Sigma - \frac{1}{2} \int_\Sigma e^{\tau\phi} |\nabla_g w|_g^2 h \cdot \nu d\Sigma \\
 &- \int_Q e^{\tau\phi} D^2\phi(\nabla_g w, \nabla_g \bar{w}) dQ \\
 &+ \frac{1}{2} \int_Q |\nabla_g w|_g^2 \operatorname{div}_0(e^{\tau\phi} h) dQ - \tau \int_Q e^{\tau\phi} |h(w)|^2 dQ \\
 &+ \int_Q [F(w) + f] e^{\tau\phi} h(\bar{w}) dQ, \tag{83}
 \end{aligned}$$

(ii)

$$a - \bar{a} = 2i(\operatorname{Im} a) = \int_\Sigma w_t \bar{w} e^{\tau\phi} h \cdot \nu d\Sigma$$

$$\int_\Sigma e^{\tau\phi} h(\bar{w}) \frac{\partial w}{\partial \nu_A} d\Sigma$$

$$\begin{aligned}
 & -i \int_Q |\nabla_g w|_g^2 \operatorname{div}_0(e^{\tau\phi} h) dQ \\
 & -i \int_Q \bar{w} \langle \nabla_g w, \nabla_g(\operatorname{div}_0(e^{\tau\phi} h)) \rangle_g dQ + \int_Q \bar{w} h(w) \frac{d(e^{\tau\phi})}{dt} dQ \\
 & + i \int_Q [F(w) + f] \bar{w} \operatorname{div}_0(e^{\tau\phi} h) dQ - \left[ \int_\Omega e^{\tau\phi} \bar{w} h(w) d\Omega \right]_0^T. \tag{84}
 \end{aligned}$$

*Proof of (i).* Once the Schrödinger equation (54) is multiplied by  $e^{\tau\phi} h(\bar{w})$ , we obtain by invoking (82) and Green’s identity (71),

$$\begin{aligned}
 ia &= - \int_Q A w e^{\tau\phi} h(\bar{w}) dQ + \int_Q [F(w) + f] e^{\tau\phi} h(\bar{w}) dQ \\
 \text{(by (71))} &= \int_\Sigma e^{\tau\phi} h(\bar{w}) \frac{\partial w}{\partial \nu_A} d\Sigma - \int_Q \langle \nabla_g w, \nabla_g(e^{\tau\phi} h(\bar{w})) \rangle_g dQ \\
 &+ \int_Q [F(w) + f] e^{\tau\phi} h(\bar{w}) dQ. \tag{85}
 \end{aligned}$$

By the identity (27) with  $H = e^{\tau\phi} h$ , we have

$$\begin{aligned}
 \langle \nabla_g w, \nabla_g(e^{\tau\phi} h(\bar{w})) \rangle_g &= D(e^{\tau\phi} h)(\nabla_g w, \nabla_g \bar{w}) + \frac{1}{2} \operatorname{div}_0(|\nabla_g w|_g^2 e^{\tau\phi} h) \\
 &- \frac{1}{2} |\nabla_g w|_g^2 \operatorname{div}_0(e^{\tau\phi} h), \tag{86}
 \end{aligned}$$

where, by identity (75), the first term on the RHS of (86) is given by

$$D(e^{\tau\phi} h)(\nabla_g w, \nabla_g \bar{w}) = \tau e^{\tau\phi} |h(w)|^2 + e^{\tau\phi} D^2 \phi(\nabla_g w, \nabla_g \bar{w}). \tag{87}$$

Inserting first (87) into (86) and then the resulting (86) into the second integral term on the right of (85), we thus obtain (83), as desired, using also the divergence theorem.

*Proof of (ii).* Using the standard divergence identity

$$\int_\Gamma \psi k \cdot \nu d\Gamma = \int_\Omega \psi \operatorname{div}_0 k d\Omega + \int_\Omega k \cdot \nabla_0 \psi d\Omega \tag{88}$$

with  $k = [e^{\tau\phi} h]$ ,  $h = \nabla_g \phi$ , see (53), and  $\psi = w_t \bar{w}$ , we compute, since  $h(\bar{w}) = \nabla_0 \bar{w} \cdot h$  by (24):

$$\int_\Sigma w_t \bar{w} e^{\tau\phi} h \cdot \nu d\Sigma = \int_Q w_t \bar{w} \operatorname{div}_0(e^{\tau\phi} h) dQ$$

$$\text{(by (24))} = \int_Q w_t \bar{w} \operatorname{div}_0(e^{\tau\phi} h) dQ + a + \int_Q e^{\tau\phi} \bar{w} h(w_t) dQ, \quad (89)$$

recalling again (82) for  $a$  in the last step. As to the last term in (89), we integrate by parts in time and obtain via (82)

$$\begin{aligned} \int_{\Sigma} w_t \bar{w} e^{\tau\phi} h \cdot \nu d\Sigma &= a - \bar{a} + \int_Q w_t \bar{w} \operatorname{div}_0(e^{\tau\phi} h) dQ \\ &- \int_Q \bar{w} h(w) \frac{d(e^{\tau\phi})}{dt} dQ + \left[ \int_{\Omega} e^{\tau\phi} \bar{w} h(w) d\Omega \right]_0^T. \end{aligned} \quad (90)$$

We next rewrite the first integral term over  $Q$  in (90). To this end, if  $m = m(x, t)$  is a real function in  $C^1(Q)$ , we may verify the identity

$$\begin{aligned} i \int_Q w_t \bar{w} m dQ &= \int_{\Sigma} \bar{w} m \frac{\partial w}{\partial \nu_{\mathcal{A}}} d\Sigma + \int_Q [F(w) + f] \bar{w} m dQ \\ &- \int_Q m |\nabla_g w|_g^2 dQ - \int_Q \bar{w} \langle \nabla_g w, \nabla_g m \rangle_g dQ. \end{aligned} \quad (91)$$

This is done by multiplying the Schrödinger equation (54) by  $\bar{w} m$  and integrating by parts using the Green's identity (71). Specializing (91) with  $m = \operatorname{div}_0(e^{\tau\phi} h)$ , we obtain

$$\begin{aligned} \int_Q w_t \bar{w} \operatorname{div}_0(e^{\tau\phi} h) dQ &= -i \int_{\Sigma} \bar{w} \operatorname{div}_0(e^{\tau\phi} h) \frac{\partial w}{\partial \nu_{\mathcal{A}}} d\Sigma \\ &+ i \int_Q |\nabla_g w|_g^2 \operatorname{div}_0(e^{\tau\phi} h) dQ + i \int_Q \bar{w} \langle \nabla_g w, \nabla_g (\operatorname{div}_0(e^{\tau\phi} h)) \rangle_g dQ \\ &- i \int_Q [F(w) + f] \bar{w} \operatorname{div}_0(e^{\tau\phi} h) dQ, \end{aligned} \quad (92)$$

which is the desired identity, to be substituted into the RHS of (90). Upon doing this, one obtains identity (84), as desired. So (ii) is proved as well.

Finally, we use the identity  $\operatorname{Re}(ia) = \frac{i}{2}(a - \bar{a})$ , with  $(ia)$  given by the expression in (83) and  $(a - \bar{a})$  given by the expression in (84): after cancellation of the term  $\frac{1}{2} \int_Q |\nabla_g w|_g^2 \operatorname{div}_0(e^{\tau\phi} h) dQ$ , we finally obtain identity (81). Theorem 4.2.1 is proved.  $\blacksquare$

**Step 2.** (Carleman estimates, first version) This is Theorem 3.3, restated.

**Theorem 4.2.2.** Assume  $(H.1) = (2)$ ,  $(H.2) = (4)$ ,  $(H.3) = (47)$ . Let  $w$  be

the following estimates hold true:

$$\begin{aligned} (BT_w)|_\Sigma + \frac{2}{\tau} \int_Q |f|^2 e^{\tau\phi} dQ + C_{\phi, \tau, T} \|w\|_{C^1([0, T]; L_2(\Omega))}^2 \\ \geq \left(\rho - \frac{C_T}{\tau}\right) \int_Q e^{\tau\phi} |\nabla_g w|_g^2 dQ - \frac{e^{-\tau\delta}}{\tau} [E(0) + E(T)] \end{aligned} \quad (93)$$

$$\geq \left(\rho - \frac{C_T}{\tau}\right) e^{-\frac{\delta\tau}{2}} \int_{t_0}^{t_1} E(t) dt - \frac{e^{-\tau\delta}}{\tau} [E(0) + E(T)], \quad (94)$$

where  $\rho > 0$  and  $\delta > 0$  are the constants in (47) and (50), respectively.  $E(t)$  is defined in (59), and finally, the boundary terms  $(BT_w)|_\Sigma$  are defined (in agreement with (58)) by

$$\begin{aligned} (BT_w)|_\Sigma = \operatorname{Re} \left( \int_\Sigma e^{\tau\phi} \frac{\partial w}{\partial \nu_{\mathcal{A}}} h(\bar{w}) d\Sigma \right) - \frac{1}{2} \int_\Sigma e^{\tau\phi} |\nabla_g w|_g^2 h \cdot \nu d\Sigma \\ + \frac{1}{2} \left| \int_\Sigma \frac{\partial w}{\partial \nu_{\mathcal{A}}} \bar{w} \operatorname{div}_0(e^{\tau\phi} h) d\Sigma - i \int_\Sigma \bar{w} w_t e^{\tau\phi} h \cdot \nu d\Sigma \right|. \end{aligned} \quad (95)$$

*Proof.* The passage from (93) to (94) simply invokes property (52) for the pseudo-convex function  $\phi$ . Thus, we prove (93). First, by (4) on  $F$ , given  $\epsilon > 0$ , we have the following estimate

$$\begin{aligned} |[F(w) + f]e^{\tau\phi} h(\bar{w})| &\geq -\frac{\epsilon}{2} |F(w) + f|^2 e^{\tau\phi} - \frac{1}{2\epsilon} e^{\tau\phi} |h(\bar{w})|^2 \\ (\text{by (4), (60)}) &\geq -\epsilon C_T |\nabla_g w|_g^2 e^{\tau\phi} - \epsilon C_T |w|^2 e^{\tau\phi} \\ &\quad - \epsilon |f|^2 e^{\tau\phi} - \frac{1}{2\epsilon} e^{\tau\phi} |h(\bar{w})|^2. \end{aligned} \quad (96)$$

Thus, invoking (96) and the key assumption of coercivity in (47), we estimate the first three terms on the right-hand side of identity (81).

$$\left| \int_Q e^{\tau\phi} D^2\phi(\nabla_g w, \nabla_g \bar{w}) dQ + \tau \int_Q e^{\tau\phi} |h(w)|^2 dQ - \operatorname{Re} \left( \int_Q [F(w) + f] e^{\tau\phi} h(\bar{w}) dQ \right) \right|$$

$$(\text{by (47), (96)}) \geq (\rho - \epsilon C_T) \int_Q e^{\tau\phi} |\nabla_g w|_g^2 dQ + \left(\tau - \frac{1}{2\epsilon}\right) \int_Q e^{\tau\phi} |h(w)|^2 dQ$$

Again, using (4) on  $F$  and (60), and the inequality  $2ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2$ , where  $a$  denotes “energy level” terms:  $\nabla_g w$ ,  $F(w)$ , as well as  $f$ , while  $b$  denotes lower-order terms, i.e.,  $w$ , we obtain the following estimate for the last three  $\int_Q$ -terms in identity (81): for any  $\epsilon > 0$ ,

$$\begin{aligned} & \left| \frac{1}{2} \int_Q \bar{w} \langle \nabla_g w, \nabla_g (\operatorname{div}_0(e^{\tau\phi} h)) \rangle_g dQ - \frac{1}{2} \int_Q [F(w) + f] \bar{w} \operatorname{div}_0(e^{\tau\phi} h) dQ \right. \\ & \left. + \frac{i}{2} \int_Q \bar{w} \frac{d(e^{\tau\phi})}{dt} h(w) dQ \right| \\ & \geq -\frac{\epsilon}{2} \int_Q e^{\tau\phi} |\nabla_g w|_g^2 dQ - C_{\tau,T,\epsilon} \int_Q e^{\tau\phi} |w|^2 dQ - \epsilon \int_Q |f|^2 e^{\tau\phi} dQ. \end{aligned} \tag{98}$$

Combining (97) with (98) we obtain the following estimate for the RHS of identity (81)

$$\begin{aligned} & \text{RHS of (81)} \geq \\ & \left( \rho - \epsilon C_T - \frac{\epsilon}{2} \right) \int_Q e^{\tau\phi} |\nabla_g w|_g^2 dQ + \left( \tau - \frac{1}{2\epsilon} \right) \int_Q e^{\tau\phi} |h(w)|^2 dQ \\ & - \epsilon C_{\phi,T} \|w\|_{C^1([0,T];L_2(\Omega))}^2 - \epsilon \int_Q e^{\tau\phi} |f|^2 dQ + \beta_{0,T}, \end{aligned} \tag{99}$$

where, using property (51) for  $\phi(x, 0)$  and  $\phi(x, T)$  and  $h(w) = \langle \nabla_g w, h \rangle_g$  via (24):

$$\begin{aligned} |\beta_{0,T}| &= \left| -\frac{i}{2} \left[ \int_{\Omega} \bar{w} e^{\tau\phi} h(w) \right]_0^T \right| \geq -\epsilon e^{-\delta\tau} \int_{\Omega} [|\nabla_g w(T)|_g^2 + |\nabla_g w(0)|_g^2] d\Omega \\ &\quad - \frac{C_{\phi}}{\epsilon} e^{-\delta\tau} \int_{\Omega} [|w(T)|^2 + |w(0)|^2] d\Omega \end{aligned} \tag{100}$$

$$= -\epsilon e^{-\delta\tau} [E(0) + E(T)] - C_{\phi,\tau,\epsilon} \|w\|_{C^1([0,T];L_2(\Omega))}^2. \tag{101}$$

In the last step we have recalled (59) for  $E(\cdot)$ .

We now select  $\tau = \frac{1}{\epsilon}$  so that  $(\tau - \frac{1}{2\epsilon}) = \frac{1}{\epsilon} > 0$ , drop the second integral term on the right-hand side of (99), combine the resulting estimate with (101), and finally obtain the desired estimate (93). ■

**Remark 4.2.1.** In the third integral over  $Q$  on the left-hand side of (97), both factors  $F(w)$  and  $h(\bar{w})$  are energy level, with  $F$  a general first-order operator. The virtue of the free parameter  $\tau$  is seen in the second term on the right-hand side of (99), in making the coefficient  $\tau - \frac{1}{2\epsilon} > 0$ , after  $\epsilon$  has been

### 5. Proof of Theorem 3.4: Carleman estimates, second version; $f = 0$

Assumptions (H.1)–(H.3) are in force throughout.

**Step 1. Lemma 5.1.** *Let  $w$  be a solution of the Schrödinger equation (54) in the class (55). Then, with reference to (59) for  $E(t)$ , we have for all  $t, s$ :*

$$E(t) = E(s) + 2\operatorname{Re} \left( \int_s^t \int_{\Gamma} \bar{w}_t \frac{\partial w}{\partial \nu_A} d\Gamma d\sigma \right) + 2\operatorname{Re} \left( \int_s^t \int_{\Omega} [F(w) + f] \bar{w}_t d\Omega d\sigma \right). \tag{102}$$

*Proof.* We multiply (54) by  $\bar{w}_t$  and integrate over  $(s, t) \times \Omega$ , obtaining by virtue of Green’s first identity, (71):

$$i \int_s^t \int_{\Omega} |w_t|^2 d\Omega d\sigma = \int_s^t \int_{\Gamma} \bar{w}_t \frac{\partial w}{\partial \nu_A} d\Gamma d\sigma - \int_s^t \int_{\Omega} \langle \nabla_g w, \nabla_g \bar{w}_t \rangle_g d\Omega d\sigma + \int_s^t \int_{\Omega} [F(w) + f] \bar{w}_t d\Omega d\sigma \tag{103}$$

$$= -\frac{1}{2}[E(t) - E(s)] + \int_s^t \int_{\Gamma} \bar{w}_t \frac{\partial w}{\partial \nu_A} d\Gamma d\sigma + \int_s^t \int_{\Omega} [F(w) + f] \bar{w}_t d\Omega d\sigma, \tag{104}$$

since

$$E(t) - E(s) = \int_{\Omega} \int_s^t \frac{\partial}{\partial \sigma} (|\nabla_g w|_g^2) d\sigma d\Omega = 2 \left( \int_s^t \int_{\Omega} \langle \nabla_g w, \nabla_g \bar{w}_t \rangle_g d\Omega d\sigma \right). \tag{105}$$

Thus, (103) yields (102), as desired. ■

**Step 2.** We next estimate the second integral term on the right-hand side of (102) by using

$$\bar{w}_t = -i\mathcal{A}\bar{w} + i[\overline{F(w)} + \bar{f}]. \tag{106}$$

**Proposition 5.2.** *Let  $f \equiv 0$ . Let  $w$  be a solution of Eqn. (54) within the class (55). Assume further (H.4) = (62); i.e., that the first-order term  $F(w)$  is given by*

$$F(w) = R(w) + rw, \quad R(w) = R \cdot \nabla_0 w \text{ by (24)}, \tag{107}$$

where  $R$  is a real vector field on  $\mathbf{R}^n$  fields and  $r : \mathbf{R}^n \rightarrow \mathbf{C}$  a function. Decompose  $R$  as

$$R \cdot \nu(x) \tag{108}$$



where, by (18),  $\nu_{\mathcal{A}}(x)$  is defined by

$$\nu_{\mathcal{A}}(x) = \sum_{i=1}^n \left( \sum_{j=1}^n a_{ij}(x) \nu_j(x) \right) \frac{\partial}{\partial x_i} = A(x) \nu(x) \tag{109}$$

to be normal of the submanifold  $\Gamma$  in the Riemann metric  $g$ . Moreover,  $W$  is a vector field on the submanifold  $\Gamma$  such that  $W(x) \in \Gamma_x$  for  $x \in \Gamma$ .

Then the following inequalities hold true: for  $T \geq t \geq s \geq 0$ ,

$$E(t) \leq E(s) + \Lambda(T) + k \int_s^t E(\sigma) d\sigma; \tag{110}$$

$$E(s) \leq E(t) + \Lambda(T) + k \int_s^t E(\sigma) d\sigma, \tag{111}$$

where

$$\Lambda(T) = 2 \int_{\Sigma} [ |w_t| + |W(w)| + |r| |w| ] \left| \frac{\partial w}{\partial \nu_{\mathcal{A}}} \right| d\Sigma + C \int_Q |w|^2 dQ, \tag{112}$$

$C, k$  are constants and  $W(w) = W \cdot \nabla_0 w$ , see (24).

*Proof.* We initially take  $f \neq 0$ , and refer to Remark 5.1 below. According to (102) we seek a bound for  $\text{Re} \left( \int_s^t \int_{\Omega} F(w) \bar{w}_t d\Omega d\sigma \right)$ . Using (106) for  $\bar{w}_t$ , we compute

$$\int_{\Omega} F(w) \bar{w}_t d\Omega = -i \int_{\Omega} \mathcal{A} \bar{w} F(w) d\Omega + i \int_{\Omega} |F(w)|^2 d\Omega + i \int_{\Omega} F(w) \bar{f} d\Omega. \tag{113}$$

Applying Green's first theorem in (71) and (107), we obtain

$$\begin{aligned} \int_{\Omega} F(w) \bar{w}_t d\Omega = & i \int_{\Gamma} F(w) \frac{\partial \bar{w}}{\partial \nu_{\mathcal{A}}} d\Gamma - i \int_{\Omega} \langle \nabla_g \bar{w}, \nabla_g (R(w)) \rangle_g d\Omega - i \int_{\Omega} r |\nabla_g w|_g^2 d\Omega \\ & - i \int_{\Omega} w \langle \nabla_g \bar{w}, \nabla_g r \rangle_g d\Omega + i \int_{\Omega} |F(w)|^2 d\Omega + i \int_{\Omega} F(w) \bar{f} d\Omega. \end{aligned} \tag{114}$$

On the other hand, invoking identity (27), we compute for the second term in (114),

$$\int_{\Omega} \langle \nabla_g \bar{w}, \nabla_g (R(w)) \rangle_g d\Omega = \int_{\Omega} \text{Re} \langle \nabla_g \bar{w}, \nabla_g (R(w)) \rangle_g d\Omega + \frac{1}{2} \int_{\Omega} |w|^2 \langle \nabla_g \bar{w}, \nabla_g (R(w)) \rangle_g d\Omega - \frac{1}{2} \int_{\Omega} |\nabla_g w|_g^2 \langle \nabla_g \bar{w}, \nabla_g (R(w)) \rangle_g d\Omega \tag{115}$$

$$= \int_{\Omega} DR(\nabla_g \bar{w}, \nabla_g w) d\Omega - \frac{1}{2} \int_{\Omega} |\nabla_g w|_g^2 \operatorname{div}_0 R d\Omega + \frac{1}{2} \int_{\Gamma} |\nabla_g w|_g^2 R \cdot \nu d\Gamma, \quad (116)$$

using the divergence (Gauss) theorem on the second term in (115). Since the vector field  $R$  is real, we obtain by (116),

$$\operatorname{Re} \left( -i \int_{\Omega} \langle \nabla_g \bar{w}, \nabla_g (R(w)) \rangle_g d\Omega + i \int_{\Omega} |F(w)|^2 d\Omega \right) = 0. \quad (117)$$

By (107) and the decomposition (108), (109), we compute

$$F(w) = R \cdot \nabla w_0 + rw = \frac{R \cdot \nu(x)}{|\nu_{\mathcal{A}}(x)|_g^2} A(x)\nu(x) \cdot \nabla w_0 + W(w) + rw. \quad (118)$$

Thus, by (19), since  $\frac{\partial \bar{w}}{\partial \nu_{\mathcal{A}}} = A(x)\nabla_0 \bar{w} \cdot \nu(x) = \nabla_0 \bar{w} \cdot A(x)\nu(x)$ , the matrix  $A$  being symmetric, (118) yields

$$\begin{aligned} & i \int_{\Gamma} F(w) \frac{\partial \bar{w}}{\partial \nu_{\mathcal{A}}} d\Gamma = \\ & i \int_{\Gamma} \frac{R \cdot \nu(x)}{|\nu_{\mathcal{A}}(x)|_g^2} |\nabla_0 w \cdot A(x)\nu(x)|^2 d\Gamma + i \int_{\Gamma} [W(w) + rw] \frac{\partial \bar{w}}{\partial \nu_{\mathcal{A}}} d\Gamma. \end{aligned} \quad (119)$$

Then, taking the real part in (119), the first term drops out since  $R$  is real and we get

$$\operatorname{Re} \left( i \int_{\Gamma} F(w) \frac{\partial \bar{w}}{\partial \nu_{\mathcal{A}}} d\Gamma \right) = \operatorname{Re} \left( i \int_{\Gamma} [W(w) + rw] \frac{\partial \bar{w}}{\partial \nu_{\mathcal{A}}} d\Gamma \right). \quad (120)$$

Finally, integrating (114) in time over  $[s, t]$ , taking the real part in the resulting expression, and invoking (117) and (120), yields

$$\begin{aligned} & \operatorname{Re} \left( \int_s^t \int_{\Omega} F(w) \bar{w}_t d\Omega d\sigma \right) = \operatorname{Re} \left( i \int_s^t \int_{\Gamma} [W(w) + rw] \frac{\partial \bar{w}}{\partial \nu_{\mathcal{A}}} d\Gamma d\sigma \right) \\ & - \operatorname{Re} \left( i \int_s^t \int_{\Omega} r |\nabla_g w|_g^2 d\Omega d\sigma + i \int_s^t \int_{\Omega} w \langle \nabla_g \bar{w}, \nabla_g r \rangle_g d\Omega d\sigma \right. \\ & \left. - i \int_s^t \int_{\Omega} F(w) \bar{f} d\Omega d\sigma \right), \end{aligned} \quad (121)$$

since vector field  $R$  is real. Estimating the right-hand side of (121), we obtain via (59) on  $E(t)$  and (4) on  $F$ ,

$$2 \left| \operatorname{Re} \left( \int_s^t \int_{\Omega} F(w) \bar{w}_t d\Omega d\sigma \right) \right| \leq 2 \int_{\Sigma} [|W(w)| + |r||w|] \left| \frac{\partial \bar{w}}{\partial \nu_{\mathcal{A}}} \right| d\Sigma + 2 \int_s^t \int_{\Omega} |r| |w|^2 d\Omega d\sigma + 2 \int_s^t \int_{\Omega} |F(w)|^2 d\Omega d\sigma \quad (122)$$

where

$$C = \frac{1}{2} \max_{x \in \Omega} |\nabla_g r|_g + \epsilon \quad \text{and} \quad k = 2 \max_{x \in \Omega} |r| + \max_{x \in \Omega} |\nabla_g r|_g + \epsilon. \quad (123)$$

Eqn. (122) gives the sought-after estimate. Inserting (122) into the right-hand side of identity (102) yields

$$\begin{aligned} E(t) - E(s) &\leq 2 \operatorname{Re} \left( \int_s^t \int_{\Gamma} \bar{w}_t \frac{\partial w}{\partial \nu_A} d\Gamma d\sigma \right) + 2 \int_{\Sigma} [|W(w)| + |rw|] \left| \frac{\partial w}{\partial \nu_A} \right| d\Sigma \\ &+ C \int_Q |w|^2 dQ + k \int_s^t E(\sigma) d\sigma + \frac{2}{\epsilon} \int_Q |f|^2 dQ \\ &+ 2 \operatorname{Re} \left( \int_s^t \int_{\Omega} f \bar{w}_t d\Omega d\sigma \right). \end{aligned} \quad (124)$$

We now set  $f \equiv 0$ , as assumed, and then (124) yields the desired inequality (110) via definition (112). Similarly for (111). The proof of Proposition 5.2 is complete.  $\blacksquare$

**Remark 5.1.** If  $f \neq 0$ , we use again (106) in the last integral term containing  $(f\bar{w}_t)$  in (124), and proceed as in (114) under the assumption that  $f \in L_2(0, T; H^1(\Omega))$ , see Triggiani (1996), Eqn. (2.3.16) and ff. for details.

**Step 3. Corollary 5.3.** *Under the assumptions of Proposition 5.2, we have for  $T \geq t \geq s > 0$ ,*

$$E(t) \leq [E(s) + \Lambda(T)]e^{k(t-s)}; \quad E(s) \leq [E(t) + \Lambda(T)]e^{k(t-s)}; \quad (125)$$

$$E(t) \geq \frac{E(0) + E(T)}{2} e^{-kT} - \Lambda(T), \quad 0 \leq t \leq T, \quad (126)$$

where  $\Lambda(T)$  and  $k$  are defined in (112) and (124), respectively.

*Proof.* To show (125), we apply the classical argument of the Gronwall's inequality to (110) and (111), where we note that the terms in the brackets are independent of  $t$  in (110), and of  $s$  in (111), respectively. Next, the inequality on the right of (125) with  $s = 0$ , and that on the left with  $t = T$  and  $s = t$ , yield then

$$E(0) \leq [E(t) + \Lambda(T)]e^{kT}; \quad E(T) \leq [E(t) + \Lambda(T)]e^{kT}. \quad (127)$$

Summing up these two inequalities in (127), we arrive at (126), as desired.  $\blacksquare$

**Step 4.** (Carleman estimates, second version) This is Theorem 3.4 restated

**Theorem 5.4.** *Let  $f = 0$ . Assume (H.1) = (2), (H.2) = (4), (H.3) = (47), and (H.4) = (62). Let  $w$  be a solution of the Schrödinger equation (54) within the class (55). Then the following one-parameter family of estimates holds true for all  $\tau > 0$  sufficiently large:*

$$\begin{aligned}
 (BT_{1,w})|_{\Sigma} + C_{\tau,T,\phi} \|w\|_{C([0,T];L_2(\Omega))}^2 & \geq e^{-\frac{\tau\delta}{2}} \left\{ \left( \rho - \frac{C_T}{\tau} \right) e^{-kT} \frac{(t_1 - t_0)}{2} - \frac{e^{-\frac{\tau\delta}{2}}}{\tau} \right\} [E(0) + E(T)] \\
 & \geq k_{\phi,\tau} [E(0) + E(T)], \text{ for some constant } k_{\phi} > 0,
 \end{aligned} \tag{128}$$

where in agreement with (65), via (58)

$$\begin{aligned}
 (BT_{1,w})|_{\Sigma} & = (BT_w)|_{\Sigma} + C_{\rho,\phi,\tau} \int_{\Sigma} [|w_t| + |W(w)| + |rw|] \left| \frac{\partial w}{\partial \nu_A} \right| d\Sigma \\
 & = \operatorname{Re} \left( \int_{\Sigma} e^{\tau\phi} \frac{\partial w}{\partial \nu_A} h(\bar{w}) d\Sigma \right) - \frac{1}{2} \int_{\Sigma} e^{\tau\phi} |\nabla_g w|_g^2 h \cdot \nu d\Sigma \\
 & \quad + \frac{1}{2} \left| \int_{\Sigma} \frac{\partial w}{\partial \nu_A} \bar{w} \operatorname{div}_0(e^{\tau\phi} h) d\Sigma - i \int_{\Sigma} \bar{w} w_t e^{\tau\phi} h \cdot \nu d\Sigma \right| \\
 & \quad + C_{\rho,\sigma} \int_{\Sigma} [|w_t| + |W(w)| + |rw|] \left| \frac{\partial w}{\partial \nu_A} \right| d\Sigma.
 \end{aligned} \tag{130}$$

*Proof.* We return to inequality (94): on its right hand side, we use inequality (126), thus obtaining

$$\begin{aligned}
 \int_{t_0}^{t_1} E(t) dt & \geq \frac{E(0) + E(T)}{2} e^{-kT} (t_1 - t_0) \\
 -2(t_1 - t_0) \int_{\Sigma} & [|w_t| + |W(w)| + |rw|] \left| \frac{\partial w}{\partial \nu_A} \right| d\Sigma \\
 - C(t_1 - t_0) \int_Q & |w|^2 dQ,
 \end{aligned} \tag{131}$$

after recalling  $\Lambda(T)$  from (119). Inserting (131) into the right side of (94) yields

### 6. Proof of Theorem 3.1: Continuous observability inequality (Dirichlet case)

Let  $\psi$  be a solution of the  $\psi$ -problem in (5) (including the B.C.  $\psi|_{\Sigma} \equiv 0$ ). We want to apply Theorem 3.4 (same as Theorem 5.4) to it.

**Step 1.** First, we deal with the values of  $|\nabla_g \psi|_g^2$  and  $h(\psi)$  on the boundary  $\Gamma$ , respectively, as required by  $(BT_w)|_{\Sigma}$  in (58).

**Lemma 6.1.** *Let  $\psi$  be the solution of problem (5) [including the B.C.  $\psi|_{\Sigma} \equiv 0$ ]. Then, in this case, the boundary term  $(BT_{1,\psi})|_{\Sigma}$  defined by (65) and (58) reduces to*

$$(BT_{1,\psi})|_{\Sigma} = (BT_{\psi})|_{\Sigma} = \frac{1}{2} \int_{\Sigma} e^{\tau\phi} \left| \frac{\partial\psi}{\partial\nu_{\mathcal{A}}} \right|^2 \frac{h \cdot \nu}{|\nu_{\mathcal{A}}|_g^2} d\Sigma, \tag{132}$$

where, via (18), we define  $\nu_{\mathcal{A}}(x)$ , as in (109):

$$\nu_{\mathcal{A}}(x) = \sum_{i=1}^n \left( \sum_{j=1}^n a_{ij}(x) \nu_j(x) \right) \frac{\partial}{\partial x_i} = A(x) \nu(x). \tag{133}$$

to be the normal of the submanifold  $\Gamma$  in the Riemannian metric  $g$ .

*Proof.* Given  $x \in \mathbf{R}^n$ , the vector  $\nabla_g \psi(x)$  has the decomposition into direct product in  $(\mathbf{R}_x^n, g(x))$  as

$$\nabla_g \psi(x) = \left\langle \nabla_g \psi(x), \frac{\nu_{\mathcal{A}}}{|\nu_{\mathcal{A}}|_g} \right\rangle_g \frac{\nu_{\mathcal{A}}}{|\nu_{\mathcal{A}}|_g} + Y(x) = \left( \frac{1}{|\nu_{\mathcal{A}}|_g^2} \frac{\partial\psi}{\partial\nu_{\mathcal{A}}} \right) \nu_{\mathcal{A}} + \frac{\partial\psi}{\partial s} s. \tag{134}$$

Here, by (133), (22), (25),

$$\langle \nabla_g \psi(x), \nu_{\mathcal{A}}(x) \rangle_g = \langle \nabla_g \psi(x), A(x) \nu(x) \rangle_g = \nabla_g \psi(x) \cdot \nu(x) = \frac{\partial\psi}{\partial\nu_{\mathcal{A}}}. \tag{135}$$

Moreover,  $Y(x) \in \mathbf{R}_x^n$  satisfies  $\langle Y(x), \nu_{\mathcal{A}} \rangle_g = 0$ ; consequently, by (22) and (133),  $Y(x) \cdot \nu(x) = \langle Y(x), \nu_{\mathcal{A}}(x) \rangle_g = 0$ , that is,  $Y(x) \in \Gamma_x$ , the tangential space of  $\Gamma$  at  $x$ . Therefore, if  $s$  denotes a unit tangential vector, then, by (24):

$$Y(x) = \langle \nabla_g \psi(x), s \rangle_g = \nabla_0 \psi(x) \cdot s = \frac{\partial\psi(x)}{\partial s} \tag{136}$$

is the tangential gradient. Thus, (135), (136) show the RHS of (134). By (134), (24), we have

$$|\nabla_g \psi|_g^2 = \langle \nabla_g \psi, \nabla_g \psi \rangle_g = \nabla_g \psi(\psi) = \frac{1}{|\nu_{\mathcal{A}}(x)|_g^2} \langle \nabla_g \psi(x), \nu_{\mathcal{A}}(x) \rangle_g^2 + Y(\psi) \tag{137}$$

$$= \frac{1}{|\nu_{\mathcal{A}}(x)|_g^2} \left| \frac{\partial\psi}{\partial\nu_{\mathcal{A}}} \right|^2 + \dots$$

since  $\psi|_\Sigma = 0$ , hence  $\nabla_0\psi \perp \Gamma$  and hence  $Y(\psi) = \nabla_0\psi \cdot Y = 0$  by (24). Similarly,  $h(x)$  has the decomposition into direct product

$$h(x) = \left\langle h(x), \frac{\nu_{\mathcal{A}}(x)}{|\nu_{\mathcal{A}}(x)|_g} \right\rangle_g \frac{\nu_{\mathcal{A}}(x)}{|\nu_{\mathcal{A}}(x)|_g} + Z(x), \tag{139}$$

where  $Z(x) \in \Gamma_x$ . Moreover, by (133), (25), (24), we have

$$\frac{\partial\psi}{\partial\nu_{\mathcal{A}}} = (A(x)\nabla_0\psi) \cdot \nu(x) = \nabla_0\psi \cdot A(x)\nu(x) = \nabla_0\psi \cdot \nu_{\mathcal{A}}(x) = \langle \nabla_g\psi, \nu_{\mathcal{A}} \rangle_g, \tag{140}$$

since the matrix  $A(x)$  is symmetric. Hence, by (24), (133), (139), (140), (22),

$$\begin{aligned} h(\psi)(x) &= \\ \langle \nabla_g\psi, h \rangle_g &= \left\langle h(x), \frac{\nu_{\mathcal{A}}(x)}{|\nu_{\mathcal{A}}(x)|_g} \right\rangle_g \langle \nabla_g\psi, \nu_{\mathcal{A}}(x) \rangle_g + \langle \nabla_g\psi, Z(x) \rangle_g \end{aligned} \tag{141}$$

$$\begin{aligned} h(\psi)(x) &= \frac{\langle h(x), \nu_{\mathcal{A}}(x) \rangle_g}{|\nu_{\mathcal{A}}(x)|_g^2} \left( \frac{\partial\psi}{\partial\nu_{\mathcal{A}}} \right) = \frac{\langle h(x), \mathcal{A}(x)\nu(x) \rangle_g}{|\nu_{\mathcal{A}}(x)|_g^2} \frac{\partial\psi}{\partial\nu_{\mathcal{A}}} \\ \text{(by (22))} \quad &= \frac{h(x) \cdot \nu(x)}{|\nu_{\mathcal{A}}(x)|_g^2} \left( \frac{\partial\psi}{\partial\nu_{\mathcal{A}}} \right), \end{aligned} \tag{142}$$

since, as before,  $\psi|_\Sigma \equiv 0$ , hence,  $\nabla_0\psi \perp \Gamma$ , and  $\langle \nabla_g\psi, Z \rangle_g = \nabla_0\psi \cdot Z = 0$ , via (24).

Finally, we return to definition (58) for  $BT|_\Sigma$  (written for  $\psi$ ) and (65) for  $BT_{1,\psi}$ ; use here  $\psi|_\Sigma \equiv 0$ , hence  $\psi_t \equiv 0$  and  $W(\psi) \equiv 0$  since  $W$  is tangential, as well as (137) and (142), to obtain

$$\begin{aligned} (BT_{1,\psi})|_\Sigma &= (BT_\psi)|_\Sigma = \\ \text{Re} \left( \int_\Sigma e^{\tau\phi} \frac{\partial\psi}{\partial\nu_{\mathcal{A}}} h(\bar{\psi}) d\Sigma \right) &- \frac{1}{2} \int_\Sigma e^{\tau\phi} |\nabla_g\psi|_g^2 h \cdot \nu d\Sigma \end{aligned} \tag{143}$$

$$= \int_\Sigma e^{\tau\phi} \left| \frac{\partial\psi}{\partial\nu_{\mathcal{A}}} \right|^2 \frac{h \cdot \nu}{|\nu_{\mathcal{A}}|_g^2} d\Sigma - \frac{1}{2} \int_\Sigma e^{\tau\phi} \left| \frac{\partial\psi}{\partial\nu_{\mathcal{A}}} \right|^2 \frac{h \cdot \nu}{|\nu_{\mathcal{A}}|_g^2} d\Sigma. \tag{144}$$

Then, (144) yields (132), as desired. ■

**Step 2. Completion of the proof of Theorem 3.1.** In the Dirichlet case, to obtain the continuous observability inequality (6) from inequality (64) of Theorem 3.4 already proved, it suffices to return to (132); since  $h(x) \cdot \nu(x) \leq 0$  for  $x \in \Gamma_0$  by assumption, we readily have from (132),

$$1 \int_\Sigma e^{\tau\phi} h(x) \cdot \nu(x) \leq \int_\Sigma \left| \frac{\partial\psi}{\partial\nu_{\mathcal{A}}} \right|^2 d\Sigma \tag{145}$$

Then (145) used on the left-hand side of inequality (64) yields (when the parameter  $\tau > 0$  is large enough) and  $f \equiv 0$ :

$$\int_0^T \int_{\Gamma_1} \left| \frac{\partial \psi}{\partial \nu_{\mathcal{A}}} \right|^2 d\Sigma + k_2 \|\psi\|_{C([0,T];L_2(\Omega))}^2 \geq k_1 E(0), \tag{146}$$

where  $k_1, k_2 > 0$  are constants.

To get the sought-after inequality (6) from (146), we only need to drop the low order term  $k_2 \|\psi\|_{C([0,T];L_2(\Omega))}^2$  in (146). This may be done, as usual, by a compactness/uniqueness argument, Lions (1988), Lasiecka, Triggiani (1987, 1991), see Remark 3.1. ■

### 7. Proof of Theorem 3.5: Main inverse inequality

We prove the specialized version of Theorem 3.5 for  $w$  being a solution of (54) within the class (55), which moreover satisfies hypothesis (66).

**Step 1. Lemma 7.1.** *Let  $w$  solve (54) and satisfy (66):  $w|_{\Sigma_0} \equiv 0$  and  $h \cdot \nu \leq 0$  on  $\Gamma_0$ .*

(a) *Then, in this case, the boundary terms  $(BT_{1,w})|_{\Sigma}$  defined by (65), (58) reduce to*

$$(BT_{1,w})|_{\Sigma} = (BT_{1,w})|_{\Sigma_0} + (BT_{1,w})|_{\Sigma_1}; \tag{147}$$

$$(BT_{1,w})|_{\Sigma_0} = (BT_w)|_{\Sigma_0} = \frac{1}{2} \int_0^T \int_{\Gamma_0} e^{\tau \phi} \frac{h(x) \cdot \nu(x)}{\nu_{\mathcal{A}}(x)} \left| \frac{\partial w}{\partial \nu_{\mathcal{A}}} \right|^2 d\Sigma \leq 0; \tag{148}$$

$$\begin{aligned} & |(BT_{1,w})|_{\Sigma_1}| \leq \\ & C \left\{ \int_0^T \int_{\Gamma_1} \left[ \left| \frac{\partial w}{\partial \nu_{\mathcal{A}}} \right|^2 + \left| \frac{\partial w}{\partial s} \right|^2 + |w_t|^2 \right] d\Sigma + \|w\|_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))}^2 \right\} \end{aligned} \tag{149}$$

for any  $\epsilon_0 > 0$ , where  $\frac{\partial w}{\partial s}$  denotes, as before, the tangential gradient (derivative) of  $w$  on  $\Gamma$ , so that  $\left| \frac{\partial w}{\partial s} \right|^2 = |\nabla_{\text{tangential}} w|_0^2$ .

(b) *Moreover, if in addition,  $w$  satisfies also  $\frac{\partial w}{\partial \nu_{\mathcal{A}}}|_{\Sigma_1} \equiv 0$ , then*

$$(BT_{1,w})|_{\Sigma_1} = (BT_w)|_{\Sigma_1} = \frac{1}{2} \int_0^T \int_{\Gamma_1} e^{\tau \phi} \left[ |w_t|^2 - \left| \frac{\partial w}{\partial s} \right|^2 \right] h \cdot \nu d\Sigma \tag{150}$$

*Proof.* We return to (65), (58): we then see that  $BT_w$  and  $BT_{1,w}$  coincide on  $\Sigma_0 = (0, T] \times \Gamma_0$ , since  $w|_{\Sigma_0} \equiv w_t|_{\Sigma_0} \equiv 0$  by assumption. We may divide  $BT_w|_{\Sigma}$  as in identity (147), where  $BT_w|_{\Sigma_0}$  is given by (148) by virtue of the

this time on  $\Sigma_0$ ). Similarly, from (65), (58), where  $h(w) = \langle \nabla_g v, \nabla_g w \rangle_g$ , we readily obtain

$$\left\{ \begin{array}{l} (BT_{1,w})|_{\Sigma_1} = (BT_w)|_{\Sigma_1} = \frac{1}{2} \int_0^T \int_{\Gamma_1} e^{\tau\phi} [ |w_t|^2 - |\nabla_g w|_g^2 ] h \cdot \nu \, d\Sigma \\ \text{when } \left. \frac{\partial w}{\partial \nu_{\mathcal{A}}} \right|_{\Sigma_1} \equiv 0; \\ | (BT_{1,w})|_{\Sigma_1} | \leq C \left\{ \int_0^T \int_{\Gamma_1} \left[ \left| \frac{\partial w}{\partial \nu_{\mathcal{A}}} \right|^2 + |\nabla_g w|_g^2 + |w_t|^2 \right] d\Sigma \right. \\ \left. + \|w\|_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))}^2 \right\}, \text{ in general,} \end{array} \right. \quad (151)$$

since  $W(w) = \langle \nabla_g w, W \rangle_g$ , by use also of trace theory applied to  $w \in \Gamma_1$ . Next, the decomposition (134) of  $\nabla_g w$  in normal and tangential components yields by virtue of (137)

$$\text{on } \Gamma_1 : |\nabla_g w|_g^2 = \begin{cases} \left| \frac{\partial^2 w}{\partial s} \right|^2, & \text{when } \left. \frac{\partial w}{\partial \nu_{\mathcal{A}}} \right|_{\Sigma_1} = 0; \\ \frac{1}{|\nu_{\mathcal{A}}(x)|_g^2} \left| \frac{\partial w}{\partial \nu_{\mathcal{A}}} \right|^2 + \left| \frac{\partial^2 w}{\partial s} \right|^2, \end{cases} \quad (152)$$

since, from (134),  $Y(x) \in \Gamma_x$ , the tangent space of  $\Gamma$  at  $x$ , we have  $Y(w) = \nabla_0 w \cdot Y = \left| \frac{\partial w}{\partial s} \right|^2$  by (24), (136). Then, (152) and (152), used in (151) and (151), yield (150) and (149), respectively. Lemma 7.1 is proved. ■

**Step 2.** The following result is taken from Triggiani (1996), Theorem 2.1.4. It is proved by micro-local analysis. It is critical in eliminating artificial geometrical conditions of the earlier literature on the controlled part  $\Gamma_1$  of the boundary in the Neumann case.

**Lemma 7.2.** *Let  $f \in L_2(Q)$  and let  $w$  be a solution of Eqn. (54) in the class (55).*

(a) *Then, for any  $\epsilon > 0$ ,  $\epsilon_0 > 0$ , and  $T > 0$ , there exists a constant  $C_{\epsilon, \epsilon_0, T} > 0$  such that*

$$\int_{\epsilon}^{T-\epsilon} \int_{\Gamma} \left| \frac{\partial w}{\partial s} \right|^2 d\Sigma \leq C_{\epsilon, \epsilon_0, T} \left\{ \int_0^T \int_{\Gamma} \left[ \left| \frac{\partial w}{\partial \nu_{\mathcal{A}}} \right|^2 + |w_t|^2 \right] d\Sigma \right. \\ \left. + \|w\|_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))}^2 + \|f\|_{H^{-\frac{1}{2}+\epsilon_0}(Q)}^2 \right\}. \quad (153)$$

(b) *Moreover, if  $w$  satisfies in addition hypothesis (66):  $w|_{\Sigma_0} \equiv 0$  and  $h \cdot \nu \leq$*



**Step 4.** We next use Lemma 7.2, (153), to eliminate the tangential derivative from the estimate (149) [or identity (150)] of the boundary terms  $(BT_{1,w})$  evaluated over  $[\epsilon, T - \epsilon] \times \Gamma_1$ .

**Proposition 7.3.** *Let  $f \in L_2(Q)$  and let  $w$  be a solution of (54) in the class (55). Moreover, let  $w$  satisfy hypothesis (66). Then, for all  $\tau > 0$  sufficiently large, there exists a constant  $k_{\phi,\tau} > 0$  such that*

$$\begin{aligned} & \int_{\epsilon}^{T-\epsilon} \int_{\Gamma_1} \left[ \left| \frac{\partial w}{\partial \nu_{\mathcal{A}}} \right|^2 + |w_t|^2 \right] d\Sigma + \frac{C_T}{\tau} \int_Q e^{\tau\phi} |f|^2 dQ + C_T \|w\|_{L_2(0,T;H^{\frac{1}{2}+\nu_0}(\Omega))}^2 \\ & \geq k_{\phi,\tau} [E(T) + E(0)]. \end{aligned} \quad (154)$$

*Proof.* We apply Theorem 3.4, estimate (64), over  $[\epsilon, T - \epsilon] \times \Gamma$  rather than  $[0, T] \times \Gamma = \Sigma$ . In so doing, we use hypothesis (66) to invoke (148) and conclude that  $(BT_{1,w})|_{[\epsilon, T - \epsilon] \times \Gamma_0} \leq 0$ . Moreover, we invoke (149) for  $(BT_{1,w})|_{[\epsilon, T - \epsilon] \times \Gamma_1}$  and use the key estimate (153). Finally, the right-hand side of (64) becomes  $k_{\phi,\tau} [E(\epsilon) + E(T - \epsilon)]$ . But

$$E(\epsilon) + E(T - \epsilon) \geq [E(0) + E(T)]e^{-k\epsilon} - 2\Lambda(T). \quad (155)$$

This can be proved as in the case of (126): by using the inequality on the right-hand side of (125) with  $s = 0$  and  $t = \epsilon$ , and the inequality on the left-hand side of (125) with  $t = T$  and  $s = T - \epsilon$ , and summing up the resulting inequalities. This yields (155). Then (154) is obtained. ■

**Step 5.** Completion of the proof of Theorem 3.5. The sought-after inequality (67) of Theorem 3.5 now follows at once from (154) of Proposition 7.3, by further majorizing its left-hand side. Theorem 3.5 is proved. ■

## 8. Proof of Theorem 3.2: Neumann case

We return to inequality (67) of Theorem 3.5(b), written for the solution  $w = \psi$  of problem (9), with the boundary integral over  $\Gamma_1$ , since, by assumption, (66) holds true:  $\psi|_{\Sigma_0} \equiv 0$  and  $h \cdot \nu \leq 0$  on  $\Gamma_0$ . Moreover, on  $\Sigma_1$ , it suffices to take  $\beta \equiv 0$  in (9), i.e.,  $\frac{\partial \psi}{\partial \nu_{\mathcal{A}}}|_{\Sigma_1} \equiv 0$ . Then, as  $f \equiv 0$ , (67) becomes the following inequality:

$$\int_{\Sigma_1} \psi_t^2 d\Sigma + k_1 \|\psi\|_{C([0,T];H^{\frac{1}{2}+\nu_0}(\Omega))}^2 \geq k_2 E(0), \quad (156)$$

where  $k_1, k_2 > 0$  are constants. Finally, by a compactness/uniqueness argument again, see Remark 3.1, we obtain the desired inequality in (10).

**Remark 8.1.** Given the  $\psi$ -problem (9), say with  $\beta \equiv 0$ , the proof of Theorem 3.5 uses (135), (137) and the first part of (139) rather than (134),

## 9. Some illustrations where Assumptions (H.1) and (H.3) on $\mathcal{A}$ hold true

**Example 9.1.** Let  $\Omega \subset \mathbf{R}^2$  be a bounded domain. Assume that  $\mathcal{A}$  is defined by

$$\begin{aligned} Au &= \frac{\partial}{\partial x} \left( \frac{1+y^6}{1+x^2+y^6} \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial x} \left( \frac{xy^3}{1+x^2+y^6} \frac{\partial u}{\partial y} \right) \\ &+ \frac{\partial}{\partial y} \left( \frac{xy^3}{1+x^2+y^6} \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1+x^2}{1+x^2+y^6} \frac{\partial u}{\partial y} \right). \end{aligned} \quad (157)$$

Set

$$A(x, y) = (a_{ij}) = \begin{pmatrix} \frac{1+y^6}{1+x^2+y^6} & \frac{xy^3}{1+x^2+y^6} \\ \frac{xy^3}{1+x^2+y^6} & \frac{1+x^2}{1+x^2+y^6} \end{pmatrix}. \quad (158)$$

Then,  $\det A(x, y) = 1/(1+x^2+y^6) > 0$ ,  $\forall (x, y) \in \mathbf{R}^2$ , and  $A(x, y)$  is strictly positive definite on the bounded domain  $\Omega$ . Thus, assumption (H.1) is verified.

The inverse of  $A(x, y)$  is

$$G(x, y) = (g_{ij}) = A^{-1}(x, y) = \begin{pmatrix} 1+x^2 & -xy^3 \\ -xy^3 & 1+y^6 \end{pmatrix}. \quad (159)$$

Consider the Riemannian manifold  $(\mathbf{R}^2, g)$ , where the Riemannian metric  $g$  is defined in the natural coordinate system  $(x, y)$  via (159) by

$$g = (1+x^2)dx dx - xy^3 dx dy - xy^3 dy dx + (1+y^6)dy dy. \quad (160)$$

Consider the surface in  $\mathbf{R}^3$  given by

$$M = \left\{ (x, y, z) \mid z = f(x, y) = \frac{1}{2}x^2 - \frac{1}{4}y^4 \right\},$$

with the induced Riemannian metric  $g_M$ . Then the (projection) map  $\Phi(x, y, z) = (x, y)$ , for any  $(x, y, z) \in M$ , determines an isometry from  $M$  to  $(\mathbf{R}^2, g)$ . The Gaussian curvature of  $(\mathbf{R}^2, g)$  at  $(x, y)$  is therefore

$$\begin{aligned} k(x, y) &= \text{the Gaussian curvature of } M \text{ at } (x, y, z) \\ &= \frac{\left( \frac{\partial^2 f}{\partial x^2} \right) \left( \frac{\partial^2 f}{\partial y^2} \right) - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2}{\left[ \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 \right]^2} = \frac{-3y^2}{(1+x^2+y^6)^2} \leq 0, \quad \forall (x, y) \in \mathbf{R}^2. \end{aligned} \quad (161)$$

Since, by (161), the Gaussian curvature is non-positive, the function defined by

$$v(x) = d_g^2(x, x_0), \quad x_0 \text{ fixed } \in \mathbf{R}^2, \tag{162}$$

i.e., as the square of the distance  $d_g(x, x_0)$ , in the Riemann metric of (160), from  $x$  to a given fixed point  $x_0 \in \mathbf{R}^2$ , is in fact strictly convex on  $(\mathbf{R}^2, g)$ , Wu, Shen and Yu (1989), p. 108. Thus, assumption (H.3) also holds true in this case.

**Example 9.2.** Let  $\Omega \subset \mathbf{R}^n$  be a bounded domain and  $a_i > 0$  constants,  $i = 1, 2, \dots, n$ . Consider the operator on  $\mathbf{R}^n$ ,

$$\mathcal{A}u = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \frac{1 + \sum_{j \neq i}^n a_j^2 x_j^2}{1 + \sum_{k=1}^n a_k^2 x_k^2} \frac{\partial u}{\partial x_i} \right) - \sum_{i \neq j} \frac{\partial}{\partial x_i} \left( \frac{a_i a_j x_i x_j}{1 + \sum_{k=1}^n a_k^2 x_k^2} \frac{\partial u}{\partial x_j} \right). \tag{163}$$

Set

$$A(x) = (a_{ij}) = \frac{1}{1 + \sum_{k=1}^n a_k^2 x_k^2} \begin{pmatrix} 1 + \sum_{i=2}^n a_i^2 x_i^2 & -a_1 a_2 x_1 x_2 & \cdots & -a_1 a_n x_1 x_n \\ -a_2 a_1 x_2 x_1 & 1 + \sum_{i \neq 2} a_i^2 x_i^2 & \cdots & -a_2 a_n x_2 x_n \\ \cdots & \cdots & \cdots & \cdots \\ -a_n a_1 x_n x_1 & -a_n a_2 x_n x_2 & \cdots & 1 + \sum_{i=1}^{n-1} a_i^2 x_i^2 \end{pmatrix}. \tag{164}$$

Then, the inverse of  $A(x)$  is

$$G(x) = (g_{ij}) = A^{-1}(x) = \begin{pmatrix} 1 + a_1^2 x_1^2 & a_1 a_2 x_1 x_2 & \cdots & a_1 a_n x_1 x_n \\ a_2 a_1 x_2 x_1 & 1 + a_2^2 x_2^2 & \cdots & a_2 a_n x_2 x_n \\ \cdots & \cdots & \cdots & \cdots \\ a_n a_1 x_n x_1 & a_n a_2 x_n x_2 & \cdots & 1 + a_n^2 x_n^2 \end{pmatrix}. \tag{165}$$

Consider the Riemannian manifold  $(\mathbf{R}^n, g)$ , where the Riemannian metric  $g$  is determined in the natural coordinate system  $x = (x_1, x_2, \dots, x_n)$  via (164) by

$$g = \sum q_i dx_i dx_i = \sum^n (\delta_{i,i} + a_i a_i x_i x_i) dx_i dx_i. \tag{166}$$

where  $\delta_{ij}$  is 1 if  $i = j$ , and 0 if  $i \neq j$ . It follows that

$$\sum_{i,j=1}^n g_{ij} \xi_i \xi_j = \sum_{i,j=1}^n (\delta_{ij} + a_i a_j x_i x_j) \xi_i \xi_j \geq |\xi|_0^2, \\ \forall x, \xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbf{R}^n. \quad (167)$$

It is easily checked from the above inequality that  $(\mathbf{R}^n, g)$  is a complete non-compact Riemannian manifold.

Let  $M$  be the hypersurface in  $\mathbf{R}^{n+1}$  given by

$$M = \left\{ [x_1, x_2, \dots, x_n, x_{n+1}] \mid x_{n+1} = \frac{1}{2} \sum_{i=1}^n a_i x_i^2 \right\}, \quad (168)$$

with the induced Riemannian metric in  $\mathbf{R}^n$ . Then, by Yao (1999), Lemma 3.1,  $M$  is of everywhere positive sectional curvature. It is easily verified from (166) that the map  $\Phi : M \rightarrow (\mathbf{R}^n, g)$ , defined by

$$\Phi(p) = x = [x_1, \dots, x_n], \quad \forall p = [x_1, \dots, x_n, x_{n+1}] \in M,$$

is an isometry between  $M$  and  $(\mathbf{R}^n, g)$ . Thus,  $(\mathbf{R}^n, g)$  itself is of everywhere positive sectional curvature. Since  $(\mathbf{R}^n, g)$  is a non-compact, complete Riemannian manifold of everywhere positive sectional curvature, then there exists a  $C^\infty$  strictly convex function  $v(x)$  on  $(\mathbf{R}^n, g)$  by Greene and Wu (1976). Assumptions (H.1) and (H.3) are verified.

## References

- BARDOS, C., LEBEAU, G. and RAUCH, J. (1992) Sharp efficient conditions for the observation, control, and stabilization of wave from the boundary. *SIAM J. Control and Optim.*, **30**, 5, (September), 1024–1065.
- DOLECKI, S. and RUSSELL, D. (1977) A general theory of observation and control. *SIAM J. Control*, **15**, 185–220.
- FURSIKOV, A.V. and IMANUVILOV, O.YU. (1996) *Controllability of Evolution Equations*, Lecture Notes N. 34, Research Institute of Mathematics, Seoul National University, Seoul, 151–742.
- GREENE, R.E. and WU, H. (1976)  $C^\infty$  convex functions and manifolds of positive curvature. *Acta Math.*, **137**, 209–245.
- HO, F.L. (1986) Observabilité frontière de l'équation des ondes. *C. R. Acad. Sci. Paris, Ser. I Math*, **302**, 443–446.
- HÖRMANDER, L. (1985) *The analysis of linear partial differential operators. III*, Springer-Verlag.
- HÖRMANDER, L. (1997) On the uniqueness of the Cauchy problem under partial analyticity assumptions. In: *Geometrical optics and related topics*,

- HORN, M.A. and LITTMAN, W. (1996a) Boundary control of a Schrödinger equation with non-constant principal part. In: *Control of Partial Differential Equations and Applications*. E. Casas, ed. Lecture Notes in Pure and Applied Mathematics, **174**, 101–106.
- HORN, M.A. and LITTMAN, W. (1996b) Local smoothing properties of a Schrödinger's equation with non-constant principal part. In: *Modelling and Optimization of Distributed Parameter Systems*, K. Malanowski, Z. Nahorski, and M. Peszyńska, eds., Chapman and Hall, 104–110.
- IMANUVILOV, O.YU. (1990) Exact controllability of hyperbolic equations, Part I & II. *Automatika*, **3**, 10–13; and **4**, 31–39 (in Russian).
- ISAKOV, V. (1998) *Inverse Problems for Partial Differential Equations*. Springer-Verlag, Applied Mathematical Sciences, **127**.
- ISAKOV, V. (1993) Carleman type estimates in an anisotropic case and applications. *J. Diff. Eqns.*, **105**, 2, 217–238.
- LASIECKA, I. and TRIGGIANI, R. (1989) Exact controllability of the wave equation with Neumann boundary control. *Appl. Math. Optim.*, **19**, 243–290. Preliminary version in Springer-Verlag LNCIS n. 100 (1987), 316–371.
- LASIECKA, I. and TRIGGIANI, R. (1991) Optimal regularity, exact controllability and uniform stabilization of the Schrödinger equation. *Diff. & Int. Eqns.*, **5**, 521–535.
- LASIECKA, I. and TRIGGIANI, R. (1992) Uniform stabilization of the wave equation with Dirichlet or Neumann feedback control without geometrical conditions, *Appl. Math. Optim.*, **25**, 189–244.
- LASIECKA, I. and TRIGGIANI, R. (1994) Carleman estimates and exact boundary controllability for a system of coupled, non-conservative second order hyperbolic equations. Marcel Dekker, *Lecture Notes in Pure and Applied Mathematics*, **188**, 215–243.
- LASIECKA, I., TRIGGIANI, R. and YAO, P.F. (1997) Exact controllability for second-order hyperbolic equations with variable coefficient-principal part and first-order terms. *Nonlinear Analysis Theory, Methods & Applications*, **30**, 1, 111–122.
- LASIECKA, I., TRIGGIANI, R. and YAO, P.F. (1999) Inverse/observability estimates for second-order hyperbolic equations with variable coefficients. *J. Math. Anal. & Appl.*, **235**, 13–57.
- LASIECKA, I., TRIGGIANI, R. and YAO, P.F. (1998) An observability estimate in  $L_2(\Omega) \times H^{-1}(\Omega)$  for second order hyperbolic equations with variable coefficients. *Control of distributed parameter and stochastic systems*, Proc. of IFIP Conf., Hangzhou. Kluwer, 71–78.
- LASIECKA, I., TRIGGIANI, R. and ZHANG, X. (1999) Non-conservative wave equations with unobserved Neumann B.C.: global uniqueness and observability in one shot. *AMS Contemporary Mathematics*, to appear.
- LEBEAU, J. (1992) Controle de l'equation de Schrödinger. *J. Math. Pures & Appl.*, **71**, 267–291.

- LIONS, J.L. (1988) *Contrôlabilité exacte, stabilisation et perturbations des systèmes distribués*. Masson, Paris, 1, Contrôlabilité exacte.
- LITTMAN, W. (1987) Near optimal time boundary controllability for a class of hyperbolic equations. Springer-Verlag LNCIS, **97**, 307–312.
- LITTMAN, W. (1992) Boundary controllability for polyhedral domains. Springer-Verlag LNCIS, **178**, 272–284.
- LITTMAN, W. and TAYLOR, S. (1992) Smoothing evolution equations and boundary control theory. *Journal d'analyse Mathématique*, **59**, 117–131.
- MACHTYNGIER, E. (1990) Contrôlabilité exacte et stabilisation frontière de l'équation de Schrödinger, *C. R. Acad. Sc. Paris*, **310**, I, 801–806.
- TATARU, D. (1992) A-priori pseudo-convexity energy estimates in domains with boundary and exact boundary controllability for conservative P.D.E.'s, Ph.D. thesis, University of Virginia, May.
- TATARU, D. (1994) A-priori estimates of Carleman type in domains with boundary. *J. Math. Pure et Appl.*, **73**, 355–387.
- TATARU, D. (1995A) Boundary controllability of conservative PDEs. *Appl. Math. Optim.*, **31**, 257–295.
- TATARU, D. (1995B) Unique continuation for solutions to P.D.E.'s, between Hörmander's theorem and Holmgren's theorem. *Comm. Part. Diff. Eqns.*, **20**, 5&6, 855–884.
- TAYLOR, A.E. and LAY, D.C. (1980) *Introduction to Functional Analysis*. John Wiley, second edition.
- TRIGGIANI, R. (1996) Carleman estimates and exact boundary controllability for a system of coupled non-conservative Schrödinger equations. Special issue *Rendiconti dell' Istituto dei Matematica dell' Università di Trieste*, **XXVIII**, 453–504, Supplement, Dedicated to the memory of Pierre Grisvard.
- YAO, P.F. (1996) On the observability inequalities for exact controllability of wave equations with variable coefficients. Preprint, and *SIAM J. Control*, to appear.
- WU, H., SHEN, C.L. and YU, Y.L. (1989) *An introduction to Riemann geometry* (in Chinese), University of Beijing, Beijing.