

Generalized J -integral method for sensitivity analysis of static shape design

by

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Abstract: The paper analyses the concept of GJ -integral. This concept is linked with the derivative of the energy functional with respect to a general perturbation of the domain and it is useful in studying many problems of sensitivity analysis. Applications of the GJ -integral method to numerous linear and nonlinear problems are indicated.

Keywords: GJ -integral, sensitivity analysis, domain perturbation, derivative of energy functional.

1. Introduction

There are many papers concerning differentiation of the potential energy functional with respect to variable domains (see e.g. Haug Edward, Choi Kyung and Komkov, 1986, Petryk and Mróz, 1986), mostly related to the shape design. The theory of calculation of material and shape derivatives in linear and unilateral boundary value problems is developed in Sokolowski and Zolesio (1992). Derivatives of energy functionals with respect to the crack length in classical linear elasticity can be found in Mazya and Nazarov (1987). With respect to the analysis of dependence of solution on the shape domain for a wide class of elastic problems we refer the reader to Khludnev and Sokolowski (1997) (see also Dauge, 1988). In the recent works, Khludnev and Sokolowski (1998a, b), the appropriate technique of finding derivatives of energy functional with respect to the crack shape for nonlinear boundary conditions is used which can be applied for purposes of sensitivity analysis. It is known that the usual calculation of

in classical sense. In this paper, we use the generalized J -integral (GJ -integral) which was first proposed in Ohtsuka (1981) to express the energy variation for the crack extension (*energy release rate*) in a 3D-elastic body. The theory of the GJ -integral is exposed in Ohtsuka (1985), its applications can be found in Ohtsuka (1982, 1991, 1996, 1997); Ohtsuka and Bochniak (1998). For a solution u of the boundary value problem corresponding to a given data f , the GJ -integral $J_\omega(u, X)$ is defined by two parameters. The first one is a domain ω , and the other is the vector field X . GJ -integral $J_\omega(u, X)$ is the sum of a surface integral $P_\omega(u, X)$ and a volume integral $R_\omega(u, X)$, i.e., $J_\omega(u, X) = P_\omega(u, X) + R_\omega(u, X)$. It can be proved that

$$J_\omega(u, X) = 0, \quad (1.1)$$

provided that function u is smooth (see Theorem 4.3). Meanwhile, $J_\omega(u, X)$ does not vanish, if u has some singularity inside of ω . For example, for a body having cracks, a solution belongs to $W^{1,2}(\Omega)$ and does not belong to $W^{2,2}(\Omega)$. The starting point in the GJ -integral method is that the variation of the potential energy $\mathcal{E}(\tau)$ with respect to the shape sensitivity parameter τ has the expression

$$\left. \frac{d\mathcal{E}(\tau)}{d\tau} \right|_{\tau=0} = -R_\Omega(u, X) + \int_\Omega \{X \cdot \nabla(f \cdot u) + (f \cdot u) \operatorname{div} X\} dx, \quad (1.2)$$

where the vector field X is obtained from shape sensitivity. In fracture mechanics the derivative $\left. \frac{d\mathcal{E}(\tau)}{d\tau} \right|_{\tau=0}$ is used to formulate rupture criteria (see Parton and Morozov, 1985). Formula (1.2) will be proved in Theorem 5.4. In this paper, we prove (1.2) with the assumptions that are weaker than the ones of Ohtsuka (1985). In (1.2), function u solves one of the variational problems given in Section 2 and the vector field X is derived from the family of mappings presented in Section 6. In the present paper we also analyse shape derivatives of solutions using the first variation of the integral $R_\omega(u, X)$ with respect to u .

2. Boundary value problems

Let Ω be a bounded domain in \mathbb{R}^N ($N = 2, 3, \dots$) which can be divided into finite number of domains with Lipschitz boundaries, and Γ be a boundary of the domain Ω . Denote by $W^{1,2}(\Omega)$ the Sobolev space of functions having the first square integrable derivatives in Ω .

Let E be a given function in $C^2(\mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^{Nm})$. We define the functional of potential energy type

$$\mathcal{E}(v; f, \Omega) = \int_\Omega \{E(x, v, \nabla v) - f \cdot v\} dx, \quad v \in V(\Omega)$$

on a closed subspace $V(\Omega)$. $V(\Omega) \subset W^{1,2}(\Omega)^m$, and consider the following

PROBLEM $P(f, V(\Omega))$: For a given $f \in L^2(\Omega)^m$, find an element $u \in V(\Omega)$ such that

$$\mathcal{E}(u; f, \Omega) \leq \mathcal{E}(v; f, \Omega) \text{ for all } v \in V(\Omega). \quad (2.3)$$

Let us denote

$$\delta E(v)[w] = \lim_{\epsilon \rightarrow 0} \epsilon^{-1} \{E(v + \epsilon w) - E(v)\} \text{ for } v, w \in C^\infty(\mathbb{R}^N)^m,$$

where $E(v) = E(x, v, \nabla v)$.

PROPOSITION 2.1 *The solution u of the problem $P(f, V(\Omega))$ satisfies the identity*

$$\int_{\Omega} \delta E(u)[v] dx = \int_{\Omega} f \cdot v dx \text{ for all } v \in V(\Omega). \quad (2.4)$$

We first introduce some notations:

$$\begin{aligned} \text{if } m > 1, \text{ then we denote } A_{ij}(x, z, p) &:= D_{p_{ij}} E(x, z, p) & (2.5) \\ &\text{for } 1 \leq i \leq m, 1 \leq j \leq N, \end{aligned}$$

$$\begin{aligned} \text{if } m = 1, \text{ then } A_{1j}(x, z, p) &:= D_{p_j} E(x, z, p) \text{ for } 1 \leq j \leq N, & (2.6) \\ B_i(x, z, p) &:= D_{z_i} E(x, z, p) \text{ for } 1 \leq i \leq m, \end{aligned}$$

where D denotes a differentiation operator. By variation of \mathcal{E} , we obtain the equation

$$\int_{\Omega} \{A_{ij}(x, u, \nabla u) D_j \varphi_i + B_i(x, u, \nabla u) \varphi_i\} dx = \int_{\Omega} f \cdot \varphi dx \quad (2.7)$$

valid for all $\varphi \in V(\Omega)$. This means that u is a weak solution of the problem

$$-Q_i u = f_i \text{ in } \Omega, \quad 1 \leq i \leq m, \quad (2.8)$$

where $-Q_i u := -D_j A_{ij}(x, u, \nabla u) + B_i(x, u, \nabla u)$ with *boundary conditions* provided by the space $V(\Omega)$. In what follows, we denote $A = (A_{ij}); 1 \leq i \leq m, 1 \leq j \leq N$, $A_i = (A_{ij}); 1 \leq j \leq N$ and $B = (B_i); 1 \leq i \leq m$.

2.1. Examples of variational problems $P(f, V(\Omega))$

In this subsection, we give typical examples of the problem $P(f, V(\Omega))$. In particular, the relation (2.9) below is the *differential equation* ($m = 1$); (2.17) is the linear *differential system* ($m = N$); (2.21) corresponds to the *nonlinear differential system* ($m = N$); (2.26) corresponds to the linear differential system ($m \neq N$); (2.10), (2.16) provide *Dirichlet boundary conditions*: (2.13), (2.20).

2.1.1. Elliptic boundary value problem

Consider the elliptic boundary value problem on Ω with the Dirichlet condition:

$$\begin{aligned} -D_i a_{ij}(x) D_j u + bu &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \quad (2.9)$$

We put $E(x, z, p) = (a_{ij} p_i p_j + bz^2)/2$ with $a_{ij} = a_{ji}$, and

$$V(\Omega) = \{v \in W^{1,2}(\Omega); v = 0 \text{ on } \Gamma\}. \quad (2.10)$$

Then the problem $P(f, V(\Omega))$ provides the weak solution of (2.9). In this case $m = 1$, and in (2.5), (2.7), we have

$$\begin{aligned} A_{1j}(x, z, p) &= D_{p_j} E(x, z, p) = a_{ij} p_i \text{ for } 1 \leq j \leq N, \\ B_1(x, z, p) &= D_z E(x, z, p) = b. \end{aligned}$$

If a_{ij} , b are smooth functions defined on \mathbb{R}^N , and there is a positive number α such that

$$\begin{aligned} a_{ij}(x) p_i p_j &\geq \alpha |p|^2, \quad \forall x \in \mathbb{R}^N, p \in \mathbb{R}^N, \\ b(x) &\geq 0 \quad \forall x \in \mathbb{R}^N, \end{aligned} \quad (2.11)$$

then the problem $P(f, V(\Omega))$ is uniquely solvable.

Next, we consider the mixed boundary value problem with the equation (2.9), which is given by the following boundary conditions

$$\begin{cases} u = 0 & \text{on } \Gamma_D, \\ \partial u / \partial n_A (:= n_i a_{ij} D_j u) = 0 & \text{on } \Gamma_N, \end{cases} \quad (2.12)$$

where $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ and $\Gamma_D \cap \Gamma_N = \emptyset$. In this case, we introduce the space

$$V(\Omega) := \{v \in W^{1,2}(\Omega); v = 0 \text{ on } \Gamma_D\}. \quad (2.13)$$

Then the problem $P(f, V(\Omega))$ is uniquely solvable, provided that the measure of Γ_D is nonzero.

2.1.2. Linear elasticity

We consider the *linear elastic field* (the case $m = N$) which is given by the following formulae

$$E(x, z, p) = \frac{1}{2} \sigma_{ij}(x, p) e_{ij}(p), \quad (2.14)$$

$$A_{ij}(x, p) = \sigma_{ij}(x, p) := c_{ijkl}(x) e_{ij}(p), \quad (2.15)$$

$$e_{ii}(p) = (p_{i,i} + p_{i,i}) / 2 \text{ for } 1 \leq i, j \leq n,$$

The variational problem $P(f; V(\Omega))$ corresponding to the space

$$V(\Omega) = \{v \in W^{1,2}(\Omega)^N; v = 0 \text{ on } \partial\Omega\}, \tag{2.16}$$

implies the boundary value problem

$$-D_j c_{ijkl} e_{kl}(u) = f_i \text{ in } \Omega, \quad i = 1, \dots, N, \tag{2.17}$$

$$u = 0 \text{ on } \partial\Omega. \tag{2.18}$$

For uniqueness of the solution to the problem $P(f, V(\Omega))$, we assume that the elements c_{ijkl} are smooth function defined on \mathbb{R}^N and satisfy the following inequality

$$c_{ijkl} \xi_{ij} \xi_{jk} \geq \alpha \xi_{ij} \xi_{ij} \quad \text{for all } \xi_{ij} \in \mathbb{R}^1; \alpha > 0. \tag{2.19}$$

The elastic field corresponding to the mixed boundary condition is given by the space

$$V(\Omega) = \{v \in W^{1,2}(\Omega)^N; v = 0 \text{ on } \Gamma_D\}. \tag{2.20}$$

2.1.3. Elasto-plasticity

Consider the case corresponding to elasto-plasticity (see Nečas and Hlaváček, 1981, Chapter 8)

$$E(v) = k\theta^2(v)/2 + \int_0^{\Gamma(v,v)} \mu(x, \sigma) d\sigma, \tag{2.21}$$

where $\theta(v) = \text{div } v$, $\Gamma(v, w) = -2\theta(v)\theta(w)/3 + 2e_{ij}(v)e_{ij}(w)$. Here $e_{ij}(v)$ denotes the infinitesimal strain tensor, i.e., $e_{ij}(v) = \{D_j v_i + D_i v_j\}/2$. To apply the result obtained in this paper, we require E to satisfy the following conditions. Assume that $k \in C^2(\mathbb{R}^N)$, $\mu \in C^2(\mathbb{R}^N \times [0, \infty))$, and suppose the existence of constants $k_0 > 0$, $k_1 > 0$ and $\mu_0 > 0$, $\mu_1 > 0$ such that

$$0 < k_0 \leq k(x) \leq k_1 < \infty, \quad |\nabla k(x)| \leq k_1 < \infty \text{ for all } x \in \mathbb{R}^N, \tag{2.22}$$

$$0 < \mu_0 \leq \mu(x, s) \leq 3k(x)/2, \tag{2.23}$$

$$|\nabla_x \mu(x, s)| \leq \mu_1 < \infty, \text{ for all } x \in \mathbb{R}^N \text{ and } s \geq 0.$$

We also assume that the inequalities

$$0 < \xi \leq \mu(x, s) + 2(\partial\mu(x, s)/\partial s)s \leq \xi_1 \tag{2.24}$$

hold with some constants ξ_1, ξ .

Let the space $V(\Omega)$ be chosen like in (2.16). Then, the problem $P(f, V(\Omega))$ implies the equation (2.17) with nonlinear Hooke's tensor

$$c_{ijkl} = \left(k - \frac{3}{2}\mu(\Gamma^2(u)) \right) \delta_{ij}\delta_{kl} + \mu(\Gamma^2(u))(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{2.25}$$

Here $\Gamma^2(u) = \Gamma(u, u)$, δ_{ij} are the elements of Kronecker's symbol, and (2.25) is derived from the consideration of generalized Hooke's law (see Nečas and

2.1.4. Micropolar elasticity

Considering the case $N \neq m$, we introduce micropolar continuum mechanics (see Eringen, 1968). For this material, $N = 3$, $m = 6$. Let $\tilde{u} = (v, \omega)$ be six-component vectors, and let $v = (v_1, v_2, v_3)$, $\omega = (\omega_1, \omega_2, \omega_3)$ be defined in the domain $\Omega \subset \mathbb{R}^3$. The linearized approximation is called the couple-stress theory, see Kupradze, Gegelia, Basheleishvili and Burchuladze (1979, p. 147), in which

$$\begin{aligned}
 2E(\tilde{u}) &= \{(3\lambda + 2\mu)/3\}|\operatorname{div} v|^2 \\
 &+ (\mu/2) \sum_{i,j} |D_j v_i + D_i v_j - (2/3)\delta_{ij} \operatorname{div} v|^2 \\
 &+ (\alpha/2) \sum_{i,j} |D_j v_i - D_i v_j + 2\varepsilon_{kji}\omega_k|^2 + \{(3\epsilon + 2\nu)/3\}|\operatorname{div} \omega|^2 \\
 &+ (v/2) \sum_{i,j} |D_i \omega_j + D_j \omega_i - (2/3)\delta_{ij} \operatorname{div} \omega|^2 \\
 &+ (\beta/2) \sum_{i,j} |D_j \omega_i - D_i \omega_j|^2,
 \end{aligned} \tag{2.26}$$

where $\lambda, \mu, \alpha, \epsilon, \nu, \beta$ are constants satisfying the conditions

$$\mu > 0, 3\lambda + 2\mu > 0, \alpha > 0, \nu > 0, 3\epsilon + 2\nu > 0, \beta > 0,$$

and ε_{kij} is the permutation tensor. If displacements and rotations are zero on Γ_D and the couple stresses are zero on Γ_N , then

$$V(\Omega) = \{\tilde{u} = (v, \omega) \in W^{1,2}(\Omega)^6 \mid \tilde{u} = 0 \text{ on } \Gamma_D\}. \tag{2.27}$$

From Ohtsuka (1985) the following estimate for $\tilde{u} \in V(\Omega)$ is obtained,

$$\int_{\Omega} E(\tilde{u}) dx \geq C_3 \|\tilde{u}\|_{W^{1,2}(\Omega)^6}^2 \tag{2.28}$$

with a constant $C_3 > 0$ independent of \tilde{u} . Under the conditions (2.26)–(2.27), the variational problem $P(f, V(\Omega))$ implies the following boundary value problem with $f = (f_1, f_2, f_3)$, $f_m = (f_4, f_5, f_6)$, for $i = 1, 2, 3$,

$$\begin{cases}
 (\mu + \alpha)\Delta u + (\lambda + \mu - \alpha)\operatorname{grad} \operatorname{div} u + 2\alpha \operatorname{rot} \omega = -f & \text{in } \Omega, \\
 (v + \beta)\Delta \omega + (\epsilon + v - \beta)\operatorname{grad} \operatorname{div} \omega \\
 \quad \quad \quad + \operatorname{rot} u - 4\alpha \omega = -f_m & \text{in } \Omega, \\
 u = 0, \omega = 0 & \text{on } \Gamma_D, \\
 \lambda n_i \operatorname{div} u + (\mu + \alpha)n_j D_i u_j + (\mu - \alpha)n_j D_j u_i \\
 \quad \quad \quad - 2\alpha \varepsilon_{ijk} n_j \omega_k = 0 & \text{on } \Gamma_N, \\
 \epsilon n_i \operatorname{div} \omega + (\mu + \beta)n_j D_i \omega_j + (\mu - \beta)n_j D_j \omega_i = 0 & \text{on } \Gamma_N.
 \end{cases} \tag{2.29}$$

In this paper, we treat the Dirichlet and mixed boundary conditions. As a matter of convenience we introduce the notations

$$V^D(\Omega) := \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma\}, \tag{2.30}$$

3. Perturbation $P(f_\tau, V_\tau(\Omega(\tau)))$ of problems $P(f, V(\Omega))$

Let a domain Ω satisfy the conditions shown at the beginning of Section 2. We consider the family of problems $P(f_\tau, V_\tau(\Omega(\tau)))$, $0 < \tau \leq T$. Let $\{\Omega(\tau)\}$, $0 < \tau \leq T$, be a family of domains in \mathbb{R}^N and, for each τ , $\Omega(\tau)$ and Ω be connected by the mapping Φ_τ . We assume the fulfilment of the following hypotheses.

(H1) The map $\Phi_\tau : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is one-to-one, $\Phi_\tau(\Omega(\tau)) = \Omega$, and Φ_τ has the positive Jacobian, $\Phi_0(x) = x$ for all $x \in \mathbb{R}^N$.

(H2) $\tau \mapsto \Phi_\tau \in C^2([0, T], W^{2,\infty}(\mathbb{R}^N)^N)$. Here $C^2([0, T], W^{2,\infty}(\mathbb{R}^N)^N)$ stands for the space of twice continuously differentiable functions with respect to τ , $0 \leq \tau \leq T$, with the values in $W^{2,\infty}(\mathbb{R}^N)^N$.

By **(H1)** and **(H2)**, the map $v(y) \mapsto \Phi_\tau^* v(x) := v(\Phi_\tau(x))$ is one-to-one from $W^{1,2}(\Omega)^m$, $y \in \Omega$, $x \in \Omega(\tau)$, onto $W^{1,2}(\Omega(\tau))^m$ and satisfies the estimate

$$C^{-1} \|v\|_{1,\Omega} \leq \|\Phi_\tau^* v\|_{1,\Omega(\tau)} \leq C \|v\|_{1,\Omega} \quad \text{for all } v \in W^{1,2}(\Omega)^m \quad (3.1)$$

with a constant C independent of τ, v . The next assumption concerns the perturbation of boundary conditions given by $V_\tau(\Omega(\tau))$, namely,

(H3) The map $\Phi_\tau^* : W^{1,2}(\Omega)^m \rightarrow W^{1,2}(\Omega(\tau))^m$ is one-to-one from $V(\Omega)$ onto $V_\tau(\Omega(\tau))$.

In Sections 3.1, 3.2 we will give examples of Φ_τ . As for the crack theory, suitable examples can be found in Mazya and Nazarov (1987); Khludnev and Sokolowski (1999).

Under the hypotheses **(H1)** - **(H3)** and for $f_\tau \in L^2(\Omega(\tau))^m$, we consider the following variational problem $P(f_\tau, V_\tau(\Omega(\tau)))$ with the parameter τ .

PROBLEM $P(f_\tau, V_\tau(\Omega(\tau)))$: For a given $f_\tau \in L^2(\Omega(\tau))^m$, find an element $u(\tau) \in V_\tau(\Omega(\tau))$ such that

$$\mathcal{E}_\tau(u(\tau); f_\tau, \Omega(\tau)) \leq \mathcal{E}_\tau(v; f_\tau, \Omega(\tau)) \quad \text{for all } v \in V_\tau(\Omega(\tau)). \quad (3.2)$$

Here

$$\mathcal{E}_\tau(v; f_\tau, \Omega(\tau)) = \int_{\Omega(\tau)} \{E(x, v, \nabla v) - f_\tau \cdot v\} dx, \quad v \in V_\tau(\Omega(\tau)).$$

The solution $u(\tau)$ of the problem $P(f_\tau, V_\tau(\Omega(\tau)))$ satisfies the following identity for each τ ,

$$\int_{\Omega(\tau)} \delta E(u(\tau))[v] dx = \int_{\Omega(\tau)} f_\tau \cdot v dx \quad \text{for all } v \in V_\tau(\Omega(\tau)). \quad (3.3)$$

We consider $f_\tau \in L^2(\Omega(\tau))^m$ for the following two cases:

$$\text{supp } f \subset \Omega(\tau) \text{ for } 0 \leq \tau \leq T. \quad (3.4)$$

In this case, we can omit the subscript τ .

$$f_\tau \text{ is the restriction } f|_{\Omega(\tau)} \text{ of } f \in L^2(\mathbb{R}^N)^m. \quad (3.5)$$

For the Dirichlet conditions the spaces $V_\tau(\Omega(\tau))$ satisfy **(H3)**, if

$$V_\tau(\Omega(\tau)) = \{v; v = 0 \text{ on } \Gamma(\tau)\}, \Gamma(\tau) = \Phi_\tau^{-1}(\Gamma). \tag{3.6}$$

3.1. Deformation in the normal direction

Let us consider the smooth boundary Γ . Let $U_\delta(\Gamma) \subset \mathbb{R}^N$ be an open neighborhood of the surface Γ , consisting of points whose distance from Γ is less than δ .

We can take δ such that for each point $x \in U_\delta(\Gamma)$ there exists a unique point $\mathcal{P}(x) \in \Gamma$ satisfying the condition $|x - \mathcal{P}(x)| = \min_{y \in \Gamma} |x - y|$. Let h be a C^∞ -function defined on Γ . We consider the surface $\Gamma_{\tau,h}$ defined by the formula

$$\Gamma_{\tau,h} = \{x + \tau h(x)n(x) \mid x \in \Gamma\},$$

and let $\Omega_{\tau,h}$ be a domain with the boundary $\Gamma_{\tau,h}$.

Let β be a function in $C_0^\infty(U_\delta(\Gamma))$ such that $\beta \geq 0, \beta = 1$ near Γ , where $C_0^\infty(U_\delta(\Gamma))$ is the set of smooth functions with compact support in $U_\delta(\Gamma)$. Setting

$$\Phi_{\tau,h}(x) = \begin{cases} x - \tau\beta(x)h(\mathcal{P}(x))n(\mathcal{P}(x)) & \text{for } x \in U_\delta(\Gamma), \\ x & \text{for } x \in \mathbb{R}^3 \setminus U_\delta(\Gamma), \end{cases} \tag{3.7}$$

we get the C^∞ -diffeomorphisms $\Phi_{\tau,h}$ from \mathbb{R}^3 onto \mathbb{R}^3 satisfying the condition $\Phi_{\tau,h}(\Omega_{\tau,h}) = \Omega$. In this case, the vector field X_h can be defined by the formula

$$X_h(x) = \frac{d}{d\tau} \Phi_{\tau,h}(x) \Big|_{\tau=0} = -\beta(x)h(\mathcal{P}(x))n(\mathcal{P}(x)). \tag{3.8}$$

The maps $\Phi_{\tau,h}$ satisfy the assumptions **(H1)**-**(H3)** since

$$V_\tau^D(\Omega(\tau)) := \{v \in W^{1,2}(\Omega(\tau))^m; v = 0 \text{ on } \partial\Omega(\tau)\}, \tag{3.9}$$

$$V_\tau^M(\Omega(\tau)) := \{v \in W^{1,2}(\Omega(\tau))^m; v = 0 \text{ on } \Gamma_D(\tau)\}, \tag{3.10}$$

$$\Gamma_D(\tau) := \Phi_\tau^{-1}(\Gamma_D).$$

3.2. Deformation in the tangential direction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2 -smooth boundary $\partial\Omega$. Assume that $\gamma = \overline{\Gamma_D} \cap \overline{\Gamma_N}$ is a smooth curve. There exists a C^2 -diffeomorphism F from $\gamma \times (-1, 1)^2$ onto a neighborhood $U(\gamma)$ of γ such that

$$\gamma = \{F(\lambda, 0, 0); \lambda \in \gamma\}, \tag{3.11}$$

$$\partial\Omega \cap U(\gamma) = \{F(\lambda, \xi, 0); \lambda \in \gamma, \xi \in (-1, 1)\},$$

$$\Omega \cap U(\gamma) = \{F(\lambda, \xi_1, \xi_2); \lambda \in \gamma, \xi_1 \in (-1, 1), \xi_2 \in (0, 1)\}.$$

We consider the curve γ of perturbation on $\partial\Omega$ and suppose that there are

curves $\gamma(\tau) := \psi_\tau(\gamma)$, $0 \leq \tau \leq T$, is the interface $\overline{\Gamma_D(\tau)} \cap \overline{\Gamma_N(\tau)}$ between the Dirichlet boundary $\Gamma_D(\tau)$ and the Neumann boundary $\Gamma_N(\tau)$. Then, we can prove existence of the function h_τ defined on γ as follows

$$\gamma(\tau) = \{F(\lambda, h_\tau(\lambda), 0); \lambda \in \gamma\}. \tag{3.12}$$

We put $(\lambda(x), \xi_1(x), \xi_2(x)) := F^{-1}(x)$ and define

$$\Phi_{\tau,\gamma}(x) := \begin{cases} F(\lambda(x), \xi_1(x) - \tau\beta(x)h_\tau(\lambda(x)), \xi_2(x)) & \text{for } x \in U(\gamma), \\ x & \text{for } x \notin U(\gamma), \end{cases} \tag{3.13}$$

where $\beta \in C_0^\infty(\mathbb{R}^N)$ satisfies the conditions

$$\beta(x) \geq 0, \beta \equiv 1 \text{ near } \gamma, \beta = 0 \text{ outside } U(\gamma). \tag{3.14}$$

Then, $\Phi_{\tau,\gamma}(x)$ satisfy the assumptions **(H1)**-**(H3)** since

$$V_\tau^M(\Omega(\tau)) := \{v \in W^{1,2}(\Omega(\tau))^m; v = 0 \text{ on } \Gamma_D(\tau)\}. \tag{3.15}$$

The vector field $X_\gamma = d\Phi_{\tau,\gamma}/d\tau|_{\tau=0}$ is constructed as follows. Consider the vector field $x_\phi := d\phi_\tau/d\tau|_{\tau=0}$ defined on γ . Next, consider the parallel displacement y_ϕ of x_ϕ along the geodesic curve on Γ normal to γ . Finally, consider the parallel displacement Y_ϕ of y_ϕ along outward normal direction to Γ . The vector field $-Y_\phi$ equals X_γ near γ .

4. Generalized J -integral (GJ -integral)

The generalized J -integral was proposed in Ohtsuka (1981); it expresses the crack extension force in the three-dimensional case. Later, in Ohtsuka (1982), the concept of GJ -integral was extended so that the theory is applicable for the sensitivity analysis of potential energy with respect to the perturbation of boundary and movement of interfaces in mixed boundary value problems. Moreover, the concept of GJ -integral includes J , L , M -integrals (see Ohtsuka 1981, Theorem 3.5).

DEFINITION 4.1 Let $\mathcal{O}(\mathbb{R}^N)$ be the set of domains ω with local Lipschitz property in \mathbb{R}^N . For a given f , let u be a solution of the problem $P(f, V(\Omega))$, and $X = (X_1, \dots, X_n)$ be a vector field defined on \mathbb{R}^N . We define GJ -integral $J_\omega(u, X)$ by the formula

$$J_\omega(u, X) := P_\omega(u, X) + R_\omega(u, X), \quad (\omega, X) \in \mathcal{O}(\mathbb{R}^N) \times W^{2,\infty}(\mathbb{R}^N)^N, \tag{4.1}$$

provided that the following integrals $P_\omega(u, X)$, $R_\omega(u, X)$ are finite,

$$P_\omega(u, X) = - \int_{\partial(\omega \cap \Omega)} \{E(u)(X \cdot n) - T(u) \cdot (X \cdot \nabla u)\} dS, \tag{4.2}$$

$$\begin{aligned} R_\omega(u, X) = & \int_{\omega \cap \Omega} \{(X \cdot \nabla_x)E(x, u, \nabla u) + f \cdot (X \cdot \nabla u)\} \\ & - \int_{\omega \cap \Omega} \{A_{ij}(x, u, \nabla u)(D_j X_k)(D_k u_i) - E(u)(\text{div } X)\} dx, \end{aligned} \tag{4.3}$$

4.1. Fundamental properties of the GJ -integral

PROPOSITION 4.2 *If*

$$\int_{\Omega} |E(x, u, \nabla u)|^2 dx < \infty, \quad \int_{\Omega} |D_{x_i} E(x, u, \nabla u)|^2 dx < \infty,$$

and $A_{ij}(x, u, \nabla u) \in L^2(\Omega)$, then by the Schwarz inequality, the integral $R_{\omega}(u, X)$ is finite.

For all examples considered in the subsection 2.1., the values of $R_{\omega}(u, X)$ are finite.

THEOREM 4.3 *Let u be a solution of the problem $P(f, V(\Omega))$. Assume that u is sufficiently smooth so that the divergence theorem*

$$\begin{aligned} \int_{\omega \cap \Omega} (X \cdot \nabla) E(u) dx &= \int_{\partial(\omega \cap \Omega)} E(u)(X \cdot n) dS \\ &- \int_{\omega \cap \Omega} E(u) \operatorname{div} X dx \end{aligned} \quad (4.4)$$

is applicable. If, moreover, Green's formula

$$\begin{aligned} \int_{\omega \cap \Omega} \delta E(u)[X \cdot \nabla u] dx &= \int_{\partial(\omega \cap \Omega)} T(u) \cdot (X \cdot \nabla u) dS \\ &- \int_{\omega \cap \Omega} \{ \operatorname{div} A(x, u, \nabla u) - B(x, u, \nabla u) \} (X \cdot \nabla u) dx \end{aligned} \quad (4.5)$$

holds for u , then we obtain

$$J_{\omega}(u, X) = 0 \text{ for all vector fields } X \in W^{2,\infty}(\mathbb{R}^N)^N. \quad (4.6)$$

Here the elements of $\operatorname{div} A(x, u, \nabla u)$ are $D_j A_{ij}(x, u, \nabla u)$, $i = 1, \dots, m$.

Proof. From the chain rule, we have

$$\begin{aligned} (X \cdot \nabla) E(u) &= (X \cdot \nabla_x) E(x, u, \nabla u) \\ &+ \delta E(u)[X \cdot \nabla u] - A_{ij}(x, u, \nabla u)(D_j X_k) D_k u_i. \end{aligned}$$

Next, from (4.4) and (4.5), it is easy to obtain

$$\begin{aligned} &\int_{\partial(\omega \cap \Omega)} E(u)(X \cdot n) dS \\ &- \int_{\omega \cap \Omega} \{ E(u) \operatorname{div} X + (X \cdot \nabla_x) E(x, u, \nabla u) - A_{ij}(x, u, \nabla u)(D_j X_k) D_k u_i \} dx \\ &= \int_{\partial(\omega \cap \Omega)} T(u) \cdot (X \cdot \nabla) u dS + \int_{\omega \cap \Omega} f \cdot (X \cdot \nabla u) dx. \end{aligned}$$

Here we use the equation $-\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) = f$ holding in Ω almost everywhere due to the regularity assumption on u . \blacksquare

COROLLARY 4.4 *Let u be a solution of the problem $P(f, V(\Omega))$. For any domains $\omega_1 \subset \omega_2 \subset \Omega$ in \mathbb{R}^N , we denote by $u|_{\omega_2 \setminus \overline{\omega_1}}$ the restriction of u to $\omega_2 \setminus \overline{\omega_1}$. If $u|_{\omega_2 \setminus \overline{\omega_1}}$ is smooth enough as required in Theorem 4.3, then*

$$J_{\omega_1}(u; X) = J_{\omega_2}(u; X). \quad (4.7)$$

Proof. The result (4.7) is easily obtained from (4.6) by replacing ω with $\omega_2 \setminus \overline{\omega_1}$. ■

4.2. Examples

We give below the forms of the GJ -integral for the problems considered in Section 2..

4.2.1. Poisson equation

$$P_\omega(u, X) = - \int_{\partial(\omega \cap \Omega)} \left\{ \frac{1}{2} |\nabla u|^2 (X \cdot n) - \frac{\partial u}{\partial n} (X \cdot \nabla u) \right\} dS,$$

$$R_\omega(u, X) = \int_{\omega \cap \Omega} \left\{ f(X \cdot \nabla u) - (\nabla u \cdot \nabla X_k) D_k u + \frac{1}{2} |\nabla u|^2 \operatorname{div} X \right\} dx.$$

4.2.2. Elliptic equations

$$P_\omega(u, X) = - \int_{\partial(\omega \cap \Omega)} \left\{ \frac{1}{2} (a_{ij} D_j u D_i u + b u^2) (X \cdot n) \right. \\ \left. - (n_i a_{ij} D_j u) (X \cdot \nabla u) \right\} dS,$$

$$R_\omega(u, X) = \int_{\omega \cap \Omega} \left\{ \frac{1}{2} ((X \cdot \nabla a_{ij}) D_j u D_i u + (X \cdot \nabla b) u^2) + f(X \cdot \nabla u) \right. \\ \left. - (a_{ij} D_j u D_i X_k) D_k u + \frac{1}{2} (a_{ij} D_j u D_i u + b u^2) \operatorname{div} X \right\} dx.$$

4.2.3. Linear elasticity

$$P_\omega(u, X) = - \int_{\partial(\omega \cap \Omega)} \left\{ \frac{1}{2} \sigma_{ij}(u) e_{ij}(u) (X \cdot n) - \sigma_{ij} n_j (X \cdot \nabla u_i) \right\} dS,$$

$$R_\omega(u, X) = \int_{\omega \cap \Omega} \left\{ \frac{1}{2} (X \cdot \nabla c_{ijkl}) e_{kl}(u) e_{ij}(u) + f_i (X \cdot \nabla u_i) \right. \\ \left. - \sigma_{::i}(u) D_i X_k D_k u + \frac{1}{2} \sigma_{::i}(u) e_{::i}(u) \operatorname{div} X \right\} dx$$

4.2.4. Elasto-plasticity

$$\begin{aligned}
 P_\omega(u, X) &= - \int_{\partial(\omega \cap \Omega)} \{E(u)(X \cdot n) - (n_j c_{ijkl}(u) e_{kl}(u))(X \cdot \nabla u_i)\} dS, \\
 R_\omega(u, X) &= \int_{\omega \cap \Omega} \left\{ (X \cdot \nabla k)(\operatorname{div} u)^2 / 2 + \int_0^{\Gamma(u, u)} X \cdot \nabla_x \mu(x, \sigma) d\sigma \right. \\
 &\quad \left. + f_i(X \cdot \nabla u_i) - c_{ijkl}(u) e_{kl}(u) D_j X_p D_p u_i + E(u) \operatorname{div} X \right\} dx,
 \end{aligned}$$

where c_{ijkl} are given by (2.25).

4.2.5. Micro-polar elasticity

$$\begin{aligned}
 P_\omega(u, X) &= - \int_{\partial(\omega \cap \Omega)} \{E(u)(X \cdot n) - (\sigma_{E,ij} n_j(u, \omega))(X \cdot \nabla u_i) \\
 &\quad - (\sigma_{R,ij}(u, \omega) n_j)(X \cdot \nabla \omega_i)\} dS, \\
 R_\omega(u, X) &= \int_{\omega \cap \Omega} \{f_i(X \cdot \nabla u_i) + \sigma_{E,ij}(u) D_j X_p D_p u_i \\
 &\quad - \sigma_{R,ij}(u) D_j X_p D_p \omega_i + E(u) \operatorname{div} X\} dx,
 \end{aligned}$$

where

$$\begin{aligned}
 \sigma_{E,ij}(u, \omega) &:= \lambda \delta_{ij} \operatorname{div} u + (\mu + \alpha) D_i u_j + (\mu - \alpha) D_j u_i - 2\alpha \varepsilon_{ijk} \omega_k, \\
 \sigma_{R,ij}(u, \omega) &:= \epsilon \delta_{ij} \operatorname{div} \omega + (v + \beta) D_i \omega_j + (v - \beta) D_j \omega_i.
 \end{aligned}$$

5. Variation of potential energy functional

In this section, we calculate the variation of potential energy functional $d\mathcal{E}(u(\tau); f_{R,\tau}, \Omega(\tau))/d\tau|_{\tau=0}$ when $\tau \mapsto \Phi_\tau$ satisfy (H1)-(H3) and $f(\tau) = f|_{\Omega(\tau)}$, $f \in W^{1,2}(\mathbb{R}^N)^N$, under the following conditions (5.1)-(5.5):

$$\begin{aligned}
 \text{The functional } v &\mapsto \int_{\Omega(\tau)} E(v) dx \\
 &\text{is Gateaux differentiable on } V(\Omega(\tau)). \tag{5.1}
 \end{aligned}$$

There is a constant $M_0 > 0$ such that

$$\begin{aligned}
 \|\delta E(x, v_1, \nabla v_1) - \delta E(x, v_2, \nabla v_2)\|_{0, \Omega(\tau)} &\leq M_0 \|v_1 - v_2\|_{1, \Omega(\tau)} \tag{5.2} \\
 &\text{for all } v_1, v_2 \in V(\Omega(\tau)).
 \end{aligned}$$

There exists a constant $M_1 > 0$ such that

$$\begin{aligned}
 |\nabla_x A_{ij}(x, z, p)|, |\nabla_x B_i(x, z, p)| &\leq M_1 (|z|^2 + |p|^2)^{1/2} \tag{5.3} \\
 |A_{ij}(x, z, p) - A_{ij}(x, \zeta, q)| &\leq M_1 (|z - \zeta|^2 + |p - q|^2)^{1/2}
 \end{aligned}$$

There is a constant $M_2 > 0$ such that

$$\begin{aligned} \nabla_p \nabla_z E(x, z, p) \xi \eta &\leq M_2 |\xi| |\eta| \\ \nabla_p \nabla_p E(x, z, p) \eta \eta' &\leq M_2 |\eta| |\eta'| \end{aligned} \tag{5.4}$$

for all $x \in \mathbb{R}^N$, $\xi \in \mathbb{R}^m$, $\eta, \eta' \in \mathbb{R}^{mN}$.

There is a constant $\alpha_0 > 0$ independent of τ such that

$$\begin{aligned} \int_{\Omega(\tau)} \{ \delta E(v+w)[w] - \delta E(v)[w] \} dx &\geq \alpha_0 \|w\|_{1, \Omega(\tau)}^2 \\ \text{for all } v, w \in V(\Omega(\tau)). \end{aligned} \tag{5.5}$$

REMARK 5.1 *The examples considered in Section 2. satisfy the conditions (5.1)-(5.5) (see Ohtsuka 1985, p. 344 for elasto-plasticity).*

PROPOSITION 5.1 *Under the assumptions (5.1)-(5.5), the problem $P(f_{R,\tau}, V_\tau(\Omega(\tau)))$ is uniquely solvable for each $f \in L^2(\mathbb{R}^N)^N$, and the functional $u \mapsto R_\Omega(u, X)$ is bounded in $V(\Omega)$.*

Note that inequality (5.5) implies the coercivity and weak lower semicontinuity of $\mathcal{E}(v; f_\tau, \Omega(\tau))$ on the space $V_\tau(\Omega(\tau))$ (see Nečas and Hlaváček, 1981), and the statement follows.

We calculate the derivative $d\mathcal{E}(u(\tau); f_{R,\tau}, \Omega(\tau))/d\tau |_{\tau=0}$ in Theorem 5.4 below. First, we provide some lemmas.

LEMMA 5.2 *Under the assumptions (H1), (H2), we have the following estimate for $f \in W^{1,2}(\mathbb{R}^N)$,*

$$\| \Phi_\tau^* f_R - f_{R,\tau} \|_{0, \Omega(\tau)} \leq C_\tau \|f\|_{1, \mathbb{R}^N} \tag{5.6}$$

with a constant $C > 0$ independent of τ .

Proof. Since the space $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,2}(\mathbb{R}^N)$, it suffices to prove (5.6) for $f \in C_0^\infty(\mathbb{R}^N)$. By the formula

$$\Phi_\tau^* f(x) - f(x) = \int_0^\tau \frac{d}{ds} \Phi_s^* f(x) ds = \int_0^\tau \left(\frac{d}{ds} \Phi_{s,i} \right) \left(\frac{\partial}{\partial x_i} f \right) (\Phi_s(x)) ds$$

and the Schwarz inequality, we get the estimate

$$| \Phi_\tau^* f(x) - f(x) |^2 \leq \tau \int_0^\tau \left| \left(\frac{d}{ds} \Phi_{s,i} \right) \left(\frac{\partial}{\partial x_i} f \right) (\Phi_s(x)) \right|^2 ds$$

with a constant $C_1 > 0$ independent of τ . Integrating next both sides over $\Omega(\tau)$ and using Fubini's theorem, we obtain

$$\begin{aligned} \int_{\Omega(\tau)} |\Phi_\tau^* f - f|^2 &\leq C_1 \tau \int_{\Omega(\tau)} \int_0^\tau |\Phi_s^*(\nabla f)|^2 ds dx \\ &= C_1 \tau \int_0^\tau \int_{\mathbb{R}^N} |\Phi_s^*(\nabla f)|^2 dx ds \leq C_2 \tau \int_0^\tau \|f\|_{1,\mathbb{R}^N}^2 ds = C_2 \tau^2 \|f\|_{1,\mathbb{R}^N}^2. \end{aligned}$$

Taking square root both sides, we arrive at (5.6). ■

LEMMA 5.3 *Let $u(\tau)$ be the solution of the problem $P(f_{R,\tau}, \Omega(\tau))$, and $u^*(\tau)$ be the solution of the problem $P(\Phi_\tau^*(f|_\Omega), \Omega(\tau))$ for a given $f \in W^{1,2}(\mathbb{R}^N)^m$. Under the hypotheses **(H1)**, **(H2)**, **(H3)** and (5.1) - (5.5), we have the estimate*

$$\|u(\tau) - u^*(\tau)\|_{1,\Omega(\tau)} \leq C\tau \|f\|_{1,\mathbb{R}^N} \tag{5.7}$$

with a constant $C > 0$ independent of τ .

Proof. Since $u(\tau) - u^*(\tau) \in V_\tau(\Omega(\tau))$, from (5.5) we have

$$\begin{aligned} &\int_{\Omega(\tau)} \{\delta E(u^*(\tau))[u(\tau) - u^*(\tau)] - \delta E(u(\tau))[u(\tau) - u^*(\tau)]\} dx \\ &\geq \alpha_0 \|u(\tau) - u^*(\tau)\|_{1,\Omega(\tau)}^2. \end{aligned} \tag{5.8}$$

From the identity (2.4), by the Schwarz inequality and (5.6), we obtain

$$\begin{aligned} &\int_{\Omega(\tau)} \{\delta E(u^*(\tau))[u(\tau) - u^*(\tau)] - \delta E(u(\tau))[u(\tau) - u^*(\tau)]\} dx \\ &= \int_{\Omega(\tau)} (\Phi_\tau^*(f|_\Omega) - f) \cdot [u(\tau) - u^*(\tau)] dx \\ &\leq C_3 \tau \|f\|_{1,\mathbb{R}^N} \|u(\tau) - u^*(\tau)\|_{1,\Omega(\tau)}. \end{aligned} \tag{5.9}$$

Combining (5.8) and (5.9), we derive (5.7). ■

THEOREM 5.4 *Let u be the solution of the problem $P((f|_\Omega), V(\Omega))$ with $f \in W^{1,2}(\mathbb{R}^N)^N$. Then, under the hypotheses **(H1)**-**(H3)** and (5.1)-(5.5) the following formula holds*

$$\begin{aligned} &d\mathcal{E}(u(\tau); f_{R,\tau}, \Omega(\tau))/d\tau |_{\tau=0} \\ &= -R_\Omega(u, X) + \int_\Omega [X \cdot \nabla(f \cdot u) + (f \cdot u)\text{div} X] dx, \end{aligned} \tag{5.10}$$

Proof. To simplify the notations, we prove (5.10) for the case of $m = 1$. The arguments used to prove the statement will be applicable for $m > 1$. For any $v \in V(\Omega)$, we put

$$a_\tau(\Phi_\tau^*u, \Phi_\tau^*v) = \int_{\Omega(\tau)} \delta E(x, \Phi_\tau^*u, \nabla(\Phi_\tau^*u))[\Phi_\tau^*v]dx.$$

Recall that $\tau \mapsto \Phi_\tau \in C^2([0, T], W^{2,\infty}(\mathbb{R}^N)^N)$. By a simple modification of the arguments used in Ohtsuka, 1985, Theorem 4.2, and by the use of the mean value theorem, we can derive

$$|a_\tau(\Phi_\tau^*u, \Phi_\tau^*v) - a_\tau(u^*(\tau), \Phi_\tau^*v)| \leq C_1\tau \|f\|_{1,\mathbb{R}^N} \|v\|_{1,\Omega} \tag{5.11}$$

for all $v \in V_\tau(\Omega(\tau))$ with a constant $C_1 > 0$ independent of τ .

By taking $\Phi_\tau^*u - u^*(\tau)$ as Φ_τ^*v in (5.11), from (5.5) we can prove the following inequality

$$\begin{aligned} \alpha_0 \|\Phi_\tau^*u - u^*(\tau)\|_{1,\Omega(\tau)}^2 &\leq a_\tau(\Phi_\tau^*u, \Phi_\tau^*u - u^*(\tau)) - a_\tau(u^*(\tau), \Phi_\tau^*u - u^*(\tau)) \\ &\leq C_1\tau \|f\|_{1,\mathbb{R}^N} \|\Phi_\tau^*u - u^*(\tau)\|_{1,\Omega(\tau)}. \end{aligned}$$

Together with (5.7), this implies the estimate

$$\|u(\tau) - \Phi_\tau^*u\|_{1,\Omega(\tau)} \leq \alpha_0^{-1}C_1\tau \|f\|_{1,\mathbb{R}^N} \tag{5.12}$$

with the constants $\alpha_0, C_1 > 0$ independent of τ . Applying the mean value theorem to the function

$$s \mapsto \int_{\Omega(\tau)} E(\Phi_\tau^*u + s(u(\tau) - \Phi_\tau^*u))dx,$$

we obtain the following equality for each $\tau > 0$,

$$\begin{aligned} &\int_{\Omega(\tau)} E(u(\tau))dx \\ &= \int_{\Omega(\tau)} E(\Phi_\tau^*u)dx + \int_{\Omega(\tau)} \delta E(\Phi_\tau^*u)[u(\tau) - \Phi_\tau^*u]dx + \Pi_0(\zeta), \end{aligned} \tag{5.13}$$

with $\Pi_0(\zeta) = \int_{\Omega(\tau)} \{\delta E(\Phi_\tau^*u + \zeta(u(\tau) - \Phi_\tau^*u)) - \delta E(\Phi_\tau^*u)\} [u(\tau) - \Phi_\tau^*u]dx$, and $\zeta \in (0, 1)$. By the assumption (5.2), there is a constant $C_2 = C_2(u)$ such that

$$|\Pi_0(\zeta)| \leq C_2 \|\Phi_\tau^*u - u(\tau)\|_{1,\Omega(\tau)}^2.$$

Since $u^*(\tau)$ is the solution of the problem $P(\Phi_\tau^*(f|\Omega), V_\tau(\Omega(\tau)))$, we have the equalities

$$\begin{aligned} \int_{\Omega(\tau)} f_{R,\tau}u(\tau)dx &= \int_{\Omega(\tau)} \Phi_\tau^*(f|\Omega)(u(\tau) - \Phi_\tau^*u)dx \\ &+ \int_{\Omega(\tau)} (f_{R,\tau} - \Phi_\tau^*(f|\Omega))u(\tau)dx + \int_{\Omega(\tau)} (\Phi_\tau^*(f|\Omega))(\Phi_\tau^*u)dx \end{aligned}$$

$$\begin{aligned}
 &= \int_{\Omega(\tau)} \delta E(u^*(\tau))[u(\tau) - \Phi_\tau^* u] dx + \int_{\Omega(\tau)} (f_{R,\tau} - \Phi_\tau^*(f|\Omega)) u(\tau) dx \\
 &+ \int_{\Omega(\tau)} (\Phi_\tau^*(f|\Omega)) (\Phi_\tau^* u) dx.
 \end{aligned}$$

Then, by letting

$$\begin{aligned}
 &\int_{\Omega(\tau)} (f_{R,\tau} - \Phi_\tau^*(f|\Omega)) u(\tau) dx = \int_{\Omega(\tau)} (f_{R,\tau} - \Phi_\tau^*(f|\Omega)) \Phi_\tau^* u dx \\
 &+ \int_{\Omega(\tau)} (f_{R,\tau} - \Phi_\tau^*(f|\Omega)) (u(\tau) - \Phi_\tau^* u) dx,
 \end{aligned}$$

and taking into account (5.6) and (5.7) we have the following equality

$$\mathcal{E}(u(\tau); f_{R,\tau}, \Omega(\tau)) = \int_{\Omega(\tau)} \{E(\Phi_\tau^* u) - f_{R,\tau}(\Phi_\tau^* u)\} dx + o(\tau). \tag{5.14}$$

Here $o(\tau)/\tau \rightarrow 0$ as $\tau \rightarrow 0$. By \mathcal{H}_τ we denote the matrix whose components $h_{i,j}$ are given by the formulae

$$h_{i,j} := D_j \Phi_{\tau,i} - \delta_{i,j} \in C^1([0, T], L^\infty(\mathbb{R}^N)^N), \quad i, j = 1, 2, \dots, n.$$

Setting

$$\Pi_1(s) = \int_{\Omega(\tau)} \{E(x, \Phi_\tau^* u, \Phi_\tau^*(\nabla u) + s\mathcal{H}_\tau \Phi_\tau^*(\nabla u))\} dx$$

and using the mean value theorem, we derive following equality,

$$\begin{aligned}
 &\Pi_1(1) = \Pi_1(0) \\
 &+ \int_{\Omega(\tau)} A(x, \Phi_\tau^* u, \Phi_\tau^*(\nabla u) + \zeta \mathcal{H}_\tau \Phi_\tau^*(\nabla u)) \mathcal{H}_\tau \Phi_\tau^*(\nabla u) dx,
 \end{aligned} \tag{5.15}$$

with $\zeta \in (0, 1)$. We can next rewrite the first term of the right-hand side of (5.15) as

$$\begin{aligned}
 &\int_{\Omega(\tau)} E(x, \Phi_\tau^* u, \Phi_\tau^*(\nabla u)) dx = \int_{\Omega(\tau)} E(\Phi_\tau(x), \Phi_\tau^* u, \Phi_\tau^*(\nabla u)) dx \\
 &+ \int_{\Omega(\tau)} \{E(x, \Phi_\tau^* u, \Phi_\tau^*(\nabla u)) - E(\Phi_\tau(x), \Phi_\tau^* u, \Phi_\tau^*(\nabla u))\} dx.
 \end{aligned} \tag{5.16}$$

Replacing $\Phi_\tau(x)$ with y , we obtain

$$\begin{aligned}
 &\int_{\Omega(\tau)} E(\Phi_\tau(x), \Phi_\tau^* u, \Phi_\tau^*(\nabla u)) dx = \int_{\Omega} E(y, u, \nabla u) dy \\
 &+ \int_{\Omega} E(y, u, \nabla u) (|D_{\Phi_\tau}(\nabla \Phi_\tau^{-1})| - 1) dy
 \end{aligned} \tag{5.17}$$

Next, it is seen that

$$\begin{aligned}
 & \tau^{-1} \left\{ \int_{\Omega(\tau)} f \Phi_\tau^* u dx - \int_{\Omega} f u dx \right\} \\
 &= \tau^{-1} \int_{\Omega(\tau)} (f - \Phi_\tau^* f) \Phi_\tau^* u dx + \tau^{-1} \int_{\Omega} f u (\det(\nabla \Phi_\tau^{-1}) - 1) dx \\
 &\rightarrow - \int_{\Omega} \{ (X \cdot \nabla f) u + f u \operatorname{div} X \} dx \\
 &= - \int_{\Omega} \{ X \cdot \nabla (f u) + f u \operatorname{div} X \} dx + \int_{\Omega} f (X \cdot \nabla u) dx \tag{5.18}
 \end{aligned}$$

as $\tau \rightarrow 0$. Since $\mathcal{H}_0 = \{0\}$, we obtain the limit

$$\begin{aligned}
 & \lim_{\tau \rightarrow 0} \tau^{-1} \int_{\Omega(\tau)} \{ \mathbf{A}(x, \Phi_\tau^* u, \Phi_\tau^*(\nabla u) + \zeta \mathcal{H}_\tau \Phi_\tau^*(\nabla u)) \mathcal{H}_\tau \Phi_\tau^*(\nabla u) \} dx \\
 &= \int_{\Omega} A_{1j}(x, u, \nabla u) \left(\frac{d}{d\tau} h_{j,k} \Big|_{\tau=0} \right) D_k u dx, \tag{5.19}
 \end{aligned}$$

where $\frac{d}{d\tau} h_{j,k} \Big|_{\tau=0} = \left(\frac{d}{d\tau} D_k \Phi_j \right)_{\tau=0} = D_k X_j$. By collecting the terms (5.15), (5.17) and (5.16), it is easy to rewrite (5.14) as

$$\begin{aligned}
 \mathcal{E}(u(\tau); f_{R,\tau}, \Omega(\tau)) &= \mathcal{E}(u; f_R, \Omega) + \int_{\Omega} E(y, u, \nabla u) \{ \det(\nabla \Phi_\tau^{-1}) - 1 \} dy \\
 &+ \int_{\Omega(\tau)} \{ \mathbf{A}(x, \Phi_\tau^* u, \Phi_\tau^*(\nabla u) + \zeta \mathcal{H}_\tau \Phi_\tau^*(\nabla u)) ((\mathcal{H}_\tau - \mathcal{H}_0) \Phi_\tau^*(\nabla u)) \} dx \\
 &+ \int_{\Omega(\tau)} \{ E(x, \Phi_\tau^* u, \Phi_\tau^*(\nabla u)) - E(\Phi_\tau(x), \Phi_\tau^* u, \Phi_\tau^*(\nabla u)) \} dx \\
 &- \left\{ \int_{\Omega(\tau)} f \Phi_\tau^* u dx - \int_{\Omega} f u dx \right\} + o(\tau)
 \end{aligned}$$

with some $0 < \zeta < \tau$. Therefore the formulae (5.13), (5.18), (5.19) provide for the proof of Theorem 5.4, i.e.,

$$\begin{aligned}
 & \lim_{\tau \rightarrow 0} \tau^{-1} \{ \mathcal{E}(u(\tau); f_{R,\tau}, \Omega(\tau)) - \mathcal{E}(u; f, \Omega) \} \\
 &= -R_\Omega(u, X) + \int_{\Omega} \{ f u \operatorname{div} X + X \cdot \nabla (f u) \} dx.
 \end{aligned}$$

COROLLARY 5.5 *If $\operatorname{supp} f$ is contained in Ω , then $f(\tau) = f$ for small τ , and we have*

$$\frac{d}{d\tau} \mathcal{E}(u(\tau); f, \Omega(\tau)) \Big|_{\tau=0} = -R_\Omega(u, X). \tag{5.20}$$

COROLLARY 5.6 *If $\text{supp } f$ is contained in Ω , and u is sufficiently smooth, as in Theorem 4.3, then*

$$\begin{aligned} & \frac{d}{d\tau} \mathcal{E}(u(\tau); f, \Omega(\tau)) \Big|_{\tau=0} = P_{\Omega}(u, X) \\ & = - \int_{\partial\Omega} \{E(u)(X \cdot n) - T(u) \cdot (X \cdot \nabla)u\} dS. \end{aligned} \quad (5.21)$$

6. Global variation of solutions by perturbations

DEFINITION 6.1 We denote the first variation of $R_{\omega}(u, X)$ with respect to u by the formula

$$\delta_u R_{\omega}(u, w; X) := \lim_{\epsilon \rightarrow 0} \epsilon^{-1} \{R_{\omega}(u + \epsilon w, X) - R_{\omega}(u, X)\}. \quad (6.1)$$

Here, for an arbitrary $\varphi \in C_0^{\infty}(\Omega)$, w is the solution of the problem $P(\varphi, V(\Omega))$.

THEOREM 6.2 *For each $f \in W^{1,2}(\mathbb{R}^N)^m$, let $u(\tau)$ be the solution of the problem $P(f_R, V_{\tau}(\Omega(\tau)))$. Assume the existence of a constant $M_3 > 0$ such that*

$$\begin{aligned} & |D_{z_i} D_{z_j} E(x, z, p)|, |D_{p_{ij}} D_{z_j} E(x, z, p)|, |D_{p_{ij}} D_{p_{kl}} E(x, z, p)| \leq M_3, \quad (6.2) \\ & \text{for all } x \in \mathbb{R}^N, z, \zeta \in \mathbb{R}^m, p \in \mathbb{R}^{mN}. \end{aligned}$$

Then, under the hypotheses **(H1)**-**(H3)**, and (5.1)-(5.5) we have the formula

$$\frac{d}{d\tau} \int_{\Omega(\tau)} u(\tau) \varphi dx \Big|_{\tau=0} = \delta_u R_{\Omega}(u, w; X), \quad \varphi \in C_0^{\infty}(\Omega). \quad (6.3)$$

Proof. Since $E \in C^2(\mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^{mN})$, by the mean value theorem, for $\zeta, \eta \in C_0^{\infty}(\mathbb{R}^N)^m$ and $\epsilon > 0$, we obtain the following relation

$$\begin{aligned} & E(x, \eta + \epsilon\zeta, \nabla(\eta + \epsilon\zeta)) = E(x, \eta, \nabla(\eta)) + \epsilon\delta E(\eta)[\zeta] \\ & + \frac{\epsilon^2}{2} \{D_{p_{ij}} D_{p_{kl}} E(x, \eta + \theta\zeta, \nabla(\eta + \theta\zeta)) D_i \zeta_j D_k \zeta_l \\ & + 2D_{p_{ij}} D_{z_k} E(x, \eta + \theta\zeta, \nabla(\eta + \theta\zeta)) \zeta_k D_i \zeta_j \\ & + D_{z_i} D_{z_j} E(x, \eta + \theta\zeta, \nabla(\eta + \theta\zeta)) \zeta_i \zeta_j\} \end{aligned} \quad (6.4)$$

with some number $0 < \theta < 1$. Let $u(\tau), w(\tau)$ be the solutions of the problems $P(f_R, V_{\tau}(\Omega(\tau)))$, $P(\varphi, V_{\tau}(\Omega(\tau)))$, respectively. From (6.4) and (3.3), we obtain

$$\begin{aligned} & \mathcal{E}(u(\tau) + \epsilon w(\tau); f_{R,\tau}, \Omega(\tau)) - \mathcal{E}(u(\tau); f_{R,\tau}, \Omega(\tau)) \\ & = -\epsilon \int_{\Omega(\tau)} \varphi \cdot u(\tau) dx + \frac{\epsilon^2}{2} \mathcal{R}(u(\tau), w(\tau)), \end{aligned} \quad (6.5)$$

$$\dots \int_{\Omega(\tau)} \dots \epsilon^2 \mathcal{P}(u, w) \quad (6.6)$$

where

$$\begin{aligned}
 \mathcal{R}(u(\tau), w(\tau)) &:= \\
 &\int_{\Omega(\tau)} \{D_{p_{ij}}D_{p_{kl}}E(x, u(\tau) + \theta_1w(\tau), \nabla(u(\tau) + \theta_1w(\tau)))D_iw_j(\tau)D_kw_l(\tau) \\
 &+ 2D_{p_{ij}}D_{z_k}E(x, u(\tau) + \theta_1w(\tau), \nabla(u(\tau) + \theta_1w(\tau)))w_k(\tau)D_iw_j(\tau) \\
 &+ \nabla_{z_i}\nabla_{z_j}E(x, u(\tau) + \theta_1w(\tau), \nabla(u(\tau) + \theta_1w(\tau)))w_i(\tau)w_j(\tau)\} dx, \\
 \mathcal{R}(u, w) &:= \int_{\Omega} \{D_{p_{ij}}D_{p_{kl}}E(x, u + \theta_2w, \nabla(u + \theta_2w))D_iw_jD_kw_l \\
 &+ 2D_{p_{ij}}D_{z_k}E(x, u + \theta_2w, \nabla(u + \theta_2w))w_kD_iw_j \\
 &+ D_{z_i}D_{z_j}E(x, u + \theta_2w, \nabla(u + \theta_2w))w_iw_j\} dx
 \end{aligned}$$

with some numbers $0 < \theta_1, \theta_2 < 1$. Combining (6.5) and (6.6), we can derive

$$\begin{aligned}
 &\int_{\mathbb{R}^N} \varphi \cdot (u(\tau) - u) dx \\
 &= -\epsilon^{-1}[\mathcal{E}(u(\tau) + \epsilon w(\tau); f_{R,\tau}, \Omega(\tau)) - \mathcal{E}(u(\tau); f_{R,\tau}, \Omega(\tau))] \\
 &+ \epsilon^{-1}[\mathcal{E}(u + \epsilon w; f_R, \Omega) - \mathcal{E}(u; f_R, \Omega)] \\
 &+ \frac{\epsilon^2}{2}[\mathcal{R}(u(\tau), w(\tau)) - \mathcal{R}(u, w)].
 \end{aligned} \tag{6.7}$$

From (6.2) and (5.12), the following estimate is obtained

$$|\mathcal{R}(u(\tau), w(\tau)) - \mathcal{R}(u, w)| \leq C_1\tau\|\varphi\|_{1,\Omega} \tag{6.8}$$

with a constant $C_1 > 0$ independent of τ . Hence, by Theorem 5.4, it follows that

$$\begin{aligned}
 &\lim_{\tau \rightarrow 0} \tau^{-1} \int_{\mathbb{R}^N} \varphi \cdot (u(\tau) - u) dx \\
 &= \epsilon^{-1} \left\{ R_{\Omega}(u + \epsilon w, X) - \int_{\Omega} [X \cdot \nabla(f \cdot u) + f \cdot \text{udiv } X] dx \right\} \\
 &- \epsilon^{-1} \left\{ R_{\Omega}(u, X) + \int_{\Omega} [X \cdot \nabla(f \cdot u) + f \cdot \text{udiv } X] dx \right\} \\
 &+ \lim_{\tau \rightarrow 0} \tau^{-1} \frac{\epsilon}{2} \{ \mathcal{R}(u(\tau), w(\tau)) - \mathcal{R}(u, w) \}.
 \end{aligned} \tag{6.9}$$

Letting $\epsilon \rightarrow 0$ in (6.9), by (6.8), we arrive at (6.3). ■

COROLLARY 6.3 *Assume that solutions u and w have the same smoothness as in Theorem 4.3, then under the hypotheses used in Theorem 6.2, we have*

$$d \int_{\Omega} \dots$$

