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Fuzzy goal programming – one notion, many meanings

by

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Abstract: A survey of fuzzy goal programming approach is presented, including three new approaches. Various approaches are classified according to the role that fuzzy numbers play in them. For each approach the corresponding model and the solution procedure are discussed.

Keywords: goal programming, fuzzy number.

1. Introduction

Goal programming has been known in the literature and has been applied successfully in practice for many years. Another already almost classical notion is "fuzziness" or "fuzzy number" (Zadeh, 1965). However, for some time already a new notion has been used, the one combining the two mentioned ones: "fuzzy goal programming". While the first two notions are rather unequivocal (although each of them comprises a lot of specific forms), "fuzzy goal programming" is used in several almost totally different meanings. The aim of this paper is to introduce a certain systematisation in this domain. This may be useful while looking for a model for a specific decision situation. Apart from some concepts introduced by other authors, three concepts proposed by the authors of this paper will be presented.

2. Goal programming and fuzziness separately

If we were to find a general definition of goal programming, it might be formulated e.g. as follows: goal programming comprises decision problems in which we have classical mathematical programming constraints and more than one objective function (more than one goal), while for each objective function the decision maker gives a target value (a goal) and its type (maximisation, minimisation, equality). In case of the maximised objective functions the decision maker will be totally satisfied if the objective function value is equal or greater than the corresponding target value, for the minimised objective functions the total satisfaction will be achieved for objective function values equal or less than the corresponding target value, for objective functions of equality type – only for objective function equal to the target value. However, as it is often impossible to attain all of the fully satisfactory values at one time, undesirable objective function values (less than the target value for maximisation, greater that the target value for minimisation, different than the target value for equality) are also accepted by the decision maker, but only to a certain extent.

In the following general goal programming formulation, (1) corresponds to the objective functions (of minimisation, maximisation and equality type respectively), and (2) to the classical constraints:

$$\begin{aligned} &\mathbf{C}_j(\mathbf{x}) \widehat{\leq} d_j (j = 1, \ldots, k_1) \\ &\mathbf{C}_j(\mathbf{x}) \widehat{=} d_j (j = k_1 + 1, \ldots, k_2) \qquad (1) \\ &\mathbf{C}_j(\mathbf{x}) \widehat{\geq} d_j (j = k_2 + 1, \ldots, k_3) \\ &\mathbf{A}(\mathbf{x}) = \mathbf{B}. \qquad (2) \end{aligned}$$

In the above formulation $\mathbf{x} = (x_i)_1^n$ is a vector of non-negative decision variables, $\mathbf{C}_j$ is the objective function (non necessarily linear) representing the j-the goal, (2) is the canonical representation of the classical mathematical programming constraints (not necessarily linear ones), and d_j $(j = 1, \ldots, k_3)$ stand for the target values.

The inequality and equality signs in (1) have the "^" sign over them, which means that the corresponding relation does not have to be fulfilled completely, that certain deviations in the undesired direction(s) are allowed.

The deviations from the target values (all of them, for the moment we do no not differentiate between the undesired and desired deviations) will be denoted in the following way:

$$d_j^+ = \max(\mathbf{C}_j(\mathbf{x}) - d_j, 0) d_j^- = \max(d_j - \mathbf{C}_j(\mathbf{x}), 0) \ (j = 1, \ldots, k_3). \qquad (3)$$

In the classical approach to goal programming it is assumed that the decision maker wants to minimise the sum (possibly a weighted one) of all the undesired deviations. Thus, the following objective function is formulated:

$$\sum_{j=1}^{k_1} w_j d_j^+ + \sum_{j=k_1+1}^{k_2} (w_j d_j^+ + w_j' d_j^-) + \sum_{j=k_2+2}^{k_3} w_j d_j^- \to \min, \qquad (4)$$

where $w_j (j = 1, \ldots, k_3)$ and $w_j' (j = k_1 + 1, \ldots, k_2)$ are positive weights.

Then, the problem with the objective function (4) and the constraints (2) and (3) is solved, or rather its equivalent form with $n + 2k_3$ positive decision variables:

$$\sum_{j=1}^{k_1} w_j d_j^+ + \sum_{j=k_1+1}^{k_2} (w_j d_j^+ + w_j' d_j^-) + \sum_{j=k_2+2}^{k_3} w_j d_j^- \to \min$$

$$\begin{aligned}&\mathbf{C}_j(\mathbf{x}) - d_j^+ + d_j^- = d_j, \quad j = 1, \ldots, k_3\\&\mathbf{A}(\mathbf{x}) = \mathbf{B}\\&\mathbf{x} \geq 0,\ d_j^+, d_j^- \geq 0\ (j = 1, \ldots, k_3).\end{aligned} \tag{5}$$

Classical goal programming includes also problems with a goal hierarchy. Each of the objective functions is then given its hierarchical position (one of the numbers $h = 1, 2 \ldots, P$, $h = 1$ corresponding to the highest hierarchical level); several objective functions may have the same hierarchical level. The set of indices $j = 1, \ldots, k_3$ such that the j-th goal has the h-th hierarchical level will be denoted as $I_h (h = 1, \ldots, P)$.

In order to solve a problem with goal hierarchy, we can use e.g. the lexicographical approach. When we adopt this approach, we assume that the goals corresponding to the highest hierarchical level have to be satisfied in the first place (to the highest possible extent). Thus we start with solving problem (5), taking into account only the deviations corresponding to the objective functions from this level. Then, having added the constraints which assure keeping the achieved objective function values from the higher hierarchical level, we formulate another problem of type (5), taking into account the objective functions from the next lower hierarchical level, etc.

It is worth noticing that the goals of equality type ($j = k_2, \ldots, k_3$) are often expressed as a conjunction of two goals – one of maximisation type and one of minimisation type, both having the target value d_j. Thus we deal with model (1) (2) satisfying the condition $k_2 = k_3$:

$$\begin{aligned}&\mathbf{C}_j(\mathbf{x}) \widehat{\leq} d_j\ (j = 1, \ldots, k_1)\\&\mathbf{C}_j(\mathbf{x}) \widehat{\geq} d_j\ (j = k_1 + 1, \ldots, k_2)\\&\mathbf{A}(\mathbf{x}) = \mathbf{B}.\end{aligned} \tag{6}$$

Some authors consider also special cases of (1) (2), with all the goals being of equality type,

$$\begin{aligned}&\mathbf{C}_j(\mathbf{x}) \widehat{=} d_j\ (j = 1, \ldots, k_3)\\&\mathbf{A}(\mathbf{x}) = \mathbf{B},\end{aligned} \tag{7}$$

of maximisation type

$$\begin{aligned}&\mathbf{C}_j(\mathbf{x}) \widetilde{\geq} d_j\ (j = 1, \ldots, k_3)\\&\mathbf{A}(\mathbf{x}) = \mathbf{B},\end{aligned} \tag{8}$$

or of minimisation type

$$\begin{aligned}&\mathbf{C}_j(\mathbf{x}) \widetilde{\leq} d_j\ (j = 1, \ldots, k_3)\\&\mathbf{A}(\mathbf{x}) = \mathbf{B}.\end{aligned} \tag{9}$$

To sum up, in the classical goal programming we have target values, linked to the individual goals (represented by the objective functions). The values of the objective functions should be as close as possible to the target values, looking from the undesired side (for goals of the equality type both sides are undesired, for goals of the maximisation or minimisation only one side), while the distance between the objective function value and the target value is measured as the absolute value of the difference of two real numbers (3). All the undesired distances for all the goals are then cumulated in one linear objective function in the form of a weighted sum, or (in the case of a goal hierarchy) in several linear objective functions. In the first case the solution of the goal programming problem reduces to solving a mathematical programming problem with one linear objective function and constraints, where the latter will be linear if both the constraints and the objective functions in the initial goal programming problem are linear. In the second case the multicriteria programming methods have to be used, e.g. the lexicographical programming. The objective functions will be linear and, if the condition mentioned with respect to the first case is fulfilled, so will the constraints of the final model.

Let us pass to the definition of fuzzy number. Here we will give a general definition and discuss several most popular examples of fuzzy numbers types. The details can be found e.g. in the book of Zimmermann (1991).

Definition 1 *A fuzzy number is an upper semicontinuous and convex function* $\mu_A : \Re^k \to [0,1]$, *denoted also as* $\widetilde{A}$. *Function* μ_A *is called membership function. For* $k = 1$ *we will simply speak about a fuzzy number, for* $k > 1$ *- about a multidimensional fuzzy number.*

Fuzzy number $\widetilde{A}$ (membership function μ_A) has two basic interpretations:

I) "possibility case": $\widetilde{A}$ can express the incomplete knowledge of the decision maker about the value of a certain magnitude A. This incomplete knowledge may occur in two basic cases: either the exact value will be known, but only in the future, or the decision maker is unable to express the given magnitude in a numerical form, but only in a linguistic form ("small", "medium", "rather big", "significant" etc.). In both cases, for each real x $\mu_A(x)$ expresses the degree to which, according to the decision maker, it is possible that x really will be the value of A. In the case of a linguistic expression, a procedure of "translating" the given expression into the language of fuzzy numbers has to be applied (see e.g. the book of Zimmermann, 1991).

II) "preference case": $\widetilde{A}$ can express the decision maker preference as to the value of a certain magnitude A. In this case, for each real x $\mu_A(x)$ expresses the degree to which the decision maker will be satisfied if x really is the value of A.

Here are the most often used fuzzy number types:

Definition 2 *a) A fuzzy number* $\widetilde{A}$ *is a trapezoidal fuzzy number if there exist* *real numbers* $a_1, a_2, a_3, a_4 (a_1 \leq a_2 \leq a_3 \leq a_4)$ *such that:*

$$\mu_A(x) = \begin{cases} 0 & \textit{for } x < a_1 \textit{ and for } x > a_4 \\ \dfrac{x - a_1}{a_2 - a_1} & \textit{for } x \in [a_1, a_2] \\ 1 & \textit{for } x \in [a_2, a_3] \\ \dfrac{a_4 - x}{a_4 - a_3} & \textit{for } x \in [a_3, a_4]. \end{cases}$$

A trapezoidal fuzzy number is unequivocally defined by four real numbers (a_1, a_2, a_3, a_4).

b) A trapezoidal fuzzy number $\widetilde{A} = (a_1, a_2, a_3, a_4)$ *is a triangular fuzzy number, if* $a_2 = a_3$. *It is unequivocally defined by the triple* (a_1, a_2, a_4).

The following definition presents two other important fuzzy number types:

DEFINITION 3 *a) Symbol* (a_1, a_2, ∞) *stands for a left-hand fuzzy number* $\widetilde{A}$ *with the following membership function:*

$$\mu_A(x) = \begin{cases} 0 & \textit{for } x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & \textit{for } x \in [a_1, a_2] \\ 1 & \textit{for } x \in [a_2, \infty]. \end{cases}$$

b) Symbol (∞, a_1, a_2) *stands for a right-hand fuzzy number* $\widetilde{A}$ *with the following membership function:*

$$\mu_A(x) = \begin{cases} 1 & \textit{for } x < a_1 \\ \dfrac{a_2 - x}{a_2 - a_1} & \textit{for } x \in [a_1, a_2] \\ 0 & \textit{for } x \in [a_2, \infty]. \end{cases}$$

The membership functions do not have to be piecewise linear. The most popular fuzzy number types with not necessarily linear membership functions are the so-called L-R fuzzy numbers (see e.g. Dubois and Prade, 1980)

The basic arithmetical operations (addition, subtraction, multiplication, division – between two fuzzy numbers and between a fuzzy number and a crisp one) are defined also on fuzzy numbers (see Zimmermann, 1991). The result is also a fuzzy number.

The following definition recalls a very important notion connected with fuzzy numbers:

DEFINITION 4 *The* λ*-cut of a fuzzy number* $\widetilde{A}$, *denoted as* A_λ, *where* $\lambda \in (0, 1]$, *is the set* $\{x \in \Re^k : \mu(x) \geq \lambda\}$.

It is easy to conclude from Definition 4 that a λ-cut is the set of all those values that are possible or preferred (according to the interpretation of the fuzzy number) at least to the extent λ.

Apart from fuzzy numbers we can also speak about fuzzy relations. They can be defined on couples of fuzzy numbers, but not only. For example, the following fuzzy relation is defined on couples formed by an interval and a real number and defines the degree to which an interval is greater than a real number:

DEFINITION 5 *Let $[a, b]$ be an interval, and c a real number. Then*

$$st([a,b] \geq c) = st([a-c, b-c] \geq 0) = \frac{\max(b-c, 0) - \max(a-c, 0)}{b-a}.$$

The following relation is an example of a fuzzy relation defined for couples of triangular fuzzy numbers. It should express the distance between two triangular fuzzy numbers.

DEFINITION 6 *Let $\widetilde{c} = (\underline{c}, c, \overline{c})$ and $\widetilde{g} = (\underline{g}, g, \overline{g})$ be two triangular fuzzy numbers. Then*

$$Dist(\widetilde{c}, \widetilde{g}) = \min\left\{1, \max\left\{\frac{g-c}{\underline{c}+\overline{g}}, \frac{c-g}{\overline{c}+\underline{g}}\right\}\right\}.$$

3. Goal programming and fuzziness together

In this section we will try to show how researchers try to combine the two notions presented in the previous section. The aim of this section is to introduce a certain systematisation into the variety of concepts bearing the same name, that occur here and there in the literature.

To start with, let us introduce a very general definition of fuzzy goal programming, that at the same time constitutes the criterion according to which we chose the concepts to be discussed in this paper. Fuzzy goal programming is thus for us any goal programming approach that in some way makes use of the notion of fuzzy number. Only such a general definition comprises all the concepts which are called in the literature "fuzzy goal programming".

It is the way fuzzy numbers are used that seems to be the best criterion according to which we can classify various fuzzy goal programming concepts.

3.1. Fuzzy numbers as a measure of the decision maker satisfaction with attaining real target values

Many authors use the notion of fuzzy number to model the decision maker satisfaction with attaining the target values, while the target values themselves and the other coefficients of the model remain crisp. Thus, we still deal with basic model (6), but provided with additional information:

- For $j = 1, \ldots, l_1$ (minimisation goals) a right-hand fuzzy number is given:

$$\mu_j^p = (\infty, d_j, d_j^p). \tag{10}$$

Numer d_j^p indicates the tolerance of the decision maker with respect to undesirable deviations over the target values connected to minimisation goals. If the undesired deviation is greater than $d_j^p - d_j$, the decision maker is satisfied to the degree 0, which means that he cannot accept such a big deviation. But each undesired deviation has a satisfaction degree smaller than 1 – it is only the values smaller than the target value d_j that fully satisfy the decision maker;

- For $j = l_1, \ldots, l_2$ (maximisation goals) a left-hand fuzzy number is given

$$\mu_j^l = (d_j^l, d_j, \infty). \tag{11}$$

The interpretation of number d_j^l is analogous to that of d_j^p.

Further on, Zimmerman (1978) interprets model (6) as the search for a vector $\mathbf{x}$, fulfilling condition (2) and maximising the total satisfaction of the decision maker with attaining the goals. He defines the total satisfaction, for a given $\mathbf{x}$ fulfilling condition (2), as

$$Z_1(\mathbf{x}) = \min\{\min_{j=1,\ldots,l_1} \mu_j^p(\mathbf{C}_j(\mathbf{x})), \min_{j=l_1,\ldots,l_2} \mu_j^l(\mathbf{C}_j(\mathbf{x}))\}. \tag{12}$$

A solution maximising (12) with constraints (1) is determined by means of an auxiliary mathematical programming problem with one linear objective function (what is more, this function is equal to one of the decision variables) and with constraints which will be linear if constraints (2) are linear and membership functions μ_j^p $(j = 1, \ldots, l_1)$ and μ_j^l $(j = l_1, \ldots, l_2)$ piecewise linear.

Ohta and Yamaguchi (1996) propose a similar approach, but for goals given in the form of ratios of linear functions. The auxiliary problem that leads to the final solution is non- linear, but it can be solved by means of linear programming methods and the halving procedure of Sakawa (1984).

Mohamed (1997) also interprets model (6) as searching a vector $\mathbf{x}$, fulfilling condition (2) and maximising the decision maker satisfaction with attaining the target values, measured by means of membership functions μ_j^p and μ_j^l, but he does not speak about "total satisfaction". Functions $\mu_j^p(\mathbf{C}_j(\mathbf{x}))$ $(j = 1, \ldots, l_1)$ and $\mu_j^l(\mathbf{C}_j(\mathbf{x}))$ $(j = l_1, \ldots l_2)$ are for him separate objective functions in the sense of classical goal programming, and all the target values are equal to 1. Problem (6) is thus reduced to the following classical goal programming problem:

$$\begin{aligned} &\mu_j^p(\mathbf{C}_j(\mathbf{x})) \widehat{\geq} 1 \ (j = 1, \ldots, l_1) \\ &\mu_j^l(\mathbf{C}_j(\mathbf{x})) \widehat{\geq} 1 \ (j = l_1 + 1, \ldots, l_2) \\ &\mathbf{A}(\mathbf{x}) = \mathbf{B}. \end{aligned} \tag{13}$$

Further on Mohamed applies the procedure described in Section 2 for problem (1) (2).

Chen (1994) and Tiwari et al. (1986) consider model (7) (equality goals), assuming that for $j = 1, \ldots, k$ a triangular fuzzy number $(d_j - s_j, d_j, d_j + r_j)$ is given (with membership function μ_j^t), expressing the satisfaction of the decision maker with goal attainment. The goal is of equality type, thus the decision maker does not accept deviations greater than s_j (below the target value) and r_j (over the target value). Chen assumes the symmetry of the triangular fuzzy numbers used, i.e. $s_j = r_j$.

Both Chen (1994) and Tiwari et al. (1986) use the concept of Zimmermann (1978), i.e. they search for a vector $\mathbf{x}$, fulfilling condition (2) and maximising the total satisfaction of the decision maker with goal attainment, defined in the following way:

$$Z_2(\mathbf{x}) = \min_{j=1,\dots,k} \mu_j^t(\mathbf{C}_j(\mathbf{x})) \tag{14}$$

Tiwari et al. (1986) propose a solution consisting in solving 2^k linear programming problems with one objective function. On the other hand, Chen solves the same problem (with the mentioned symmetry assumption) by means of one linear programming problem with one objective function.

Wang and Fu (1997) consider problem (1)(2), using again the Zimmermann's concept: for each goal the membership functions described above are given: μ_j^p $(j = 1, \dots, k_1)$, μ_j^{tr} $(j = k_1, \dots, k_2)$ and μ_j^l $(j = k_2, \dots, k_3)$. Functions μ_j^{tr} $(j = k_1, \dots, k_2)$ can be triangular (like μ_j^t), but also trapezoidal, which means that instead of one single target value d_j $(j = k_1, \dots, k_2)$ we can have a target interval $\lfloor \underline{d}_j, \overline{d}_j \rfloor$ $(j = k_1, \dots, k_2)$. The number μ_j^{tr} takes then the form $(d_j - s_j, \underline{d}_j, \overline{d}_j, d_j + r_j)$. What is more, Wang and Fu also allow for membership functions which are not piecewise linear. The "sides" of the triangular and trapezoidal membership functions can be replaced by power functions, and the power value depends on the tolerance degree of the decision maker with respect to the deviations (e.g. concave functions will be used by decision makers whose dissatisfaction grows while going further away from the target value quicker than the dissatisfaction of the decision makers who would use piecewise linear membership functions). Apart from that, problem (1) (2) should have a goal hierarchy $(h = 1, \dots, P)$.

Wang and Fu also use the notion of the total decision maker satisfaction with attainment of goals, but they do it with respect to a given hierarchical level:

$$Z_h(\mathbf{x}) = \min \left\{ \begin{array}{l} \min_{j=1,\dots,k_1} \mu_j^p(\mathbf{C}_j(\mathbf{x})), \min_{j=k_1,\dots,k_2} \mu_j^{tr}(\mathbf{C}_j(\mathbf{x})), \min_{j=k_2,\dots,k_3} \mu_j^l(\mathbf{C}_j(\mathbf{x})); \\ j\text{– th goal has the } h\text{th hierarchical level} \end{array} \right\}. \tag{15}$$

Wang and Fu solve then the following problem: they search for a vector $\mathbf{x}$, fulfilling condition (2), such that it would satisfy to the degree 1 (in the sense of function (15)) the goals from the hierarchical levels $h = 1, \dots P_0$ for as big P_0 $(P_0 \leq P)$ as possible, and – if this biggest P_0 fulfils condition $P_0 < P$ – that would maximise $Z_{(P_0+1)}(\mathbf{x})$ (the latter value is also searched for). This problem is solved by means of a mathematical programming problem with one linear objective function and constraints which are linear if the membership functions used are piecewise linear.

Gen et al. (1997) use fuzzy numbers to determine the decision maker satisfaction in problem (1) (2) in the same way as Wang and Fu. However, they do not assume the linearity of objective functions (1) or constraints (2) and they do not introduce any goal hierarchy. They search for a solution maximising the total decision maker satisfaction defined in the following way:

$$Z_3(\mathbf{x}) = \sum_{j=1}^{k_1} t_j \mu_j^p(\mathbf{C}_j(\mathbf{x})) + \sum_{j=k_1}^{k_2} t_j \mu_j^{tr}(\mathbf{C}_j(\mathbf{x})) + \sum_{j=k_2}^{k_3} t_j \mu_j^l(\mathbf{C}_j(\mathbf{x})) \qquad (16)$$

where t_j $(j = 1, \ldots, k_3)$ are positive, crisp weights, given by the decision maker. This solution is determined by means of a genetic algorithm proposed by the authors.

Rao et al. (1992) act in a similar way, but with respect to problem (9). They also allow for nonlinear objective functions and constraints, and the satisfaction of the decision maker with the attainment of the individual goals is defined in their paper in the same way. But the total satisfaction degree (the objective function maximised in the auxiliary problem that is finally solved) can be expressed in 3 following ways:

$$Z_4(\mathbf{x}) = \sum_{j=1}^{k_1} \mu_j^p(\mathbf{C}_j(\mathbf{x})) \qquad (17)$$

$$Z_5(\mathbf{x}) = \sum_{j=1}^{k_1} t_j \mu_j^p(\mathbf{C}_j(\mathbf{x})) \qquad (18)$$

$$Z_6(\mathbf{x}) = \sum_{j=1}^{k_1} (\mu_j^p(\mathbf{C}_j(\mathbf{x})))^2 \qquad (19)$$

(the weights in (18) are given by the decision maker).

Rao et al. (1992) also consider problems with goal hierarchy. For such problems they suggest to choose one of objective functions (17), (18), (19) and to maximise it at each hierarchy level (for a fixed h $(h = 1, \ldots, P)$) (the sum will be executed for $j \in I_h$). They use here the lexicographic approach:

For a fixed h and for each $j \in I_h$, $\mu_j^{p^*}$ denotes the value of function $\mu_j^p(\mathbf{C}_j(\mathbf{x}))$ for the solution found while considering level h. While ascending to level $h+1$, Rao et al. (1992) add the constraints $\mu_j^p(\mathbf{C}_j(\mathbf{x})) = \mu_j^{p^*}$ $(j \in I_h)$.

A similar use of fuzzy numbers in goal programming is proposed by Mohandas et al. (1990). However, they also consider another use of fuzzy numbers, and so their proposal is presented in Section 3.2.

Sasaki et al. (1990) ask the decision maker to give the target values d_j $(j = 1, \ldots, k_3)$ in an interactive way (for problem (7), assuming the linearity of the objective function and the constraints). However, they start with solving k_3 couples of the single-criterion linear programming problems. For each $j = 1, \ldots, k_3$ the following two problems are solved:

$$\begin{aligned} &\mathbf{C}_j(\mathbf{x}) \to \max \\ &\mathbf{A}(\mathbf{x}) = \mathbf{B} \end{aligned} \tag{20}$$

and

$$\begin{aligned} &\mathbf{C}_j(\mathbf{x}) \to \min \\ &\mathbf{A}(\mathbf{x}) = \mathbf{B}. \end{aligned} \tag{21}$$

Let $C_j^{\max}$ and $C_j^{\min}$ ($j = 1, \ldots, k_3$) denote, respectively, the optimal values of the objective function of problems (20) and (21). The decision maker is asked to give the target values d_j ($j = 1, \ldots, k_3$), fulfilling the condition $C_j^{\min} \leq d_j \leq C_j^{\max}$ ($j = 1, \ldots, k_3$). Further on, the authors use two fuzzy measures of the decision maker satisfaction with the attainment of the target values:

$$\mu_j^-(\mathbf{C}_j(\mathbf{x})) = \frac{\mathbf{C}_j(\mathbf{x}) - C_j^{\min}}{d_j - C_j^{\min}} \text{ (for } C_j^{\min} \leq \mathbf{C}_j(\mathbf{x}) \leq d_j) \tag{22}$$

$$\mu_j^+(\mathbf{C}_j(\mathbf{x})) = \frac{C_j^{\max} - \mathbf{C}_j(\mathbf{x})}{C_j^{\max} - d_j} \text{ (for } d_j \leq \mathbf{C}_j(\mathbf{x}) \leq C_j^{\max}) \tag{23}$$

where (22) measures the decision maker's satisfaction with the attainment of the target values in the case when the value of the corresponding objective function is smaller than the target value, and (23) measures this satisfaction in the other case.

For each collection of target values d_j ($j = 1, \ldots, k_3$) given by the decision maker, it is proposed to search, by means of the lexicographic method, a solution maximising the value of two following functions:

$$Z_7(\mathbf{C}_j(\mathbf{x})) = \sum_{j=1}^{k_3} t_j^- \mu_j^-(\mathbf{C}_j(\mathbf{x})) \tag{24}$$

$$Z_8(\mathbf{C}_j(\mathbf{x})) = \sum_{j=1}^{k_3} t_j^+ \mu_j^+(\mathbf{C}_j(\mathbf{x})), \tag{25}$$

where t_j^- and t_j^+ ($j = 1, \ldots, k_3$) are positive weights, given by the decision maker. If the solution obtained does not satisfy the decision maker, he is asked to give new target values d_j ($j = 1, \ldots, k_3$).

The final auxiliary problems solved in course of the algorithm are the single criterion linear programming problems.

3.2. Fuzzy numbers as goal weights

Fuzzy numbers in goal programming can also play the role of goal weights. In this situation we deal with the first interpretation of fuzzy number, and more exactly with the case when the exact values will never be known – the decision maker is simply unable to express them in the form of a crisp number and uses, instead, a linguistic expression.

Mohandas et al. (1990) use fuzzy numbers to measure the satisfaction with the attainment of crisp goals in a way that combines the approaches proposed by Mohamed (1987), Gen (1984) and Rao et al. (1992). They do it with respect to problem (8), reformulated in such a way that all the target values, as well as all the right hand sides of the constraints are equal to 0. They search for a solution which is non-dominated with respect to the (maximised) objective functions:

$$F_j(\mathbf{x}) = \mu_j^l(\mathbf{C}_j(\mathbf{x}))\ (j = 1, \dots k_3). \tag{26}$$

i.e. such a solution that the improvement of any of the objective functions (26) deteriorates the value of another one.

In order to find such a solution, Mohandas et al. (1990) use a heuristic algorithm, that makes use of the following modification of expression (16) (this modification is to determine the total decision maker's satisfaction with the attainment of the target values)

$$\widetilde{F}_1(\mathbf{x}) = \sum_{j=1}^{k_3} \widetilde{t}_j \mu_j^l(\mathbf{C}_j(\mathbf{x})), \tag{27}$$

where $\widetilde{t}_j$ $(j = 1, 2, \dots, k_3)$ are triangular, left-hand or right-hand fuzzy numbers, given by the decision maker. They should express the importance of the individual goals in the decision maker eyes and originally are given in a linguistic form.

Once (27) is defined, the realisation of the algorithm can be started. In each step the fuzzy gradient of (27) is determined

$$\frac{\partial \widetilde{F}_1(\mathbf{x})}{\partial(\mathbf{x})} = \left[\frac{\partial \widetilde{F}_1(\mathbf{x})}{\partial(x_1)}, \dots, \frac{\partial \widetilde{F}_1(\mathbf{x})}{\partial(x_n)}\right] \tag{28}$$

whose coordinates are found in the following way:

$$\frac{\partial \widetilde{F}_1(\mathbf{x})}{\partial(x_i)} = \sum_{j=1}^{n} \widetilde{t}_j \frac{\partial}{\partial x_i}(\mu_j^l(\mathbf{C}_j(\mathbf{x})))\ (i = 1, \dots n). \tag{29}$$

Coordinates (28) are thus fuzzy numbers (which come into being by means of the multiplication of the fuzzy weights with crisp derivate values and the addition of the fuzzy results). For a fixed i $(i = 1, \dots, n)$ the coordinate $\frac{\partial \widetilde{F}_1(\mathbf{x})}{\partial(x_i)}$ should indicate the influence of decision variable x_i on the total degree of satisfaction with the attainment of the goals. One should attempt to improve the value of this decision variable x_i, whose influence is the highest one. As this influence is evaluated by means of fuzzy numbers $\frac{\partial \widetilde{F}_1(\mathbf{x})}{\partial(x_i)}$, and fuzzy numbers are often incomparable, Mohandas et al. (1990) determine for each fuzzy coordinate a real characteristic, thanks to which the choice of the best improvement direction is unequivocal (although not necessarily the best one). The new solution is accepted if it is better or at least not worse than the already found ones with respect to each of objective functions (26). The algorithm terminates in the moment when gradient (28) does not lead to any new accepted solution.

3.3. Fuzzy numbers as a tool to determine crisp target values in classical goal programming

In the classical goal programming formulation it is assumed that target values d_j $(j = 1, \dots, k_3)$, as well as the goal hierarchy, are already given. But their determination is not easy. Too much subjectivity may lead to solutions that will not be of much use. According to Mohanty and Vijayaraghavan (1995), this is above all the consequence of the fact that the target values are often determined disregarding mutual relations among individual goals. For example, if two goals are in conflict in the sense that the improvement of the value of one of them always deteriorates the value of the other one, and the target values for both goals will be close to each other, big deviations cannot be avoided. Their "distribution" in the final solution may get out of control: one of the goals may assume almost the whole deviation and the other one may get a very small deviation or none at all. Such a solution will in fact disregard the first goal.

In order to avoid such a situation, Mohanty and Vijayaraghavan (1995) determine the target values d_j $(j = 1, \dots, k_3)$ and goal hierarchy in the classical goal programming approach using in particular fuzzy numbers. Their starting point is problem (8) with values d_j $(j = 1, \dots, k_3)$ assumed to be unknown at the beginning.

The procedure of searching for the target values starts by solving the problems (20) and (21) for $(j = 1, \dots, k_3)$. For each goal $j = 1, \dots, k_3$ a left-hand fuzzy number is defined with the following membership function:

$$\mu_{C_j}(\mathbf{x}) = (C_j^{\min}, C_j^{\max}, \infty). \tag{30}$$

This fuzzy number determines the decision maker's satisfaction with the value of the given objective function with respect to the best value of the objective function attained disregarding the other goals. The decision maker is totally satisfied only if this globally best value is achieved, and less satisfied with smaller values. Of course, when we consider all the goals at the same time, it is usually not possible to attain value 1 of this satisfaction degree for all of them. This is a consequence of the fact that goals are usually in conflict to some extent.

Further on, considering the gradients of individual objective functions and their mutual position and applying a special procedure, Mohanty and Vijayaraghavan (1995) determine (for each goal $j = 1, \dots, k_3$) the degree w_j to which the given goal is not in conflict with the other ones. Numbers w_j $(j = 1, \dots, k_3)$

take values from 0 to 1; where 0 means being totally in conflict with the other goals, and 1 means being in no conflict at all.

Assuming that the target values of the individual goals should be in harmony with mutual relations between goals and that the decision maker can expect to be satisfied with the value of the given goal only to the extent to which this goal is not in conflict with the other goals, the authors suggest to determine target values d_j $(j = 1, \ldots, k_3)$ by means of the following equations system:

$$\mu_{C_j}(d_j) = w_j \ (j = 1, \ldots, k_3). \tag{31}$$

In this way the classical goal programming problem (8) is fully formulated – the crisp target values have been determined. Moreover, Mohanty and Vijayaraghavan (1995) suggest to derive the goal hierarchy from values w_j $(j = 1, \ldots, k_3)$: the bigger w_j, the higher the hierarchy level (the smaller h value) of a given goal, goals having the same w_j values belonging to the same hierarchy level.

3.4. Fuzzy numbers as target values

The authors of the present paper have proposed three goal programming models (together with corresponding solution procedures) for problem (1) (2) and the case where all the coefficients are crisp and only the goals (the target values) are fuzzy. Thus, we consider the following problem:

$$\begin{aligned}
&\mathbf{C}_j(\mathbf{x}) \widehat{\leq} \widetilde{d}_j \ (j = 1, \ldots, k_1) \\
&\mathbf{C}_j(\mathbf{x}) \widehat{=} \widetilde{d}_j \ (j = k_1 + 1, \ldots, k_2) \\
&\mathbf{C}_j(\mathbf{x}) \widehat{\geq} \widetilde{d}_j \ (j = k_2 + 1, \ldots, k_3) \\
&\mathbf{A}(\mathbf{x}) = \mathbf{B}.
\end{aligned} \tag{32}$$

where $\widetilde{d}_i$ $(i = 1, \ldots, k_3)$ are L-R fuzzy numbers with membership functions $\mu_{d_j}(d_j)$. In Chanas and Kuchta (2002) we considered two situations. The first situation corresponds to the preference case. We assume that the decision maker can choose the exact value d_i of each goal $\widetilde{d}_i$ $(i = 1, \ldots, k_3)$, and the value $\mu_{d_j}(d_j)$ expresses, for each real number d_j, the satisfaction of decision maker with the fact that $\widetilde{d}_i = d_j$. He wants to find such values d_i $(i = 1, \ldots, k_3)$ that would give him the maximal overall satisfaction with the selected goals and he also wants to determine the solution vector $\mathbf{x}$ that corresponds to this "best" choice of goals.

The decision maker cannot simply choose those values that satisfy him most because those values may correspond to "ambitious goals", which might be unattainable for him and his team. Thus, he looks for such goals which would give him a compromise between his satisfaction with the goals and the attainability of the goals (the magnitude of the deviations). The overall satisfaction with the goals is defined as $SAT(\{d_i\}_{i=1}^{k_3}) = \min_{i=1,\ldots,k_3}\{\mu_{d_i}(d_i)\}$.

This problem is finally reformulated as the following bicriterial problem and solved by any method of bicriterial programming, e.g. by looking for Pareto solutions.

$$\lambda \to \max$$
$$\sum_{i=1}^{k_1} w_i d_i^+ + \sum_{i=k_1+1}^{k_2} (w_i d_i^+ + w_i' d_i^-) + \sum_{i=k_2+2}^{k_3} w_i d_i^- \to \min$$
$$\mathbf{C}_i \mathbf{x} - d_i^+ + d_i^- = d_i, i = 1, \ldots, k_3$$
$$\underline{d}_i - \alpha_i L^{-1}(\lambda) \leq d_i \leq \overline{d}_i + \beta_i L^{-1}(\lambda)$$
$$\mathbf{Ax} = \mathbf{B}$$
$$d_i^+, d_i^- \geq 0 \ (i = 1, \ldots, k_3),$$

where w_i $(i = 1, \ldots, n_3)$ and w_i' $(i = k_1 + 1, \ldots, k_2)$ are positive weights.

Another interpretation of (32), discussed in Chanas and Kuchta (2002), corresponds to the "possibility case". Now the decision maker is not able to choose the exact value of each goal, because the goals will be fixed by someone else, on whom the decision maker has no influence. However, the decision maker possesses some knowledge about the goals: he knows the possibility distributions of the crisp values that can be the values of the goals. Hence, the value $\mu_{d_i}(d_i)$ expresses, for each real number d_i, the possibility of d_i of being the final value of the i-th goal.

In this case with little room for manoeuvre the decision maker might just be interested in the possibility distributions of the deviations from the (for the moment unknown) exact goal values. In other words, he might want to know what deviations may occur and how possible they are, in order to be able to evaluate his risk of the deviations being too big. In particular, the information about the possibility distribution of the minimal value of the total deviation may be of use to him.

This possibility distribution is found by solving the following parametric linear programming problem (linear parametric programming methods are enough to solve it).

$$\sum_{i=1}^{k_1} w_i d_i^+ + \sum_{i=k_1+1}^{k_2} (w_i d_i^+ + w_i' d_i^-) + \sum_{i=k_2+2}^{k_3} w_i d_i^- \to \min$$
$$\mathbf{C}_i \mathbf{x} - d_i^+ + d_i^- = d_i, i = 1, \ldots, k_3$$
$$\underline{d}_i - \alpha_i L^{-1}(\lambda) \leq d_i \leq \overline{d}_i + \beta_i L^{-1}(\lambda)$$
$$\mathbf{Ax} = \mathbf{B}$$
$$d_i^+, d_i^- \geq 0 (i = 1, \ldots, k_3),$$

where w_i $(i = 1, \ldots, n_3)$ and w_i' $(i = 1, \ldots, n_2)$ are positive weights and λ is a parameter from the interval $[0, 1]$.

In Chanas and Kuchta (2001) the problem (32) is interpreted again in the context of the possibility case. The goals are unknown and the decision maker wants to minimise the sum of deviations from them. These deviations of crisp objective functions values from fuzzy goals are defined as crisp numbers, in the following way:

$$D_j^+ = D^+(\mathbf{C}_j\mathbf{x}, \widetilde{d}_j), \quad D_j^- = D^-(\mathbf{C}_j\mathbf{x}, \widetilde{d}_j), \text{ where}$$

$$D^+(r, A) = \begin{cases} 0 & \text{if } r \leq \overline{a} \\ \int_{\overline{a}}^{r} (1 - \mu_A(z))dz & \text{elsewhere} \end{cases},$$

$$D^-(r, A) = \begin{cases} 0 & \text{if } r \geq \underline{a} \\ \int_{r}^{\underline{a}} (1 - \mu_A(z))dz & \text{elsewhere} \end{cases}.$$

Then, the following problem is solved:

$$\sum_{j=1}^{k_1} w_j D_j^+ + \sum_{j=k_1+1}^{k_2} (w_j D_j^+ + w_j' D_j^-) + \sum_{j=k_2+2}^{k_3} w_j D_j^- \to \min$$

$$\mathbf{Ax} = \mathbf{B}.$$

This is a nonlinear problem, but it can still be effectively solved for special cases of L-R fuzzy numbers representing the goals, using quadratic and generally nonlinear programming methods.

3.5. Fuzzy numbers as objective function and constraint coefficients and as target values

Kuwano (1966) considers the following multicriteria problem:

$$\widetilde{\mathbf{C}}_j\mathbf{x} \to \max\ (j = 1, \ldots k_3) \tag{33}$$

$$\widetilde{\mathbf{Ax}} \leq \widetilde{\mathbf{b}}, \quad \mathbf{x} \geq \mathbf{0}, \tag{34}$$

where the coefficients in (33) are triangular fuzzy numbers $(c_{1,i}^j, c_{2,i}^j, c_{3,i}^j)$ $(i = 1, \ldots, n), (j = 1, \ldots, k_3)$, and the coefficients in (34) are fuzzy numbers of any type. The fuzziness of the coefficients means that they are not known exactly (possibility case).

Problem (33) (34) is not a goal programming problem, but Kuwano tries to solve it using a goal programming approach. Originally, the target values are not given. Kuwano suggests to determine for problem (33) (34) a fuzzy multidimensional target value. It is a kind of an ideal solution (usually unattainable), constructed on the basis of solutions of the following single criterion problems P_k^α $(k = 1, \ldots l)$

$$\mathbf{c}_1^j\mathbf{x} \to \max\ (j = 1, \ldots k_3)$$

$$\mathbf{A}^\alpha \mathbf{x} \leq \mathbf{b}^\alpha, \quad \mathbf{x} \geq \mathbf{0}$$

where $\mathbf{c}_1^j$ is a vector $(c_{1,i}^j)_{i=1}^n$ is the vector of the left-hand ends of the α-cuts of the coordinates of vector $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{b}}^\alpha$ is the vector of the right-hand ends of the α-cuts of the coordinates of vector $\widetilde{\mathbf{b}}$. Such an α is fixed (assuming that it exists), for which all the problems P_k^α $(k = 1, \dots l)$ have a unique optimal solution $x_k^*(\alpha)$. On the basis of solutions $x_k^*(\alpha)$, the ideal target value is determined. It is called the α-optimal value of problem ((33) (34)) and is denoted as $\mathbf{Z}(\alpha)$. It is a multidimensional fuzzy number, defined in space $\Re^{k_3}$.

Further on, in space $\Re^{k_3}$ and for each vector $\mathbf{x}$, another multidimensional fuzzy number is defined, denoted $\widetilde{\mathbf{C}}\mathbf{x}$ and expressing the fuzzy value of the objective function vector for $\mathbf{x}$. It has the following membership function:

$$\mu_{\widetilde{C}x}((y_j)_{j=1}^{k_3}) = \min\{\mu_j(y_j) \ (j = 1, \dots, k_3)\},$$

where μ_j is the membership function of fuzzy number $\widetilde{\mathbf{C}}_j\mathbf{x}$.

Solution of problem (33) (34) is defined as a feasible solution $\mathbf{x}$ minimising the distance D^β between fuzzy numbers $\widetilde{\mathbf{C}}\mathbf{x}$ and $\mathbf{Z}(\alpha)$. Kuwano defines the distance D^β between two multidimensional fuzzy numbers in the following way:

$$D^\beta(\widetilde{\mathbf{M}}, \widetilde{\mathbf{L}}) = \sup_{\beta \leq \gamma \leq 1} d_H(\widetilde{\mathbf{M}}(\gamma), \widetilde{\mathbf{N}}(\gamma)), \tag{35}$$

where $\widetilde{\mathbf{M}}$, $\widetilde{\mathbf{L}}$ are fuzzy numbers in space $\Re^{k_3}$, β is a fixed number from interval $[0, 1]$, $\widetilde{\mathbf{M}}(\gamma)$, $\widetilde{\mathbf{N}}(\gamma)$ are, respectively, γ-cuts of fuzzy numbers $\widetilde{\mathbf{M}}$, $\widetilde{\mathbf{L}}$, and d_H is a measure of the distance between subsets of $\Re^{k_3}$, defined in the following way:

$$d_H(A, B) = \max\{\max(d(a, B) : a \in A), \max(d(b, A) : b \in B)\}$$

(where d – any distance in $\Re^{k_3}$).

It is then enough to solve a single-criterion linear programming problem, in which the (minimised) objective function is equal to D^β $(\widetilde{\mathbf{C}}\mathbf{x}, \mathbf{Z}(\alpha))$.

3.6. Fuzzy numbers as the coefficients of the objective function and constraints, as target values and as decision variables

Kono and Yamaguchi (1992) consider the following linear problem, in which the target values, the objective function coefficients as well as the decision variables are fuzzy numbers.

$$\sum_{i=1}^n \widetilde{c}_i^j \widetilde{x}_i \widetilde{\gtrsim} \widetilde{d}_j \ (j = 1, \dots, k_3). \tag{36}$$

The authors solve the following auxiliary problem for each $\lambda \in (0, 1]$:

$$\sum_{i=1}^n c_i^{j\lambda} x_i^\lambda \gtrsim d_i^\lambda \ (j = 1, \dots, k_3). \tag{37}$$

Problem (37) is then reformulated, using classical (see e.g. Moore, 1996) operations on intervals:

$$\sum_{i=1}^{n} c_i^{j\lambda} x_i^{\lambda} - d_i^{\lambda} \widetilde{\geq} 0 \; (j = 1, \ldots, k_3). \tag{38}$$

For each $\lambda \in (0, 1]$, the left hand side of (38) will be an interval. In order to compare intervals with real numbers, Kono and Yamaguchi use the fuzzy relation from Definition 2.5 and solve the following problem (with some additional constraints, preventing the intervals to be too wide), maximising the degree of fulfilment of the fuzzy constraints from (38):

$$\begin{aligned} &\alpha \to \max \\ &st\Big(\sum_{i=1}^{n} c_i^{j\lambda} x_i^{\lambda} \geq d_i^{\lambda}\Big) \geq \alpha (j = 1, \ldots, k_3). \end{aligned} \tag{39}$$

The solutions (λ-cuts of the decision variables) determined in this way for individual $\lambda \in (0, 1]$ (with some additional constraints), constitute the fuzzy solution of problem (36).

Ohta and Yamaguchi (1996) apply this approach to the case when the objective functions are ratios of linear expressions.

Ramik (2000) solves the following problem:

$$\sum_{i=1}^{n} \widetilde{c}_i^j \widetilde{x}_i \widehat{=} \widetilde{d}_j \; (j = 1, \ldots, k_3),$$

in which the decision variables $(\widetilde{x}_1, \ldots \widetilde{x}_n)$ are triangular "symmetric" fuzzy numbers $(x - \xi x, x, x + \xi x)$, where $x \in \Re$, and ξ is a fixed number from interval $(0, 1)$, the target values $\widetilde{d}_j$ $(j = 1, \ldots, k_3)$ and coefficients $\widetilde{c}_i^j$ $(i = 1, \ldots n;\ j = 1, \ldots, k_3)$ are any triangular fuzzy numbers.

The author minimises the value of the measure of the deviation of the fuzzy objective functions from fuzzy goals, defined as:

$$\overline{D}\Big(\Big(\sum_{i=1}^{n} \widetilde{c}_i^1 \widetilde{x}_i, \ldots, \sum_{i=1}^{n} \widetilde{c}_i^{k_3} \widetilde{x}_i\Big), (\widetilde{d}_1, \ldots \widetilde{d}_{k_3})\Big) = \max_{j=1,\ldots k_3} Dist\Big(\sum_{i=1}^{n} \widetilde{c}_i^j \widetilde{x}_i, \widetilde{d}_i\Big),$$

where the deviation $Dist$ is defined according to Definition 2.6.

The problem defined in this way is then reduced to a single-criterion quadratic programming problem (with quadratic constraints), in which the minimised objective function is a decision variable, equal to $\overline{D}((\sum_{i=1}^{n} \widetilde{c}_i^1 \widetilde{x}_i, \ldots, \sum_{i=1}^{n} \widetilde{c}_i^{k_3} \widetilde{x}_i)$, $(\widetilde{d}_1, \ldots \widetilde{d}_{k_3}))$.

4. Combination of stochastic programming with fuzzy goal programming

Mohamed (1992) uses fuzzy numbers to solve the goal programming problem (8), in which the target values d_j $(j = 1, \ldots, k_3)$ are random variables with probability distributions $F_j(y) = P(d_j \leq y)$. Problem (8) is reformulated in the form of the following fuzzy goal programming problem:

$$\begin{array}{l} P(\mathbf{C}_j(\mathbf{x}) \geq d_j) \tilde{\geq} \alpha_j \ (j = 1, \ldots, k_3) \\ \mathbf{A}(\mathbf{x}) = \mathbf{B}, \end{array} \tag{40}$$

where $\alpha_j (j = 1, \ldots k_3)$ are new target values, given by the decision maker.

Further on, fuzzy numbers are used to measure the degree of achieving the goals formulated in such a way. The decision maker links to each goal α_j $(j = 1, \ldots k_3)$ a left hand fuzzy number $\nu_j^l = (\alpha_j^l, \alpha_j, \infty)$, where α_j^l is the smallest probability still tolerated by the decision maker. Fuzzy numbers ν_j^l are defined on values $P(\mathbf{C}_j(\mathbf{x}) \geq d_j) (j = 1, \ldots, k_3)$.

Further on, by making use of the properties of probability distributions, the problem is formulated in the following form of classical goal programming approach:

$$\frac{\mathbf{C}_j(\mathbf{x}) - F_j^{-1}(\alpha_j^l)}{F_j^{-1}(\alpha_j) - F_j^{-1}(\alpha_j^l)} \widehat{\geq} 1 \ (j = 1, \ldots, k_3) \tag{41}$$

$$\mathbf{A}(\mathbf{x}) = \mathbf{B},$$

and solved by means of classical methods.

Having obtained the optimal solution $\mathbf{x}^*$ of problem (41), we have to calculate values $\nu_j^l(P(\mathbf{C}_j(\mathbf{x}^*) \geq d_j))$ $(j = 1, \ldots, k_3)$. The vector of these values is supposed to be the final measure of the decision maker's satisfaction with the solution obtained.

5. Conclusions

The idea of fuzzy goal programming comprises many distinct concepts and notions. The difference consists mainly in the role of fuzziness. Fuzzy numbers can serve e.g. as a measure of the decision maker's satisfaction with attainment of crisp goals, but they can also represent goals, other coefficients of the model or the values of decision variables. The present paper offers a survey of various approaches (including three of the authors themselves). It may help the decision maker in choosing the right model in a given decision situation or give an inspiration for further research in the area of "fuzzy goal programming".

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