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Fast controls and minimum time

by

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Abstract: A relationship between the minimum time function and the minimum norm control presented in Cârjă (1993) is further analysed. A new approach to obtaining the estimates for fast controls is given.

Keywords: minimum time function, minimum norm, controllability, rank condition.

1. Introduction

Consider a linear control system

$$x'(t) = Ax(t) + Bu(t) \tag{1}$$

where A generates a C_0-semigroup, $S(t)$, $t \geq 0$, on a Banach space X and $B \in L(U, X)$, U being also a Banach space. For $p \in [1, \infty]$ suppose that the control system (1) is L^p null-controllable for every $T > 0$, that is, for each time $T > 0$ and $\eta \in X$ there exists $u \in L^p(0, T; U)$ such that $x(0) = \eta$ and $x(T) = 0$, where $x(\cdot)$ is the mild solution of (1), i.e.,

$$x(t) = S(t)x(0) + \int_0^t S(t-s)Bu(s)\, ds.$$

By the open mapping theorem (see e.g., Cârjă, 1993, Zabczyk, 1992), if we prescribe $\rho > 0$, there exists $\alpha(T) > 0$ such that all points from the ball of radius $\alpha(T)$ can be transfered to zero by controls $u \in L^p(0, T; U)$ with $\|u\|_{L^p(0,T;U)} \leq \rho$. In other words, where

$$H(T)u = \int_0^T S(T-s)Bu(s)\,ds$$

and $B(0,r)$ stands for the closed ball of center zero and radius r.

This, in particular, implies the continuity in 0 of the minimum time function defined by

$$\mathcal{T}(\eta) = \inf\{T;\ x(0)=\eta,\ x(T)=0,\\ u \in L^p(0,T;U),\ \|u\|_{L^p(0,T;U)} \leq \rho\}; \tag{3}$$

see Cârjă (1993).

As remarked there, more precise estimates of $\alpha(T)$ for T small give more precise estimates for the minimum time function arround the origin.

Our aim here is to show that there is a complete equivalence between the following problems:

(a) estimates of $\alpha(T)$ in (2) for T small;

(b) estimates of the minimum time function;

(c) estimates of the minimum L^p norm for T small.

In the next theorem $C_T(x) := \inf\{\|u\|; u \in L^p(0,T;U),\ S(T)x = H(T)u\}$.

THEOREM 1.1 (i) *Suppose there exists a function* $\alpha : [0,T] \to \mathbb{R}$ *strictly increasing, continuous, with* $\alpha(0)=0$, *and such that* (2) *is verified for* $T \leq T_1$. *Then* $\mathcal{T}(x) \leq \alpha^{-1}(\|x\|)$ *for* $\|x\| \leq \alpha(T_1)$.

(ii) *Suppose there exists a function* $\beta : [0,a] \to \mathbb{R}$ *strictly increasing, continuous, with* $\beta(0)=0$, *such that* $\mathcal{T}(x) \leq \beta(\|x\|)$ *for* $\|x\| \leq a$. *Then for each* $x \in X$ *and each* T *with* $T \leq \beta(a)$ *we have*

$$C_T(x) \leq \frac{\rho}{\beta^{-1}(T)}\|x\|.$$

(iii) *Suppose there exists a function* $\varphi : (0,T_2] \to (0,+\infty)$ *such that* $C_T(x) \leq \varphi(T)\|x\|$, *for each* $x \in X$ *and each* $T \in (0,T_2]$. *Then we have* (2) *for* $T \in (0,T_2]$ *with* $\alpha(T) = \rho/(2\varphi(T))$.

Proof. (i) Take x with $\|x\| \leq \alpha(T_1)$. There exists $T \leq T_1$ such that $\|x\| = \alpha(T)$. By (2) there exists u with $\|u\| \leq \rho$ such that $S(T)x = H(T)u$, hence $\mathcal{T}(x) \leq T$, that is $\mathcal{T}(x) \leq \alpha^{-1}(\|x\|)$.

(ii) Take $T \leq \beta(a)$. There exists $b \leq a$ such that $T = \beta(b)$. For $x \in X$, $x \neq 0$, and $\varepsilon \in (0,1)$, define $y = \varepsilon b\frac{x}{\|x\|}$, observe that $\|y\| < b$ and conclude that $\mathcal{T}(y) < \beta(b) = T$. This shows that there exists a control with $\|u\| \leq \rho$ such that $S(T)y = H(T)u$. Clearly $S(T)x = H(T)\frac{\|x\|}{\varepsilon b}u$, hence

$$C_T(x) \leq \frac{\rho}{\varepsilon\beta^{-1}(T)}\|x\|.$$

(iii) Let x be such that $\|x\| \leq \frac{\rho}{2\varphi(T)}$. We have $\|x\| < \frac{\rho}{\varphi(T)}$, $\varphi(T)\|x\| < \rho$ and $C_T(x) < \rho$. This implies that there exists a control u such that $S(T)x = H(T)u$ and $\|u\| < \rho$, hence the conclusion. ■

REMARK 1.1 We note that (i) and (iii) above were first proved by the second author in Cârjă (1993). We also note that the referees brought our attention to the paper Gozzi and Loreti (1999) where a similar relationship between the problems (b) and (c) above is presented.

We give now an application in the finite dimensional setting. In this case A and B are constant matrices $n \times n$ and $n \times m$ respectively. Assume that the system is controllable, so that the matrix

$$[B, AB, \ldots, A^{n-1}B]$$

has the rank n; see Lee and Markus (1967). Let k be the minimal exponent giving

$$\mathrm{rank}[A, AB, \ldots, A^k B] = n. \tag{4}$$

If this is the case, then for every $\xi, \eta \in \mathbb{R}^n$ and for every time $T > 0$ there exists an L^∞ control u that, applied to (1) gives $x(0) = \eta$, $x(T) = \xi$. Gyurkovics (1984) proved the following result:

THEOREM 1.2 *For the finite dimensional control system* (1), *let k be the minimal exponent giving the rank condition* (4) *and let $\mathcal{T}$ be the minimum time function defined by* (3) *with $p = \infty$. Then there exists $\omega > 0$ such that*

$$\mathcal{T}(x) \leq \omega \|x\|^{\frac{1}{k+1}} \tag{5}$$

for $\|x\|$ small.

Combining Theorems 1.1 and 1.2 we deduce

COROLLARY 1.1 *For the finite dimensional control system* (1), *there exists $\gamma > 0$ computable from A and B, such that for every $\eta \in \mathbb{R}^n$ and for every T small there exists an L^∞ control u that transfers η to 0 in time T and such that*

$$\|u\|_{L^\infty(0,T;U)} \leq \gamma T^{-k-1} \|\eta\|,$$

where k is the minimal exponent giving the rank condition.

This result has been proved recently by Seidman and Yong (1997) using methods somewhat along the lines of the previous paper of Seidman (1989)

2. A new approach

We present here a completely different approach in obtaining estimates for the minimum L^p norm problem for T small (estimates for fast controls, in the terminology of Seidman) in the finite dimensional setting as well as in some infinite dimensional cases. It allows us to get also estimates of the transfer control for T large. It is based on a nice result of Triggiani (1992) which gives an explicit formula for a transfer control u (suboptimal), involving A and B. Let us present that result (see also Zabczyk, 1992).

Let $A \in L(X)$ and $B \in L(U, X)$ be bounded operators on the Banach space X, and from the Banach space U to X, respectively, such that

$$\text{span}\{BU, ABU, \ldots, A^k BU\} = X, \tag{6}$$

for some nonnegative integer k. Define the linear bounded operator $Q : \times_{k+1} U \to X$ by

$$Q(u_0, u_1, \ldots, u_k) = Bu_0 + ABu_1 + \cdots + A^k Bu_k.$$

Assume that $\ker Q$ has a closed complement (in Hilbert spaces this is alaways true). Then, since Q is surjective, it is well known that there exists a right inverse for Q, i.e., there exists $E \in L(X, \times_{k+1} U)$ such that $QE = I$ (the identity operator in X). This implies the existence of $k+1$ linear bounded operators $E_i \in L(X, U)$, $i = 0, 1, \ldots, k$, such that

$$BE_0 + ABE_1 + \cdots + A^k BE_k = I. \tag{7}$$

Let φ be a function of class C^k from $[0, T]$ into $\mathbb{R}$ such that

$$\varphi^{(i)}(0) = \varphi^{(i)}(T) = 0, \ i = 0, 1, \ldots, k; \ \int_0^T \varphi(s)\, ds = 1. \tag{8}$$

Here $\phi^{(i)}$ means the derivative of order i.

THEOREM 2.1 *Assume* (6) *and let* φ *be a function that satisfies* (8). *Then the control*

$$u(s) = E_0 \psi(s) + E_1 \psi'(s) + \cdots + E_k \psi^{(k)}(s), \ s \in [0, T],$$

applied to the dynamical system (1), *where*

$$\psi(s) = S(s - T)(\xi - S(T)\eta)\varphi(s), \ s \in [0, T],$$

transfers η *to* ξ *at time* T.

Our main result is

THEOREM 2.2 *Assume condition* (6). *Then there exists* $\gamma > 0$ *that depends only on* A *and* B *such that for each* $\eta, \xi \in X$ *and for each* $T > 0$, *the* C^∞ *control* u *given by Theorem* 2.1 *satisfies*

$$\|u(t)\| \le \gamma \max\{T^{-k-1}, T^{-1}\} \|S(t - T)(\xi - S(T)\eta)\|, \ t \in [0, T] \tag{9}$$

Proof. We take in Theorem 2.1 the function $\varphi(t) = h(t)/\int_0^T h(s)\,ds$, where $h(t) = t^{k+1}(T-t)^{k+1}$, $t \in [0,T]$.

First, observe that

$$\int_0^T h(s)\,ds = c_1 T^{2k+3} \tag{10}$$

where c_1 depends only on k.

In the next step we show that

$$|h^{(i)}(t)| \le c\, T^{2k+2-i},\ t \in [0,T],\ i = 0,1,\ldots,k, \tag{11}$$

where c depends only on k. This fact follows by an induction argument on k, writing

$$h(t) = f(t)^{k+1},\ t \in [0,T],$$

with, of course, $f(t) = t(T-t)$, and using the formula

$$(f^k f)^{(i)} = (f^k)^{(i)} f + i(f^k)^{(i-1)} f' + \frac{i(i-1)}{2}(f^k)^{(i-2)} f''.$$

By (10) and (11) we obtain

$$|\varphi^{(i)}(t)| \le \frac{c}{c_1} T^{-i-1},\ t \in [0,T],\ i = 0,1,\ldots,k,$$

therefore, if $T \le 1$ we have

$$|\varphi^{(i)}(t)| \le \frac{c}{c_1} T^{-k-1},\ t \in [0,T],\ i = 0,1,\ldots,k,$$

while if $T > 1$ we have

$$|\varphi^{(i)}(t)| \le \frac{c}{c_1} T^{-1},\ t \in [0,T],\ i = 0,1,\ldots,k.$$

Now, the result follows from Theorem 2.1 taking into account that $S(t)'x = AS(t)x$, so that all the derivatives up to the order k of $S(t-T)x$ are bounded above by $\alpha\|S(t-T)x\|$ with α depending only on A. The proof is complete. ■

REMARK 2.1 If $\xi = 0$ we obtain the statement given in Corollary 1.1, because for $0 < T \le 1$ the max in (9) is T^{-k-1} and $\|S(t)\|$ is bounded by a constant independent of T.

Taking into account that for every $p \in [1,\infty]$ we have

$$\|u\|_{L^p(0,T;U)} \le T^{1/p}\|u\|_{C(0,T;U)},$$

COROLLARY 2.1 *Assume* (6). *Then for every* $p \in [1, \infty]$ *there exists a* C^∞ *control* u *that, applied to the dynamical system* (1), *transfers* η *to* 0 (*or* 0 *to* η) *in time* T *and satisfies*

$$\|u\|_{L^p(0,T;U)} \leq \gamma T^{-k-1+\frac{1}{p}} \|\eta\|$$

for T *sufficiently small, where* γ *is independent of* T *and* η.

This agrees with the result of Seidman and Yong (1997) where estimates for the minimum norm control were investigated.

The following theorem shows that the estimate given in Theorem 2.2 is sharp.

THEOREM 2.3 *Assume* (6) *and let* $p \in [1, \infty]$. *Consider* $\eta \in X$ *and suppose* $E_k \eta \neq 0$, *where the operator* E_k *is given by* (7). *Then there exist* $\gamma > 0$, *that depends only on* A, B *and* η, *and a* C^∞ *control* u *that, applied to the dynamical system* (1), *transfers* η *to* 0 *(or* 0 *to* η*) at time* T, *and satisfies*

$$\|u\|_{L^p(0,T;U)} \geq \gamma T^{-k-1+\frac{1}{p}},$$

for T *small enough.*

Proof. We show that the control u given by Theorem 2.1 with $\xi = 0$ (the other case is similar) satisfies

$$\int_0^T \|u(t)\| \, dt \geq \gamma T^{-k} \tag{12}$$

with some $\gamma > 0$. If this holds true, then the result follows by the Hölder's inequality which gives

$$\int_0^T \|u(t)\| \, dt \leq T^{1/q} \|u\|_{L^p(0,T;U)},$$

for $p \in (1, \infty]$, where $1/p + 1/q = 1$.

Let us prove (12). The control u given by Theorem 2.1 may be written as

$$u(s) = \varphi^{(k)}(s) E_k S(s)\eta + \varphi^{(k-1)}(s) R_{k-1}(s)\eta + \cdots + \varphi(s) R_0(s)\eta$$

where $R_i(\cdot)$ are continuous.

Since

$$\int_0^T \varphi^{(i)}(s) \, ds = \omega_i T^{-i}, \; i = 0, 1, \ldots, k$$

where $\omega_i > 0$ and depends only on i, by a mean value theorem we have and

$$\lim_{T\to 0}\int_0^T T^k\|\varphi^{(k-1)}(s)R_{k-1}(s)\eta+\ldots+\varphi(s)R_0(s)\eta\|\,ds=0.$$

Taking into account that

$$\int_0^T T^k\|u(s)\|\,ds\geq\int_0^T T^k\varphi^{(k)}(s)\|E_kS(s)\eta\|\,ds$$
$$-\int_0^T T^k\|\varphi^{(k-1)}R_{k-1}(s)\eta+\cdots+\varphi(s)R_0(s)\eta\|\,ds,$$

the proof is complete. ■

REMARK 2.2 If $\xi=0$, formula (9) gives estimates for the set of initial states which can be transfered to zero by bounded controls, on a given interval. In particular it shows that if $\|S(t)\|$ is bounded for $t\geq 0$ then, for T large, a C^∞ control u transfering η to 0 can be obtained such that

$$\|u(t)\|\leq\bar{\gamma}T^{-1}\|\eta\|,\ t\in[0,T] \tag{13}$$

with $\bar{\gamma}$ independent of T and η. Moreover,

$$\|u\|_{L^p(0,T;U)}\leq\bar{\gamma}T^{-1+\frac{1}{p}}\|\eta\|$$

for T large. This shows that, if $\tilde{u}_T$ is an L^p minimum norm control with $p>1$, we get

$$\lim_{T\to\infty}\|\tilde{u}_T\|_{L^p(0,T;U)}=0.$$

This also gives the well-known result that if $S(t)$ is bounded for $t\geq 0$, then null-controllability with controls in L^p, $p\in(1,\infty]$, implies null controllability in finite time with bounded controls. In this situation the domain of the minimum time function is the whole space X.

REMARK 2.3 It is well known that, in the finite dimensional case, there exists a minimal integer $k\geq 0$ such that

$$X_k:=\operatorname{span}\{BU,ABU,\ldots,A^kBU\}$$

is the reachable set, i.e.,

$$\operatorname{Range}H(T)=X_k$$

for all $T>0$. All the above estimates remain true for $\xi,\eta\in X_k$. To get these facts we work in X_k instead of X. It is important to note that, in the proof of the corresponding variant of Theorem 1.1 in this situation, we use the fact that

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