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Dynamic consistency under private information[1]

by

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Abstract: Time inconsistency is often demonstrated in the context of the global Stackelberg solution of a two-person, two-stage dynamic game. The loop model of dynamic games is used. The recommended solution to the problem is for both players to adopt open-loop strategies. We recast the problem in an imperfect information framework. In contrast to the standard result, (i) a consistency in the outcomes of the Nash game and the associated Stackelberg game is shown and (ii) it is proved that under feedback information patterns, both players prefer to play the Stackelberg game rather than the associated Nash game.

Keywords: time consistency, imperfect information

1. Introduction

In an appraisal of the state of modern macroeconomics, Hahn and Solow (1995) lament the fact that the subject has not benefited from the vast progress that has been made in the study of asymmetric information and strategic interaction. This charge motivates the treatment of the problem of time inconsistency below. The issue must perforce be cast in the language of dynamic systems theory. The loop model of dynamic games is used. The model allows for situations where agents possess imperfect information about the value of the state vector in each period. This is captured by a *private* state-measurement equation. In other words, the framework can also be said to include the phenomenon of asymmetric information. In an optimal control model with a finite number of players, the state vector is both exogenous and endogenous to the players. In the former sense, the information represented by the value of the state in each period is fed back into the strategies of the players. If the observation of

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the state is imperfect, it can be assumed that there is a decrease in imperfect information about the state from period to period for reasons to do with rational learning and so on. It is also simultaneously true that players by means of their *joint* actions engender the state. Information is endogenous. In the language of the economics of information, it can therefore be taken as an axiom that truth-telling is a dominating strategy. The players cannot do better than reveal their private information about the state vector as this information is fed back into their strategies. Both these aspects of information are encapsulated in an assumption that is critical to the argument.

Time inconsistency is shown in the framework of a two-agent, two-period dynamic game. The cost functionals of the players is assumed to be additively separable. The problem of minimising the cost functionals is not fully specified until the beliefs that each agent has of the decision rules of the other are described. The common way of specifying these rules and of defining an equilibrium is the (feedback) Nash equilibrium (NE). The definition of the solution leads to a recursive derivation that involves the solution of two static games. Starting at stage two, a single-act game is solved. The feedback equilibrium solution depends only on the value of the state at that stage. The optimal controls are then substituted into the constrained optimisation problem of stage one. The dependence on the initial stage is only at stage one. If the optimisation process is repeated when the second period actually arrives, assuming the optimal values of the state vector and actions obtained from the optimisation process in the initial period, the same result for the solution in the second period is obtained. The feedback NE is time-consistent.

The next equilibrium concept to be considered in the economics literature is the Global Stackelberg Solution. The concept is suitable for the class of decision problems wherein the leader has the ability to announce her decisions at all of her possible decision sets ahead of time. The motivation for a transition from a regime in which both players are symmetrically poised to one in which there is a hierarchy in decision making is easy to provide from examples in economics. There is a vigorous debate underway about the choice of monetary arrangements and whether the leadership of a central bank is desirable. There is also intense scholarly discussion concerning whether exchange rate regimes necessarily involve a hegemon like Germany in the case of Europe. We shall assume that the Nash costs are part of the common knowledge of the game. The Global Stackelberg Solution is predicated on the leader committing herself irrevocably to the actions dictated by those strategies. However, in the absence of a commitment technology, in the second period, the leader has an incentive to depart from the previously optimal action and deploy a different action. In the macroeconomic illustrations, the payoffs to the government (leader) thereby increase and the payoffs to the follower (private sector) fall. The time-consistent solution is suboptimal. Since the inconsistency arises because of the dynamic information that is forthcoming in each period, the suggested solution to the problem is to employ constant strategies. In terms of the hoary debate in macroeconomics, the case is made for rules as against discretion.

This claim is shown not to hold if the imperfect observation variant of the state-space model is used. The result is unsurprising if it is recalled that the subject is considered against the broad background of the classical/Keynesian divide in macroeconomics. New Keynesian theory as, for instance, exemplified by the work of Stiglitz and his associates, can be regarded as dynamic games played under information asymmetry. Agents, it is assumed, possess useful private information that cannot be conceivably be communicated to governments. However, the object of the research strategy is to show that even in classical environments outcomes are inefficient both in the static as well as in the dynamic sense. If information in the possession of agents is imperfect, there is no reason to assume that governments have information partitions that are necessarily coarser than those of private agents. On the other hand, it might be assumed that governments have an informational advantage with regard to *system-wide* variables (Correa, 1997, p.103).

It could be argued that the form of private contracts permits agents to vary price not on the basis of private information alone but also on the basis of public information. For example, interest rate contracts might be related to a publicly observed rate administered by the monetary authorities. In that case, it can be shown that there is scope for policy intervention that makes *both* leader (monetary authorities) and follower (private sector) better off (Correa, 1997, p.107). Indeed, there are potential generalisations of New Keynesian models in the framework elaborated below. For instance, there is a bias in these stories in favour of one-sided asymmetric information. A major asymmetry, for example, is posited between businessmen and banks because the former know more about the conditions and prospects for their own companies. By the same token, however, the condition of the balance-sheets of banks is information not revealed to firms. This realisation is germane to appraisals of financial crises in Asian economies. As a result of terminating years of interest rate repression, the discretionary power of banks to raise intermediate margins increased. Banks were tempted to misallocate resources. Underfunded banks were tempted to finance high return/high risk projects. The proportion of non-performing loans in bank portfolios rose. The implicit cost had to be borne by debtors still meeting their obligations. In the following account asymmetric information will be taken to mean bilateral private information.

The following account of the loop model is from Basar and Jan Olsder (1995, pp.225-228). A final section is a summary.

2. The main results

DEFINITION 2.1 *A* two-person, two-stage dynamic game *consists of the following, where superscripts distinguish the players, subscripts the stages.*
(i) *An index set* $\mathrm{N} = \{1, 2\}$ *called the* players' set.
(ii) *An index set* $\mathrm{K} = \{1, 2\}$ *denoting the* stages *of the game.*

(iii) *An infinite set* X *with some topological structure called the* state space *of the game, to which the state of the game* x_k *belongs for all* $k \in \mathbf{K}$ *and* $k = 3$.

(iv) *An infinite set* U_k^i *with some topological structure defined for each stage* $k \in \mathbf{K}$ *and each player* $i \in \mathbf{N}$ *called the* control set *of player* i *at stage* k. *Its elements are permissible actions* u_k^i *of player* i *at stage* k.

(v) *A function* $f_k : X \times U_k^1 \times U_k^k \to X$, *defined for* $k \in \mathbf{K}$ *so that* $x_{k+1} = f_k(x_k, u_k^1, u_k^2)$, $k \in \mathbf{K}$ *for some* $x_1 \in X$ *called the* initial state *of the game. The difference equation above is called the* state equation *of the dynamic game.*

(vi) *A set* Y_k^i *with some topological structure defined for each stage* $k \in \mathbf{K}$ *and each player* $i \in \mathbf{N}$ *called the* observation set *of player* i *at stage* k, *to which the observation* y_k^i *of player* i *belongs at stage* k.

(vii) *A function* $h_k^i : X \to Y_k^i$, *defined for* $k \in \mathbf{K}$ *and* $i \in \mathbf{N}$ *so that* $y_k^i = h_k^i(x_k)$, $k \in \mathbf{K}$ *and* $i \in \mathbf{N}$ *which is* the private state-measurement equation *of player* i *concerning the value of* x_k.

(viii) *A finite set* η_k^i, *defined for each stage* $k \in \mathbf{K}$ *and each player* $i \in \mathbf{N}$ *as a subcollection of* $\{y_1^1, \ldots, y_k^1; y_1^2, \ldots, y_k^2; u_1^1, \ldots, u_{k-1}^1; u_1^2, \ldots, u_{k-1}^2\}$, *which determines the information gained and recalled by player* i *at stage* k *of the game.*

(ix) *A set* N_k^i, *defined for each stage* $k \in \mathbf{K}$ *and player* $i \in \mathbf{N}$ *as an appropriate subset of* $\{(Y_1^1 \times \ldots \times Y_k^1) \times (Y_1^2 \times \ldots \times Y_k^2) \times (U_1^1 \times \ldots \times U_{k-1}^1) \times (U_1^2 \times \ldots \times U_{k-1}^2)\}$ *compatible with* $\eta_k^i N_k^i$ *is called the* information space *of player* i *at stage* k, *induced by her information* η_k^i.

(x) *A prespecified class* Γ_k^i *of mappings* $\gamma_k^i : N_k^i \to U_k^i$ *which are the permissible* strategies *of player* i *at stage* k. *The aggregate mapping* $\gamma^i = \{\gamma_1^i, \gamma_2^i\}$ *is a* strategy *for player* i *in the game. The class* Γ^i *of all such mappings* γ^i *so that* $\gamma_k^i \in \Gamma_k^i$, $k \in \mathbf{K}$, *is called the* strategy space *of player* i.

(xi) *A composite mapping* $J^i : \Gamma^1 \times \Gamma^2 \to R$, *for each player* $i \in \mathbf{N}$ *called the cost functional of player* i .

The possible information structures of relevance are classified in the following definition.

DEFINITION 2.2 *Player* i*'s information structure is called*
(i) closed-loop imperfect state *if* $\eta_k^i = \{y_1^i, \ldots, y_k^i\}$, $k \in \mathbf{K}$
(ii) feedback imperfect state *if* $\eta_k^i = \{y_k^i\}$, $k \in \mathbf{K}$

The comments on information in the first section are summarised in the following fundamental assumption. It is stated for player 1 and for any strategy γ^{2^*} of player 2. The assumption naturally extends to the case when the roles of the players are reversed.

ASSUMPTION 2.1 $$\min_{\gamma_{k+1}^1 \in \Gamma_{k+1}^1} J^1(\gamma^1, \gamma^{2^*}) \le \min_{\gamma_k^1 \in \Gamma_k^1} J^1(\gamma^1, \gamma^{2^*}),\ k \in \mathbf{K}$$

Recall that the assumption reflects a decrease in imperfect information about the value of state variable in the second stage. It also embodies the observation mentioned in the introduction that the players can do worse than reveal their *bilateral* private information about the state vector as this information is fed back into their strategies.

The popular equilibrium solution is given by

DEFINITION 2.3 *A pair of strategies* $\{\gamma^{1^*}, \gamma^{2^*}\}$ *with* $\gamma^{i^*} \in \Gamma^i$, $i = 1, 2$ *is said to constitute a* Nash equilibrium solution *if, and only if the following inequalities are satisfied for all* $\{\gamma^i \in \Gamma^i;\ i = 1, 2\}$*:*

$$\begin{aligned} J^{1^*} &\equiv J^1(\gamma^{1^*}, \gamma^{2^*}) \leq J^1(\gamma^1, \gamma^{2^*}) \\ J^{2^*} &\equiv J^2(\gamma^{1^*}, \gamma^{2^*}) \leq J^2(\gamma^{1^*}, \gamma^2) \end{aligned}$$

For the purpose of the Stackelberg equilibrium solution, the following definitions will be required where **P1** is the leader and **P2** the follower.

DEFINITION 2.4 *The set*

$$R^2(\gamma^1) = \{\xi \in \Gamma^2 : J^2(\gamma^1, \xi) \leq J^2(\gamma^1, \gamma^2), \forall \gamma^2 \in \Gamma^2\}$$

is the rational reaction set *of* **P***2 to the strategy* $\gamma^1 \in \Gamma^1$ *of* **P***1. If* $R^2(\gamma^1)$ *is a singleton for each* $\gamma^1 \in \Gamma^1$ *then there exists a mapping* $T^2 : \Gamma^1 \to \Gamma^2$ *such that* $\gamma^2 \in R^2(\gamma^1)$ *implies* $\gamma^2 = T^2\gamma^1$.

DEFINITION 2.5 *A strategy* $\gamma^{1^*} \in \Gamma^1$ *is called a* Stackelberg equilibrium strategy *for the leader if*

$$\max_{\gamma^2 \in R^2(\gamma^{1^*})} J^1(\gamma^{1^*}, \gamma^2) = \min_{\gamma_1 \in \Gamma^1} \max_{\gamma^2 \in R^2(\gamma^1)} J^1(\gamma^1, \gamma^2)$$

Let the above quantity be denoted by J^{1^*} which is the Stackelberg cost of the leader. When the optimal response of the follower is unique for every strategy of the leader, the definition simplifies to

$$J^1(\gamma^{1^*}, T^2\gamma^{1^*}) = \min_{\gamma^1 \in \Gamma^1} J^1(\gamma^1, T^2\gamma^1) \equiv J^{1^*}$$

DEFINITION 2.6 *Any element* $\gamma^{2^*} \in R^2(\gamma^{1^*})$ *is an* optimal strategy *for the follower that is* in equilibrium *with* γ^{1^*}. *The pair* $\{\gamma^{1^*}, \gamma^{2^*}\}$ *is a* Stackelberg solution *for the game and the pair* $(J^1(\gamma^{1^*}, \gamma^{2^*}), J^2(\gamma^{1^*}, \gamma^{2^*}))$ *is the corresponding* Stackelberg equilibrium outcome.

A fundamental role is played in the theory of differential games by the notion of "representations of strategies along trajectories". It enables the construction of equivalence classes of equal open-loop value strategies in the general class of closed-loop strategies. The procedure is follows. First determine the set Γ^G of all elements of Γ which are strategies that depend only on the initial state x_1 and

the discrete parameter k. Γ^G is the class of all permissible open-loop controls in Γ. (The superscript G is for global). Now let $\tilde{\gamma}^G = \{\tilde{\gamma}_k^G(x_1^G); k \in \mathbf{K}\}$ be a chosen element of Γ^G which generates, by substitution into the state equation, a unique trajectory $\{\tilde{x}_k^G, k \in \mathbf{K}\}$. Then consider all elements $\gamma^G = \{\gamma_k^G(\cdot), k \in \mathbf{K}\}$ of Γ^G with the properties
(i) γ^G generates the same trajectory as $\tilde{\gamma}^G$.
(ii) $\gamma_k^G(\tilde{x}_k^G, \tilde{x}_{k-1}^G, \ldots, \tilde{x}_1^G, x_1^G) \equiv \tilde{\gamma}_k^G(x_1^G), k \in \mathbf{K}$.
The subset of Γ thus constructed constitutes an equivalence class of representations all of which have the same open-loop value $\tilde{\gamma}^G$. The construction leads to an uncountable number of elements in each equivalence class. The main reason for this nondenumerable property of equivalence classes is that in deterministic problems dynamic information patterns involving memory exhibit a redundancy in information. This gives rise to an infinity of different representations of the same open-loop policy.

A common property of Stackelberg solutions obtained under dynamic information is that they are also Nash equilibrium solutions (Basar & Olsder, 1995, p.384). A formal proof is as follows.

PROPOSITION *Under dynamic information, the Stackelberg solution is also a Nash equilibrium solution.*

Proof. In the familiar manner, starting at the last level K, solve a single-act game for each information set of the first-acting player at each level of play and then appropriately concatenate all the equilibrium strategies thus obtained. Suppose that the minimization of $J^2(\gamma^{1^*}, \cdot)$ over $R^2(\gamma^{1^*})$ is given by γ^{2^*}. Minimization of the same expression over $\Gamma^2 \supseteq R^2(\gamma^{1^*})$, an increase in deterministic information does not result in a change in the value of the cost functional because of the phenomenon of "representations of strategies along trajectories". In other words,

$$\begin{aligned} J^2(\gamma^{1^*}, \gamma^{2^*}) &\leq J^2(\gamma^{1^*}, \gamma^2), \forall \gamma^2 \in \Gamma^2 \text{ and} \\ J^1(\gamma^{1^*}, \gamma^{2^*}) &\leq J^1(\gamma^1, \gamma^{2^*}), \forall \gamma^1 \in \Gamma^1 \end{aligned}$$

■

Indeed, in the case of the leader it is well known that if the rational reaction set is a singleton, the Stackelberg cost for the leader is, in fact, lower than the Nash equilibrium cost. It is important to recall that this sufficiency condition cannot be relaxed. In particular, if the game admits a unique Nash equilibrium solution and a unique "cheating" strategy for the leader and if the rational reaction set of the follower is a singleton, the result might not hold for the leader.

The dynamic game described in Definition 1 is now solved in the following manner. Solve the optimization problem for player 1 at the first stage and obtain the Nash equilibrium for both stages. Denote this by $(\gamma_{1(1)}^{1N}, \gamma_{2(1)}^{1N})$. Let the Nash costs associated with the Nash game for the player be denoted by

B. Now consider the optimisation process when the second stage effectively arrives, assuming that the strategies in the first period have taken on the values $\gamma^{1N}_{1(1)}, \gamma^{2N}_{1(1)}$. The Nash equilibrium for player 1 is given by $\gamma^{1N}_{2(2)}$. Call the Nash costs associated with this game A. By virtue of the assumption

$$A \leq B$$

Conduct a similar exercise, this time with the first player as Stackelberg leader. Let the Stackelberg costs for the leader incurred by solving the leader-follower game at the first stage be called D. The Stackelberg costs for the leader as a result of playing the Stackelberg game at the second stage is denoted by C. From the assumption,

$$C \leq D$$

It is well known that if the optimal reaction set of the follower is a singleton for any announced strategy of the leader, the leader never prefers to play the Nash game than the associated Stackelberg game. In other words,

$$D \leq B; \quad C \leq A$$

An analogous sequence for the second player can be carried out. A star will distinguish player 2. In the Nash case,

$$A^* \leq B^*$$

As Stackelberg follower,

$$C^* \leq D^*$$

Since the Stackelberg solutions obtained under dynamic information are also Nash equilibrium solutions,

$$D^* \leq B^*; \quad C^* \leq A^*$$

The symmetry of the two sets of outcomes is in contrast to the asymmetry in the traditional account. In the familiar story, the Stackelberg solution is a non-equilibrium one unless the leader ties her hands in advance to an announced strategy. The follower, as in the Nash equilibrium concept, considers the leader's strategy as independent of her own decisions. As a result, the optimization problem of the follower is an optimal control problem and her Stackelberg equilibrium is time-consistent. The leader, on the other hand, takes into account the influence of her strategy on that of the follower. In the macroeconomic illustrations, the payoffs to the government (leader) from cheating increase and the payoffs to the private sector (follower) fall. Distinguishing the cheating solution by the superscript c and recalling that the usual account is in terms of perfect information,

$C^c \leq D$ for the leader and

$D^* \leq C^{C^*}$ for the follower

In the present framework, however, *both* players gain from a decrease in imperfect information and *both* the Nash and Stackelberg equilibrium solutions are shown to be stable. In particular, the follower also prefers to play the Stackelberg game indicating that the notion of leadership is relative. When both players benefit from the leadership of one of them, there is no reason for any player to deviate from the corresponding Stackelberg solution that was computed under mutual agreement. Such a solution is called *concurrent* (Basar & Olsder, 1995, p.192).

3. Conclusion

There are two types of hierarchical, multi-stage decision problems in the state-space representation of dynamic games. In the prior commitment mode of play, decisions are made at the beginning. If the leader cannot 'tie her hands' to the actions dictated by her strategies, her optimal policy is time-inconsistent. She will be tempted to renege on her commitment at every stage based on the actual information that will be available at that stage. It is possible that the follower will be worse off than would be the case if the players were playing a Nash game. According to the delayed commitment solution, on the other hand, the leader waits to employ the dynamic information that will be forthcoming at every stage before announcing her action. The solution is time-consistent. We show that under the imperfect information representation of the loop model of dynamic games, the delayed commitment solution is not inferior to the prior commitment solution for both leader and follower. In terms of the classic controversy in macroeconomics, the case is made for discretion over rules.

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