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Duality theory for state-constrained control problems governed by a first order PDE system

by

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Abstract: In this paper we prove weak and strong duality results for optimal control problems with multiple integrals, first–order partial differential equations and state constraints. We formulate conditions under which the sequence of canonical variables y^ϵ in the ϵ-maximum principle, proved in Pickenhain and Wagner (2000), form a maximizing sequence in the dual problem.

Keywords: control problems with multiple integrals, necessary and sufficient optimality conditions, duality theory.

1. Introduction

1.1. Problem Formulation

We consider the following optimal control problem (P) with first-order partial differential equations and state constraints:

$$J(x,u) = \int_\Omega f_0(t,x(t),u(t))\,dt + \sum_{k=1}^{s} \int_\Omega f_k(t,x(t))\,d\alpha_k(t) \to \text{Min!} \tag{1}$$

subject to $x \in W_p^{1,n}(\Omega)$, $u \in L_p^r(\Omega)$ $(p > m)$, satisfying a.e. on Ω:

state equations

$$x_{i;t_j}(t) = g_{ij}(t,u(t)), \quad i = 1,\ldots,n;\ j = 1,\ldots,m, \tag{2}$$

control restrictions

$$u(t) \in \mathrm{U}, \quad \mathrm{U} \in \mathrm{Comp}\,(\mathbb{R}^r) \setminus \{\emptyset\}, \tag{3}$$

boundary conditions

$$x(t) = \varphi(t) \text{ for all } t \in \Gamma \subset \Omega;\ \Gamma \text{ compact},\ \Gamma \neq \emptyset, \tag{4}$$

and *state constraints*

The formulation of (P) includes *state-constrained problems of Dieudonné–Rashevsky type* if $f_k = 0$ for all $k > 0$, see Cesari (1969), as well as *state-constrained deposit problems* for $f_0 = 0$, $f_k(t, x(t)) = x_k(t)$, see Klötzler and Pickenhain (1993). Convexity assumptions are needed for the derivation of existence results as well as necessary and sufficient optimality conditions. If these assumptions do not hold, we construct the standard relaxation $(\overline{P})$ (or convexification) of (P) using the Young measures, namely: Minimize

$$\bar{J}(x,\mu) = \int_\Omega \int_{\mathrm{U}} f_0(t, x(t), v)\, d\mu_t(v)\, dt + \sum_{k=1}^{s} \int_\Omega f_k(t, x(t))\, d\alpha_k(t) \tag{6}$$

subject to $x \in W_p^{1,n}(\Omega)$, $\mu \in \mathcal{M}_{\mathrm{U}}$, satisfying a.e. on Ω:

state equations

$$x_{i;t_j}(t) = \int_{\mathrm{U}} g_{ij}(t, v)\, d\mu_t(v), \quad i = 1, \ldots, n;\ j = 1, \ldots, m, \tag{7}$$

generalized control restrictions

$$\operatorname{supp} \mu_t \subseteq U,\ U \in \operatorname{Comp}(\mathbb{R}^r) \setminus \{\emptyset\} \text{ for all } t \in \Omega \tag{8}$$

boundary conditions

$$x(t) = \varphi(t) \text{ for all } t \in \Gamma \subset \Omega;\ \Gamma \text{ compact},\ \Gamma \neq \emptyset, \tag{9}$$

and *state constraints*

$$x(t) \in \overline{X(t)} \Leftrightarrow h_l(t, x(t)) \leq 0 \text{ for all } t \in \Omega, \quad l = 1, \ldots, w. \tag{10}$$

Our basic assumptions for (P) and $(\bar{P})$ are the following:

(V1) We have $m \geq 2$ and $m < p < \infty$. $\Omega \subset \mathbb{R}^m$ is a compact Lipschitz domain (in the strong sense, see Morrey, 1966). Then, functions $x \in W_p^{1,n}(\Omega)$ are continuously representable, and functions $x \in W_\infty^{1,n}(\Omega)$ have Lipschitz representatives on Ω, Alt (1992) (p. 185, Theorem 5.5).

(V2) The functions f_0, f_k, g_{ij}, h_l and φ are continuous w.r.t. all their arguments; $f_0(t,\cdot,\cdot)$, $f_k(t,\cdot)$, $g_{ij}(t,\cdot)$ and $h_l(t,\cdot)$ are continuously differentiable w.r.t. ξ resp. (ξ, v) for all $t \in \Omega$.

(V3) $\alpha_k \in rca(\Omega)$ are signed regular measures on the σ-algebra of the Borel sets on Ω.

(V4) The set of feasible solutions (x, u) of (P) is denoted by Z, and Z is nonempty.

We emphasize *two special types of boundary conditions:*

$$x(t_0) = x_0 \text{ for fixed } t_0 \in \Omega, \text{ i.e. } \Gamma = \{\, t_0 \}; \tag{11}$$

1.2. Outline and main result of the paper

The main topic of this paper is to prove duality results for the problem $(\overline{P})$. We do this in two essential steps. First we introduce class-qualified problems as in Pickenhain and Wagner (1999, 2000):

DEFINITION 1.1 *For (P) as well as $(\overline{P})$ and $k \in \mathbb{N}_0$ we obtain the class-qualified problem*

$$(\mathrm{P})_{\mathcal{B}^k} \quad (1)\text{–}(5) \quad \textit{and } x_{i;t_j} \textit{ possesses one representative from } \mathcal{B}^k(\Omega) \tag{13}$$

and the class-qualified relaxed problem

$$(\overline{P})_{\mathcal{B}^k} \quad (6)\text{–}(10) \quad \textit{and } x_{i;t_j} \textit{ possesses one representative from } \mathcal{B}^k(\Omega). \tag{14}$$

In Pickenhain and Wagner (2000) we derived Pontryagin's maximum principle for problems (P) and $(\overline{P})$. It was pointed out that, in the general case, the multipliers corresponding to the state equations (2), respectively (7), cannot be taken from $(L_p^{nm})^*$, $1 \leq p < \infty$, but from the space $(L_\infty^{nm})^*$. To avoid this situation and to obtain more regular multipliers, it was proposed to restrict the feasible domain on elements (x, u), respectively (x, μ), having representatives *of first Baire class* for $(x_{i;t_j})_{ij}$ as well as for $(g_{ij}(\cdot, u(\cdot)))_{ij}$, respectively $(\int_{\mathrm{U}} g_{ij}(\cdot, v)\, d\mu_\cdot(v))_{ij}$. In this way the maximum principle for $(\overline{P})_{\mathcal{B}^k}$ was shown in Pickenhain and Wagner (1999) [Theorem 3.4.] wherein the multipliers corresponding to (7) are Radon measures. Furthermore, in Pickenhain and Wagner (2001A,B) sufficient conditions were proved under which the minimal value of (P) and $(\overline{P})$ remains unchanged if a Baire class qualification is added to the problem. This was done for problems without state contraints. In the present paper this result is extended to the case of state-constrained problems.

In the second step we formulate a dual problem with dual variables as Radon measures. We use the ϵ-maximum principle for $(\overline{P})$ to prove strong duality results for $(\overline{P})$ and $(\overline{P})_{\mathcal{B}^k}$, $k = 0, 1, 2$.

1.3. Notations

We abbreviate the m-dimensional Lebesgue measure of A by $|\mathrm{A}|$, the closure of A by $\overline{\mathrm{A}}$ and the actual zero element by 0. $C^{k,n}(\Omega)$, $L_p^n(\Omega)$ and $W_p^{k,n}(\Omega)$ $(1 \leq p \leq \infty)$ denote the spaces of n-dimensional vector functions on Ω, whose components are k-times continuously differentiable, respectively belong to $L_p(\Omega)$ or to the Sobolev space of $L_p(\Omega)$-functions having weak derivatives up to k^{th} order in $L_p(\Omega)$. The subspace of $C^{k,n}(\Omega)$-functions with compact support is denoted by $\mathring{C}^{k,n}(\Omega)$; instead of $C^{0,1}(\Omega)$ we write shortly $C^0(\Omega)$. For the classical and weak partial derivatives of x_i w.r.t. t_j we use the same notation: $x_{i;t_j}$. The Banach space of Radon measures (signed regular measures with the total set of nonnegative Radon measures is $rca_+(\Omega, \mathcal{B})$. Due to the compactness of Ω, $rca(\Omega, \mathcal{B})$ is isomorphical to the dual space $(C^0(\Omega))^*$, see Dunford and Schwarz (1988) (p. 265, Theorem 3), so that each linear, continuous functional on $C^0(\Omega)$ can be represented by an integral w.r.t. a Radon measure $\nu \in rca(\Omega, \mathcal{B})$.

DEFINITION 1.2 (Generalized controls) *A family $\mu = \{\mu_t \mid t \in \Omega\}$ of probability measures $\mu_t \in rca(\Omega, \mathcal{B}_{\mathrm{U}})$ acting on the σ-algebra $\mathcal{B}_{\mathrm{U}}$ of the Borel sets of* U *is called a generalized control on* U *if for any continuous function $f \in C^0(\Omega \times \mathrm{U})$ the function $h_f: \Omega \to \mathbb{R}$ with $h_f(t) = \int_{\mathrm{U}} f(t,v)\, d\mu_t(v)$ is measurable, see Gamkrelidze (1978) (p. 23). Two families μ', μ'' can be identified iff $\mu'_t \equiv \mu''_t$ for a.e. $t \in \Omega$.*

The set of all generalized controls on U is denoted by $\mathcal{M}_{\mathrm{U}}$. We equip $\mathcal{M}_{\mathrm{U}}$ with the following topology:

$$\{\mu^N\} \to \mu^* \Leftrightarrow \lim_{N\to\infty} \int_\Omega \int_{\mathrm{U}} f(t,v)\, d\mu_t^N(v)\, dt = \int_\Omega \int_{\mathrm{U}} f(t,v)\, d\mu_t^*(v)\, dt \tag{15}$$

for all $f \in C^0(\Omega \times \mathrm{U})$. By compactness of Ω and U, each family $\{\mu_t\}$ is finite in the sense of Gamkrelidze (1978, p. 21 ff.), and each function h_f generated by some $\mu \in \mathcal{M}_{\mathrm{U}}$ is bounded and, consequently, integrable on Ω. The set $\mathcal{M}_{\mathrm{U}}$ is convex, see Gamkrelidze (1978, p. 25), and sequentially compact in the topology introduced above, see Kraut and Pickenhain (1990, p. 391, Theorem 4). Upon the definition of the set-valued maps $\widetilde{\mathrm{G}}(t): \Omega \to \mathcal{P}(\mathbb{R}^{nm})$ and $\mathcal{M}_{\mathrm{U}}(t): \Omega \to \mathcal{P}(rca(\mathrm{U}, \mathcal{B}_{\mathrm{U}}))$ as

DEFINITION 1.3

$$\widetilde{G}(t) = \Big\{ z \in L_p^{nm}(\Omega) \,\Big|\, z_{ij} = \int_{\mathrm{U}} g_{ij}(t,v)\, d\mu_t(v) \ \text{a.e. on } \Omega; \ \mu \in \mathcal{M}_{\mathrm{U}}(t) \Big\} \tag{16}$$

and

$$\mathcal{M}_{\mathrm{U}}(t) = \{\mu_t \in rca(\mathrm{U}, \mathcal{B}_{\mathrm{U}}) \mid \mu_t \geq 0,\ \mu_t(\mathrm{U}) = 1\} \tag{17}$$

The state equations (7) can be reformulated as differential inclusions:

$$(7) \Leftrightarrow (x_{i;t_j}(t))_{ij} \in \widetilde{\mathrm{G}}(t) \quad \text{for a.e. } t \in \Omega. \tag{18}$$

DEFINITION 1.4 (Baire classification) *We call a continuous function ψ defined on the compact set $\Omega \subset \mathbb{R}^m$ from 0^{th} Baire class and write $\psi \in \mathcal{B}^0(\Omega)$. The limit functions of everywhere pointwise convergent sequences $\{\psi^K\}$, $\psi^K \in \mathcal{B}^0(\Omega)$, form the first Baire class $\mathcal{B}^1(\Omega)$; the limit functions of everywhere pointwise convergent sequences $\{\psi^K\}$, $\psi^K \in \mathcal{B}^1(\Omega)$, form the second Baire class $\mathcal{B}^2(\Omega)$,* Obviously, $\mathcal{B}^0(\Omega) \subset \mathcal{B}^1(\Omega) \subset \mathcal{B}^2(\Omega) \subset \ldots$ holds. Note that each finite function contained in any Baire class is measurable, see Carathéodory (1968, p. 404, Theorem 4); conversely, any measurable, essentially bounded function on Ω agrees a.e. with some function of second Baire class, see Carathéodory (p. 406, Theorem 5). Each Baire class is closed under (pointwise) addition and multiplication of finite functions, see Carathéodory (p. 397, Theorems 6 and 7).

Defining functionals $H_l: C^{0,n}(\Omega) \to \mathbb{R}$ by

$$H_l(x) = \mathrm{Max}_{t\in\Omega}\, h_l(t, x(t)), \quad 1 \le l \le w,$$

the state constraints (5) can be expressed as follows:

$$h_l(t, x(t)) \le 0 \;\forall\, t \in \Omega \;\Leftrightarrow\; H_l(x) \le 0, \quad l = 1, \ldots, w. \tag{19}$$

2. Comparison of minimal values

We prove now the following conditions for the coincidence of the infima of $(\overline{P})$ and $(\overline{P})_{\mathcal{B}^k}$:

THEOREM 2.1 *Let $(\overline{P})$ be given under assumptions* (V1)–(V4). *We assume further that:*

($\overline{\mathrm{V5}}$) *There exists a ball $\mathrm{K}(0, \omega_1)$ which is a subset of $\widetilde{\mathrm{G}}(t)$ for all $t \in \Omega$.*

($\overline{\mathrm{V6}}$) *$(\overline{P})$ admits a feasible solution $(x, \mu) \in (C^{1,n}(\Omega) \cap W^{2,n}_\infty(\Omega)) \times \mathcal{M}_{\mathrm{U}}$ with $(x_{i;t_j}(t))_{ij} \in \widetilde{\mathrm{G}}(t)$, $0 < \omega_0 \le \mathrm{Dist}\,((x_{i;t_j}(t))_{ij}, \partial\widetilde{\mathrm{G}}(t))$ for a.e. $t \in \Omega$ and $x(t) \in \overline{X(t)}$, $0 < \omega_1 \le \mathrm{Dist}\,(x(t), \partial\overline{X(t)})$ for all $t \in \Omega$, the set $\overline{X(t)}$ is assumed to be convex $\forall t \in \Omega$.*

($\overline{\mathrm{V7}}$) *The functions g_{ij} satisfy (independently of $v \in \mathrm{U}$) Lipschitz conditions of the type*
$|g_{ij}(t', v) - g_{ij}(t'', v)| \le L_{ij} \cdot |t' - t''| \;\forall\, t',\, t'' \in \Omega$ with $L_{ij} > 0$.

Then, $(\overline{P})$ admits a minimizing sequence $\{(x^N,\, \mu^N)\}$ with representatives of 0^{th} Baire class for $x^N_{i;t_j}$, and the minimal values of $(\overline{P})$ and $(\overline{P})_{\mathcal{B}^k}$, $k = 0, 1, \ldots$ coincide. Moreover, the (x^N, μ^N) can be determined in such a way that the relaxed state equations (7) are satisfied everywhere on Ω.

Proof. We consider the sequence of problems $(\overline{P}_m)$, $m \in \mathbb{N}$, where the state constraints (10) are replaced by

$$x(t) \in X_m(t), \tag{20}$$

with the open set $X_m(t)$ and

$$\xi \in X_m(t) \;\Leftrightarrow\; \xi \in X(t) \quad \text{and} \quad \mathrm{Dist}\,(\xi, \partial\overline{X(t)}) > \frac{1}{m}. \tag{21}$$

Assumption ($\overline{\mathrm{V6}}$) guarantees that $X_m(t) \ne \emptyset \quad \forall t \in \Omega$ and $m \ge m_0$. It follows now that

$$\nu_m := \inf(\overline{P}_m) \ge \inf(\overline{P}_{m+1}) \ge \inf(\overline{P}) = \nu \tag{22}$$

for all $m \geq m_0$. Let (x^*, μ^*) be an existing optimal solution of $(\overline{P})$, see Pickenhain and Wagner (2000), Theorem 2.2. Then the sequence $\{(x^M, \mu^M)\}_{M=1}^{\infty}$ with

$$x^M = \frac{1}{M}x + \left(1 - \frac{1}{M}\right)x^*, \quad \mu^M = \frac{1}{M}\mu + \left(1 - \frac{1}{M}\right)\mu^* \tag{23}$$

and (x, μ) from assumption $(\overline{\text{V6}})$ is admissible to $(\overline{P})$, since $\overline{X(t)}$ is convex and it easily follows

$$\lim_{M \to \infty} \bar{J}(x^M, \mu^M) = \bar{J}(x^*, \mu^*). \tag{24}$$

From assumption $(\overline{\text{V6}})$ and $(\overline{\text{V7}})$ we obtain for the ball $K_{\omega_1}(x(t))$ around $x(t)$:

$$K_{\omega_1}(x(t)) \subseteq \overline{X(t)} \quad \forall t \in \Omega \tag{25}$$

and

$$\operatorname{conv}\{K_{\omega_1}(x(t)), \{x^*(t)\}\} \subseteq \overline{X(t)}. \tag{26}$$

Therefore

$$K_r(x^M(t)) \subseteq \overline{X(t)} \quad \text{for } r = \frac{1}{M}\omega_1 \tag{27}$$

and we obtain that (x^M, μ^M) is admissible for $(\overline{P}_m)$ with $m > \frac{M}{\omega_1}$. Together with (22) we find

$$\kappa = \lim_{M \to \infty} \bar{J}(x^M, \mu^M) \geq \lim_{M \to \infty} \kappa_{m(M)} \geq \kappa \tag{28}$$

and

$$\lim_{m \to \infty} \kappa_m = \kappa. \tag{29}$$

Now we apply Theorem 1.3 from Pickenhain and Wagner (2001B). Then $(\overline{P}_m)$ has an optimal solution (x_m^*, μ_m^*), see Pickenhain and Wagner (2000), and admits a sequence $\{(x_m^N, \mu_m^N)\}_{N=1}^{\infty}$ with the following properties:

A) $x_m^N \in C^{1,n}(\Omega)$, $\lim_{N \to \infty} \|x_m^N - x_m^*\|_{C^{0,n}((\Omega)} = 0$,

B) $\lim_{N \to \infty} \|x_{m_{i;t_j}}^N - x_{m_{i;t_j}}^*\|_{L_1(\Omega)} = 0 \; \forall i, j$

C) $\{(x_m^N, \mu_m^N)\}_{N=1}^{\infty}$ satisfies the conditions (7), (8), (9) of $(\overline{P})$

D) $|\bar{J}(x_m^N, \mu_m^N) - \kappa_m| < \frac{1}{m}$ for all $N \geq N_0(m)$

E) For $\{(x_m^N, \mu_m^N)\}$ the relaxed state equations (7) are satisfied everywhere on Ω.

From A) it follows that $x_m^N \in \overline{X(t)}$ for all $t \in \Omega$ and $N \geq N_1(m)$.
Let $N^*(m) := \max(N_0(m), N_1(m))$. Then the diagonal sequence $\{(x_m^{N^*(m)}, \mu_m^{N^*(m)})\}$ is admissible for $(\overline{P})$ and together with D) and (29) we obtain that it is a minimizing sequence. ■

3. Duality theorems

3.1. Weak duality results

We consider the problem $(\overline{P})$ under the assumptions (V1) – (V4) and the additional assumptions of Theorem 2.1. Then, the infima of the problems $(\overline{P})$, $(\overline{P})_{\mathcal{B}^0}$ and $(\overline{P})_{\mathcal{B}^1}$ coincide. Therefore we can restrict us to formulate a dual problem to $(\overline{P})_{\mathcal{B}^0}$ with regular Borel measures as dual variables.

We define the following sets:

DEFINITION 3.1

$$\begin{aligned} \mathrm{X}_0 = \Big\{(x,\mu) \in W^{1,n}_{\infty}(\Omega) \times \mathcal{M}_{\mathrm{U}} \,\Big|\, & x_{i;t_j} \in \mathcal{B}^0(\Omega), \forall i,j,\ z_{ij} \in \mathcal{B}^0(\Omega), \\ & z_{ij} = \int_{\mathrm{U}} g_{ij}(t,v)\, d\mu_t(v), \\ & x(t) = \varphi(t)\ \forall t \in \Gamma, \\ & \|x\| \leq \rho,\ \rho \textit{ sufficiently large}\Big\}, \end{aligned} \tag{30}$$

$$\begin{aligned} \mathrm{X}_1 = \Big((x,\mu) \in W^{1,n}_{\infty}(\Omega) \times \mathcal{M}_{\mathrm{U}} \,\Big|\, & x_{i;t_j}(t) = \int_{\mathrm{U}} g_{ij}(t,v)\, d\mu_t(v); \\ & \forall i,\ j,\ \forall t \in \Omega; \\ & h_l(t, x(t)) \leq 0\ \forall l,\ \forall t \in \Omega\Big\}, \end{aligned} \tag{31}$$

$$\mathrm{Y}_0 = (rca\,(\Omega, \mathcal{B})^{nm}, rca_+(\Omega, \mathcal{B})^{w}). \tag{32}$$

The Lagrange functional $\Phi\colon \mathrm{X}_0 \times \mathrm{Y}_0 \to \mathbb{R}$ is defined by

$$\begin{aligned} \Phi(x,\mu,\nu,\sigma) = \bar{J}(x,\mu) &+ \sum_{i,j} \int_{\Omega} \Big[x_{i;t_j}(t) - \int_{\mathrm{U}} g_{ij}(t,v)\, d\mu_t(v)\Big]\, d\nu_{ij}(t) \\ &+ \sum_{l=1}^{w} \int_{\Omega} h_l(t, x(t))\, d\sigma_l(t). \end{aligned} \tag{33}$$

The duality construction in the sense of Fenchel–Rockafellar, see Pickenhain and Wagner (2001A), is used for the construction of a dual problem.

LEMMA 3.1 *Let $(\overline{P})$ be given under the assumptions of this chapter. Then the functional Φ fulfilles the equivalence requirement*

$$\inf_{(x,\mu)\in \mathrm{X}_0 \cap \mathrm{X}_1} \bar{J}(x,\mu) = \inf_{(x,\mu)\in \mathrm{X}_0} \sup_{(\nu,\sigma)\in \mathrm{Y}_0} \Phi(x,\mu,\nu,\sigma). \tag{34}$$

Proof. We first remark that due to the compactness of U the infimum of $(\overline{P})$ is unchanged if x is restricted to the ball $\|x\| \leq \rho$ with ρ sufficiently large, see Pickenhain and Wagner (1999, p. 222, Lemma 2.1.(2)). Let $(x,\mu) \in \mathrm{X}_0$ with

$$x_{i_0;t_{j_0}}(t_0) - \int g_{i_0 j_0}(t_0, v)\, d\mu_{t_0}(v) > 0 \text{ or } h_{l_0}(t_1, x(t_1)) > 0$$

for at least on t_0 or $t_1 \in \Omega$ and indices i_0, j_0, l_0. Then we construct the sequence of measures $\nu^N \in (rca\,(\Omega, \mathcal{B}))^{nm}$ with $\nu^N_{i_0,j_0} = N \cdot \delta_{t_0}$ and $\nu^N_{ij} = 0$ for $i \neq i_0$ or $j \neq j_0$, $\sigma^N \in (rca_+(\Omega, \mathcal{B}))^w$ with $\sigma^N_{l_0} = N \cdot \delta_{t_1}$ and $\sigma^N_l = 0$ for $l \neq l_0$. Then we obtain

$$\lim_{N\to\infty} \Phi(x, \mu, \nu^N, \sigma^N) = +\infty \tag{35}$$

and

$$\sup_{Y_0} \Phi(x, \mu, \nu, \sigma) = \begin{cases} \bar{J}(x, \mu) & |(x, \mu) \in X_0 \text{ satisfies (7), (10) } \forall t \in \Omega, \\ (+\infty) & | \text{ else.} \end{cases} \tag{36}$$

By applying Theorem 2.1 we find a minimizing sequence $\{(x^N, \mu^N)\}_{N=1}^{\infty}$ in $(\overline{P})_{\mathcal{B}^0}$ satisfying (7) everywhere on Ω. We finally get that along this sequence

$$\begin{aligned} &\inf_{(x,\mu)\in X_0\cap X_1} \bar{J}(x, \mu) = \lim_{N\to\infty} \bar{J}(x^N, \mu^N) \\ &= \lim_{N\to\infty} \sup_{(\nu,\sigma)\in Y_0} \Phi(x^N, \mu^N, \nu, \sigma) = \inf_{(x,\mu)\in X_0} \sup_{(\nu,\sigma)\in Y_0} \Phi(x, \mu, \nu, \sigma) \end{aligned} \tag{37}$$

holds and the proof is complete. ■

THEOREM 3.1 *Under the assumptions of Theorem 2.1 the following problem* $(\overline{D})$ *is weakly dual to each of the problems* $(\overline{P})$, $(\overline{P})_{\mathcal{B}^0}$ *and* $(\overline{P})_{\mathcal{B}^1}$,

$$\begin{aligned} L(\nu, \sigma) &= \inf_{(x,\mu)\in X_0} \Phi(x, \mu, \nu, \sigma) \\ &= \inf_{(x,\mu)\in X_0} \Big[\bar{J}(x, \mu) + \sum_{i,j} \int_\Omega \Big[x_{i;t_j}(t) - \int_U g_{ij}(t, v)\, d\mu_t(v)\Big]\, d\nu_{ij}(t) \\ &\qquad + \sum_l \int_\Omega h_l(t, x(x)\, d\sigma_l(t)\Big] \to Max! \end{aligned} \tag{38}$$

$$(\nu, \sigma) \in Y_0 = (rca\,(\Omega, \mathcal{B})^{nm}, rca_+(\Omega, \mathcal{B})^w). \tag{39}$$

Proof. By Theorem 2.1 we have

$$\inf(\overline{P}) = \inf(\overline{P})_{\mathcal{B}^0} = \inf(\overline{P})_{\mathcal{B}^1}, \tag{40}$$

and by Lemma 3.1. we get

$$\inf(\overline{P})_{\mathcal{B}^0} = \inf_{(x,\mu)\in X_0\cap X_1} \bar{J}(x, \mu), \tag{41}$$

$$\inf_{(x,\mu)\in X_0\cap X_1} \bar{J}(x, \mu) = \inf_{(x,\mu)\in X_0} \sup_{(\nu,\sigma)\in Y_0} \Phi(x, \mu, \nu, \sigma) \tag{42}$$

and finally from the well-known inequality

$$\inf_{(x,\mu)\in X_0} \sup_{(\nu,\sigma)\in Y_0} \Phi(x, \mu, \nu, \sigma) \geq \sup_{(\nu,\sigma)\in Y_0} \inf_{(x,\mu)\in X_0} \Phi(x, \mu, \nu, \sigma) \tag{43}$$

we obtain

$$\inf(\overline{P}) \geq \sup(\overline{D}). \;■ \tag{44}$$

3.2. Strong duality results

By strong duality we understand the coincidence of the supremum of $(\overline{\mathrm{D}})$ with the infimum of $(\overline{P})$.

THEOREM 3.2 *We consider $(\overline{P})$ under the assumptions of Theorem 2.1. Moreover:*

(V8) Let $f_0(t,\cdot,v)$, $h_l(t,\cdot)$ be convex for all $t \in \Omega$ and $v \in U$.

(V9) By α_k^+ and α_k^- we denote the positive and negative part in the Jordan decomposition of the measures α_k. Let $f_k(t,\cdot)$ be convex for all $t \in \mathrm{co}\,(\mathrm{supp}\,\alpha_k^+)$ and concave for all $t \in \mathrm{co}\,(\mathrm{supp}\,\alpha_k^-)$.

(V10) For the existing global minimizer (x^*,μ^*) of $(\overline{P})$ we have: For each active index l (i.e. $H_l(x^*) = 0$) there exists a feasible process (x^l,μ^l) with $H_l'(x^*, x^l - x^*) < 0$ (i.e. "(x^*,μ^*) can be strongly varied").

Then the problem (D) *is strongly dual to each of the problems $(\overline{P})$, $(\overline{P})_{\mathcal{B}^0}$ and $(\overline{P})_{\mathcal{B}^1}$.*

Proof. As an essential ingredient of the proof we use the ϵ-maximum principle for $(\overline{P})$, with multipliers from $L_q^{nm}(\Omega)$ which was shown in Pickenhain and Wagner (2000):

Let (x^*,μ^*) be a global minimizer of the problem $(\overline{P})$ under the assumptions of the Theorem. Then for arbitrary $\varepsilon > 0$ there exist multipliers $y^\varepsilon \in L_q^{nm}(\Omega)$ $(1/p + 1/q = 1)$ and $\sigma_l^\varepsilon \in rca_+(\Omega,\mathcal{B})$, $1 \le l \le w$, satisfying the ε-maximum condition (in integrated form)

$$(\mathcal{M})_\varepsilon\colon \quad \varepsilon - \int_\Omega \int_{\mathrm{U}} f_0(t,x^*(t),v)[d\mu_t^*(v) - d\mu_t(v)]dt$$
$$+ \sum_{i,j} \int_\Omega \int_{\mathrm{U}} g_{ij}(t,v)[d\mu_t^*(v) - d\mu_t(v)]y_{ij}^\varepsilon(t)\,dt \ge 0 \tag{45}$$

for all $\mu \in \mathcal{M}_{\mathrm{U}}$ as well as the inequality ("perturbed canonical inequality")

$$(\mathcal{K})_\varepsilon\colon \quad \Big| \sum_{i,j} \int_\Omega y_{ij}^\varepsilon(t)\zeta_{i;t_j}(t)\,dt$$
$$+ \int_\Omega \nabla_\xi^{\mathrm{T}} \Big[\int_{\mathrm{U}} f_0(t,\xi,v)\,d\mu_t^*(v) \Big]_{\xi = x^*(t)} \zeta(t)\,dt$$
$$+ \sum_k \int_\Omega \nabla_\xi^{\mathrm{T}} [f_k(t,\xi)]_{\xi = x^*(t)} \zeta(t)\,d\alpha_k(t)$$
$$+ \sum_l \int_\Omega \nabla_\xi^{\mathrm{T}} [h_l(t,\xi)]_{\xi = x^*(t)} \zeta(t)\,d\sigma_l^\varepsilon(t) \Big| \le \varepsilon \|\zeta\|_{W_p^{1,n}(\Omega)} \tag{46}$$

for all test functions $\zeta \in W_p^{1,n}(\Omega)$ with $\zeta|\Gamma \equiv 0$. If the boundary conditions

Moreover, the measures σ_l^ε satisfy

$$(\mathcal{C})_\varepsilon: \quad \operatorname{supp} \sigma_l^\varepsilon \subseteq \{t \in \Omega \mid h_l(t, x^*(t)) = 0\}$$
$$\Rightarrow \int_\Omega h_l(t, x^*(t))\, d\sigma_l^\varepsilon(t) = 0, \quad 1 \le l \le w. \tag{47}$$

Taking in the ϵ-Maximum principle $\varepsilon = 1/N$, $N \in \mathbb{N}$, we obtain multipliers $y^\epsilon = y^N \in L_q^{nm}(\Omega)$ $(1/p + 1/q = 1)$ and $\sigma^\epsilon = \sigma^N$. If y^N is assumed to be a density of an absolutely continuous measure ν^N, then (ν^N, σ^N) is admissible to $(\overline{D})$. From (V8) and (V9) we get the following inequalities $\forall\, t \in \Omega$, $\forall\, \xi \in \mathbb{R}^n$, $\forall\, v \in U$,

$$f_0(t, \xi, v) \ge f_0(t, x^*(t), v) + \nabla_\xi^{\mathrm{T}} f_0(t, x^*(t), v)(\xi - x^*(t)), \tag{48}$$

$$f_k(t, \xi) \ge f_k(t, x^*(t)) + \nabla_\xi^{\mathrm{T}} f_k(t, x^*(t))(\xi - x^*(t))$$
$$\forall\, t \in \operatorname{co}(\operatorname{supp} \alpha_k^+), \tag{49}$$

$$f_k(t, \xi) \le f_k(t, x^*(t)) + \nabla_\xi^{\mathrm{T}} f_k(t, x^*(t))(\xi - x^*(t))$$
$$\forall\, t \in \operatorname{co}(\operatorname{supp} \alpha_k^-), \tag{50}$$

$$h_l(t, \xi) \ge h_l(t, x^*(t)) + \nabla_\xi^{\mathrm{T}} h_l(t, x^*(t))(\xi - x^*(t)). \tag{51}$$

Therefore,

$$\begin{aligned}
&\int_\Omega \int_{\mathrm{U}} f_0(t, x(t), v)\, d\mu_t(v)\, dt + \sum_k \int_\Omega f_k(t, x(t))\, d[\alpha_k^+ - \alpha_k^-](t) \\
&\ge \int_\Omega \int_{\mathrm{U}} (f_0(t, x^*(t), v) + \nabla_\xi^{\mathrm{T}} f_0(t, x^*(t), v)(x - x^*)(t))\, d\mu_t(v)\, dt \\
&+ \sum_k \int_\Omega (f_k(t, x^*(t)) + \nabla_\xi^{\mathrm{T}} f_k(t, x^*(t))(x - x^*)(t)) d[\alpha_k^+ - \alpha_k^-](t).
\end{aligned} \tag{52}$$

Now we estimate the value of the dual objective for (ν^N, σ^N):

$$\begin{aligned}
L(\nu^N, \sigma^N) = \inf_{(x,\mu) \in \mathrm{X}_0} \Big[&\int_\Omega \int_{\mathrm{U}} f_0(t, x(t), v)\, d\mu_t(v)\, dt \\
&+ \sum_k \int_\Omega f_k(t, x(t))[d\alpha_k^+(t) - d\alpha_k^-(t)] \\
&+ \sum_{i,j} \int_\Omega \Big[x_{i;t_j}(t) - \int_{\mathrm{U}} g_{ij}(t, v)\, d\mu_t(v) \Big] y_{ij}^N(t)\, dt \\
&+ \sum_k \int_\Omega h_l(t, x(t)) d\sigma_l^N(t) \Big] \\
\ge \inf_{(x,\mu) \in \mathrm{X}_0} \Big[&\int_\Omega \int_{\mathrm{U}} (f_0(t, x^*(t), v)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_k \int_\Omega (f_k(t, x^*(t)) \\
&+ \nabla_\xi^{\mathrm{T}} f_k(t, x^*(t))(x(t) - x^*(t)))[d\alpha_k^+(t) - d\alpha_k^-(t)] \\
&+ \sum_{i,j} \int_\Omega \Big[x_{i;t_j}(t) - \int_{\mathrm{U}} g_{ij}(t, v)\, d\mu_t(v)\Big] y_{ij}^N(t)\, dt \\
&+ \sum_l \int_\Omega (h_l(t, x^*(t)) + \nabla_\xi^{\mathrm{T}} h_l(t, x^*(t))(x - x^*)(t))\, d\sigma_l^N(t)\Big].
\end{aligned}
$$

By rearranging the sum on the right hand side of the last inequality we obtain

$$
\begin{aligned}
L(\nu^N, \sigma^N) \geq \inf_{(x,\mu)\in \mathrm{X}_0} \Big[&\int_\Omega \int_{\mathrm{U}} f_0(t, x^*(t), v)\, d\mu_t^*(v)\, dt \\
&+ \sum_k \int_\Omega f_k(t, x^*(t))\, d\alpha_k(t) + S_1 + S_2 + S_3 + S_4 \Big]
\end{aligned} \tag{53}
$$

with

$$
S_1 = \sum_{i,j} \int_\Omega \Big(x_{i;t_j}^*(t) - \int_{\mathrm{U}} g_{ij}(t, v)\, d\mu_t^*(v)\Big) y_{ij}^N(t)\, dt, \tag{54}
$$

$$
\begin{aligned}
S_2 = &\sum_{i,j} \int_\Omega (x_{i;t_j}(t) - x_{i;t_j}^*(t))\, y_{ij}^N(t)\, dt \\
&+ \int_\Omega \nabla_\xi^{\mathrm{T}} \Big[\int_{\mathrm{U}} f_0(t, x^*(t), v)(x(t) - x^*(t))\Big)\, dt \\
&+ \sum_k \int_\Omega \nabla_\xi^{\mathrm{T}} f_k(t, x^*(t))(x(t) - x^*(t))\, d\alpha_k(t) \\
&+ \int_\Omega \nabla_\xi^{\mathrm{T}} h_l(t, x^*(t))(x(t) - x^*(t))\, d\sigma_l^N(t),
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
S_3 = &- \int_\Omega \int_{\mathrm{U}} f_0(t, x^*(t), v)[d\mu_t^*(v) - d\mu_t(v)]\, dt \\
&+ \sum_{i,j} \int_\Omega \int_{\mathrm{U}} g_{ij}(t, v)[d\mu_t^*(v) - d\mu_t(v)] y_{ij}^N(t)\, dt
\end{aligned} \tag{56}
$$

and

$$
S_4 = \sum_l \int_\Omega h_l(t, x^*(t))\, d\sigma_l^N(t). \tag{57}
$$

S_1 vanishes since (x^*, μ^*) is admissible for $(\overline{P})$. It follows from the perturbed canonical inequality (46) that

$$
S_2 \geq -|S_2| \geq -\frac{1}{N}\|x - x^*\| \tag{58}
$$

and from the ϵ-Maximum condition (45) we get

$$S_3 \geq -\frac{1}{N}. \tag{59}$$

S_4 vanishes since the complementary condition (47) is fulfilled. The estimations in (53), (58) and (59) yield together

$$L(\nu^N, \sigma^N)) \geq \bar{J}(x^*, \mu^*) - \frac{1}{N} - \frac{1}{N} \sup_{(x,\mu)\in X_0} \|x - x^*\| \tag{60}$$

and

$$\begin{aligned} \bar{J}(x^*, \mu^*) &= \min(\overline{P}) = \inf(\overline{P})_{\mathcal{B}^0} = \inf(\overline{P})_{\mathcal{B}^1} \geq \sup(\overline{\mathrm{D}}) \\ &\geq L(\nu^N) \geq \bar{J}(x^*, \mu^*) - \frac{1}{N} - \frac{2}{N}\rho \end{aligned} \tag{61}$$

$\forall N \in \mathbb{N}$. This finally gives with (61) for $N \to \infty$

$$\inf(\overline{P}) = \inf(\overline{P})_{\mathcal{B}^0} = \inf(\overline{P})_{\mathcal{B}^1} = \sup(\overline{\mathrm{D}}) \tag{62}$$

and the proof is complete. ■

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