Control and Cybernetics

vol. 27 (1998) No. 1

Discrete-time Markovian jump linear systems

by

K. Szajowski[*,] and W.L. de Koning[*]**

[*] Faculty of Information Technology and Systems,
Delft University of Technology,
P.O. Box 5031, 2628 CD Delft, The Netherlands,
e-mail: `w.l.dekoning@math.tudelft.nl`
[**] Institute of Mathematics,
Wrocław University of Technology,
Wybrzeże Wyspiańskiego 27, PL-50-370 Wrocław, Poland,
e-mail: `szajow@im.pwr.wroc.pl`

Abstract: The paper considers a problem of optimal control of a linear system with the parameters dependent on the states of a Markov chain. The cost criterion is quadratic in the controls and states of the system. The criterion parameters also depend on the states of the Markov chain. Two models of observation of the Markov chain are adopted - delay for one step and no delay. It is shown that under appropriate mean square detectability and stabilizability conditions the infinite horizon optimal control problem for the general case of Markovian jump linear quadratic systems has a unique mean square stabilizing solution. Necessary and sufficient conditions are given to determine if a system is mean square stabilizable.

Keywords: jump linear systems, Markov chain, optimal control, delayed observation, stabilizability, detectability

1. Introduction

The paper concerns linear discrete time systems with quadratic criteria both having parameters dependent on a Markov chain. Such systems were studied by Ji and Chizeck (1990). The case of i.i.d. parameters was considered by many authors. For references see Ji and Chizeck (1990) and de Koning (1982). The aim of the paper is to find appropriate stabilizability and detectability conditions which assure a unique solution for which the control system is stable. Related results concerning the stability of such systems were obtained by Costa and Fragoso (1993) and Fragoso, Costa and de Souza (1993). The control of such systems were considered recently by Blom (1991) and Yang and Bar-Shalom (1991). A general system discribed by a linear difference equation in a

Hilbert space with three types of disturbances, control-dependent noise, state dependent noise and purely additive noise was considered by Zabczyk (1975). However, these results were based on other assumptions about the parameters or knowledge taken into account in the construction of the control. Here, two different models of information available at each moment are considered. In the first one, similarly as in Ji and Chizeck (1990), the knowledge of all past controls, the past and present states of the system and Markovian parameters is assumed. The other one is based on the assertion that only the values of the Markovian parameters are known, i.e. there is a delay of one step in the observation of these random parameters. A generalization of the stochastic and mean square (ms-)stability for the considered control problems is given (see Zabczyk, 1981, and Klamka, 1991 for basic definition and results). Relations between these two kinds of stability are investigated. To formulate the condition of stability for the optimal control in the infinite horizon case ms-detectability is introduced, which generalizes the ms-detectability for the systems with i.i.d. parameters considered by de Koning (1983). Necessary and sufficient conditions for the existence of the solution of the infinite-horizon control problem are established. Conditions for the ms-stability are given. The differences between these two available models of information are pointed out. The results for the model with delayed observation of the Markov parameter are a generalization of the control problem with i.i.d. parameters considered by de Koning (1983). Suggestions of computational methods of verification of the obtained conditions are given. The stochastic controllability of considered systems (see paper by Klamka and Le Si Dong, 1990, for related results) will be subject of other paper.

The organization of the paper is as follows. In the next section, preliminary notations and definitions are given. The finite horizon optimal control problem is solved in Section 3. Next in Section 4, problems of stability of the system are considered. Section 5 is devoted to the solution of the infinite horizon optimal control problem.

2. Notations and formulation of the problem

At the beginning of this section some notations are introduced and basic mathematical facts are mentioned. Next, the problem of optimal control for systems with randomly varying description in the finite and infinite horizon case is formulated.

Scalars are denoted by lower case Greek letters, column vectors by lower case italic letters, matrices by capital Greek or italic letters. Capital scripts will be used to denote spaces and bold italic letters for operators. Exceptions from these rules will be noted. The transpose of vectors or matrices is denoted by superscript T.

Let $\Re^n$ denote the Euclidean n-space. For $x, y \in \Re^n$, the inner product $< x, y >= x^T y$ and the norm $|x| = (x^T x)^{1/2}$. $\mathcal{M}^{mn}$ denotes the space of all real $m \times n$ matrices with norm $\|A\| = \sup_{|x|=1} |Ax|$, where $A \in \mathcal{M}^{mn}$ and $x \in \Re^n$. Denote $\mathcal{M}^{nn}$ by $\mathcal{M}^n$ and let $\mathcal{S}^n \subset \mathcal{M}^n$ be a space of the real symmetric matrices. The zero and identity element in $\mathcal{M}^n$ are $\boldsymbol{O}$ and $\boldsymbol{I}$, respectively. Matrix $A \in \mathcal{S}^n$ is called *non-negative definite* if for every $x \in \Re^n$ we have $x^T A x \geq 0$. If in addition $x^T A x = 0$ implies $x = 0$, matrix A is called *positive definite*.

The set of all non-negative definite matrices of $\mathcal{S}^n$ is denoted by $\mathcal{K}^n$. Put $\mathbb{E} = \{1, 2, \cdots, s\}$. We will denote by $\mathcal{M}^{mn}_{\mathbb{E}}$ the set of functions defined on $\mathbb{E}$ with values in $\mathcal{M}^{mn}$. Denote by $\mathbb{I}$ the element of $\mathcal{M}^n_{\mathbb{E}}$ such that $\mathbb{I}(i) = \boldsymbol{I}$ and let $\Theta \in \mathcal{M}^n_{\mathbb{E}}$ be such that $\Theta(i) = \boldsymbol{O}$. For $f \in \mathcal{M}^{mn}_{\mathbb{E}}$ we define norm $\|f\|_* = \max_{r \in \mathbb{E}} \{\|f(r)\|\}$. Space $\mathcal{S}^n_{\mathbb{E}}$ with norm $\|\cdot\|_*$ is a Banach space.

The paper deals with a discrete time linear system with Markovian jumps, modeled by

$$x_{k+1} = A(r_k)x_k + B(r_k)u_k, \tag{1}$$

where $k = 0, 1, \cdots, N$, $x_k \in \Re^n$, $u_k \in \Re^m$, $A \in \mathcal{M}^n_{\mathbb{E}}$, $B \in \mathcal{M}^{nm}$. N can be finite or infinite. It is assumed that x_0 is given and $\{r_k\}_{k=0}^N$ is a homogenous Markov chain defined on a fixed probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{E}$. Let $(p_i)_{i \in \mathbb{E}}$ and $(p_{ij})_{i,j \in \mathbb{E}}$ denote the initial and transition probabilities for this Markov chain, respectively. Let $\vec{u} = (u_0, u_1, \ldots, u_{N-1})$. In the finite horizon case, system (1) is considered with the cost criterion

$$J_N(\vec{u}, x_0) = \mathbf{E}\{\sum_{i=0}^{N-1} [x_i^T Q(r_i) x_i + u_i^T R(r_i) u_i] + x_N^T H(r_N) x_N | x_0\} \tag{2}$$

and in the infinite horizon case with the cost criterion

$$J(\vec{u}, x_0) = \mathbf{E}[\sum_{i=0}^{\infty} x_i^T Q(r_i) x_i + u_i^T R(r_i) u_i | x_0] \tag{3}$$

where $Q, H \in \mathcal{K}^n_{\mathbb{E}}$, $R \in \mathcal{K}^m_{\mathbb{E}}$.

Denote $r^k = (r_0, r_1, \ldots, r_k)$ and $z^k = (x_0, u_0, \ldots, x_{k-1}, u_{k-1}, x_k)$. The different classes of admissible controls can be considered for system (1) with criterion (2) or (3). We focus our attention on two different classes of strategies:

(D0) $u_i = g_i(z^i, r^i)$ – the control at moment i is based on information about the states, controls and states of the Markov chain up to moment i;

(D1) $u_i = g_i(z^i, r^{i-1})$ – uses the same information about the states and controls of the system as in (D0) but there is a one step delay in the observation of the state of the Markov chain.

DEFINITION 1 *Assume that the control laws $\{g_i\}$ belong to class (D0) ((D1)) and the system is given by (1) with criterion (2). The problem of finding the control sequence $\vec{u}^*$ which minimizes $J_N(\vec{u}, x_0)$ for all x_0 and determining the minimal value $J_N(\vec{u}^*, x_0)$ is called the finite horizon optimal control problem without delay in the observation of the Markov chain (with a delay in the observation of the Markov chain for one step).*

DEFINITION 2 *The infinite horizon optimal control problem without delay in the observation of the Markov chain (with a delay for one step in the observation of the Markov chain) for system (1) and criterion (3) is to determine the control* $\vec{u}^*$ *in class (D0) ((D1)) which minimizes* $J(\vec{u}, x_0)$ *and to find the minimal value* $J(\vec{u}^*, x_0)$.

Related optimal control problems were considered in de Koning (1982, 1983), Ji and Chizeck (1990). Ji and Chizeck (1990) considered the case of control policies (D0). The problem with delayed observation of the Markov chain parameter is a new one. A unified approach to the problems is proposed. A comparison of the solution of the problems for classes of strategies (D0) and (D1) allows us to underline the differences between the two models. For i.i.d. random variables $\{r_n\}_{n=0}^N$, the knowledge of $r_0, \ldots, r_{k-1}$ has no influence for the *posterior* distribution of r_k. Taking this into account, the results of the paper for the class of control policies (D1) are a generalization of the considerations in De Koning's papers (1982, 1983) for r_k with discrete distribution.

3. Finite horizon optimal control

In this section the finite horizon optimal control problem for system (1) with cost criterion (2) is solved for both cases of admissible sets of controls (D0) and (D1). The results were obtained with the following dynamic programming principle (DPP) for systems with the Markovian parameters (see e.g. Kumar and Varaiya, 1986, for DPP in an i.i.d. parameter case).

Let

$$x_{k+1} = f_k(x_k, u_k, r_k), \tag{4}$$

for $k = 0, 1, \ldots, N-1$, where $x_k \in \Re^n$ and r_k is the Markov chain defined in Section 2. An admissible control law is: any sequence $\vec{g} = (g_0, g_1, \ldots, g_N)$ such that $u_k = g_k(z^k, r^k)$. Denote by $\mathcal{G}$ the set of admissible controls. The cost criterion for system (4) and given $\vec{g} \in \mathcal{G}$ is defined as

$$J_N(\vec{g}, x_0) = \mathbf{E}\{\sum_{i=0}^{N-1} c_k(x_k, u_k, r_k) + c_N(x_N, r_N)|x_0\}$$

For $\vec{g} \in \mathcal{G}$ define the cost-to-go at moment n by

$$J_{n,N}(\vec{g}, x^{g^n}, r^n) = \mathbf{E}\{\sum_{i=n}^{N-1} c_k(x_k^g, u_k^g, r_k) + c_N(x_N^g, r_N)|x^{g^n}, r^n\},$$

where x_i^g and u_i^g are the state and the control process corresponding to control law $\vec{g}$, respectively. An admissible control law $\vec{g} \in \mathcal{G}$ is called Markovian if g_k depends only on (x_k, r_k). $\mathcal{G}_M$ denotes a class of Markovian policies. We can formulate the following DPP, Kumar and Varaiya (1986):

LEMMA 1 *(DPP) Define recursively the function*

$$\begin{aligned} w_N(x,r) &= c_N(x,r) \\ w_n(x,r) &= \inf_u\{c_n(x,u,r)+\mathbf{E}[w_{n+1}(x_{n+1},r_{n+1})|x_n=x,r_n=r]\} \end{aligned} \quad (5)$$

for $n = N-1,\ldots,0$.

(i) *For arbitrary* $\vec{g} \in \mathcal{G}$ *we have* $w_n(x_n^g, r_n) \le J_{n,N}(\vec{g}, x^{gn}, r^n)$ *a.e. and* $J_N(\vec{g}, x_0) \ge \mathbf{E}w_0(x_0, r_0)$.

(ii) *If* $\vec{g} \in \mathcal{G}_M$ *is optimal, then the infimum in (5) at moment* n *is attained at* $u_n = g_n(x,r)$. *We have then* $w_n(x_n^g, r_n) = J_{n,N}(\vec{g}, x^{gn}, r^n)$ *a.e. and* $J^* = \inf_{\vec{g}\in\mathcal{G}} J_N(\vec{g}, x_0) = \inf_{\vec{g}\in\mathcal{G}} \mathbf{E}J_{0,N}(\vec{g}, x_0, r_0) = \mathbf{E}w_0(x_0, r_0)$.

(iii) *If for each* n *the infimum at state* x_n^g *in (5) is attained by* $u_n = g_n(x_n^g, r_n)$ *i.e.*

$$w_n(x_n^g, r_n) = c_n(x_n^g, g_n(x_n^g, r_n), r_n) + \mathbf{E}[w_{n+1}(x_{n+1}^g, r_{n+1})|x_n^g, r_n] \text{ a.e}$$

then $\vec{g}$ *is optimal in* $\mathcal{G}$.

Let us consider the case of controls (D0) for the system described by (1). For $A \in \mathcal{M}_{\mathbb{E}}^{mn}, B \in \mathcal{M}_{\mathbb{E}}^{n}$ we adopt the following convention: $A(r)B(r) = [AB](r)$. The point of the following lemma is in the preliminary calculation of criterion (2) for the given controls.

LEMMA 2 *Suppose*

$$u_i = -L(r_i)x_i \quad (6)$$

where: $L \in \mathcal{M}_{\mathbb{E}}^{mn}$, *then for every* x_0

$$J_N(\vec{u}, x_0) = x_0^T \mathbf{E}\mathcal{D}_L^N H(r_0)x_0$$

Operator $\mathcal{D}_L : S_{\mathbb{E}}^n \to S_{\mathbb{E}}^n$ *is defined by*

$$\mathcal{D}_L X(r) = \mathcal{E}_L X(r) + Q_L(r)$$

and

$$\mathcal{E}_L X(r) = \Psi_L^T(r)\mathbf{E}[X(r_1)|r_0 = r]\Psi_L(r)$$

where $Q_L(r) = Q(r) + [L^T RL](r)$ *and* $\Psi_L(r) = A(r) - [BL](r)$.

Proof. By (6) we can write (2) as

$$J_N(\vec{u}, x_0) = \mathbf{E}\{\sum_{i=0}^{N-1} x_i^T Q_L(r_i)x_i + x_N^T H(r_N)x_N|x_0\}$$

and

$$J_{n,N}(\vec{u}, x^n, r^n) = \mathbf{E}\{\sum_{i=n}^{N-1} x_i^T Q_L(r_i)x_i + x_N^T H(r_N)x_N|x^n, r^n\}.$$

Using state equation (1) and the Markov property of parameter r_n,

$$J_{n,N}(\vec{u},x^n,r^n)=\mathbf{E}\{\sum_{i=n}^{N-1}x_i^TQ_L(r_i)x_i+x_N^TH(r_N)x_N|x_n,r_n\}.$$

We have $J_{N,N}(\vec{u},x^N,r^N)=x_N^T\boldsymbol{\mathcal{D}}^0H(r_N)x_N$. Suppose for backward induction

$$J_{n+1,N}(\vec{u},x^{n+1},r^{n+1})=x_{n+1}^T\boldsymbol{\mathcal{D}}_L^{N-n-1}H(r_{n+1})x_{n+1}.$$

We have

$$\begin{aligned}
J_{n,N}(\vec{u},x^n,r^n) &= \mathbf{E}[\sum_{i=n}^{N-1}x_i^TQ_L(r_i)x_i+x_N^TH(r_N)x_N|x^n,r^n]\\
&= \mathbf{E}\{\mathbf{E}[\sum_{i=n}^{N-1}x_i^TQ_L(r_i)x_i+x_N^TH(r_N)x_N|x^{n+1},r^{n+1}]|x^n,r^n\}=\\
&= x_n^TQ_L(r_n)x_n\\
&\quad+\mathbf{E}\{\mathbf{E}[\sum_{i=n+1}^{N}x_i^TQ_L(r_i)x_i+x_N^TH(r_N)x_N|x^{n+1},r^{n+1}]|x^n,r^n\}=\\
&\overset{(1)}{\underset{(6)}{=}} x_n^TQ_L(r_n)x_n+\mathbf{E}[x_n^T\Psi_L^T(r_n)\boldsymbol{\mathcal{D}}_L^{N-n-1}H(r_{n+1})\Psi_L(r_n)x_n|x^n,r^n]=\\
&= x_n^T[Q_L(r_n)+\boldsymbol{\mathcal{E}}_L\boldsymbol{\mathcal{D}}_L^{N-n-1}H(r_n)]x_n=x_n^T\boldsymbol{\mathcal{D}}_L^{N-n}H(r_n)x_n
\end{aligned}$$

For $n=0$ we get $J_N(\vec{u},x_0)=\mathbf{E}J_{0,N}(\vec{u},x_0,r_0)=x_0^T\mathbf{E}\boldsymbol{\mathcal{D}}_L^NH(r_0)x_0$. ∎

Denote $\bar{X}(r)=\mathbf{E}[X(r_1)|r_0=r]$ for $X\in\mathcal{S}_{\mathbb{E}}^n$.

PROPOSITION 1 *Solution $\vec{u}^*$ of the finite horizon optimal control problem without a delay in the observation of the Markov chain is given recursively as follows*

$$u_n^* = -\boldsymbol{\mathcal{L}}\boldsymbol{\mathcal{D}}_*^{N-n-1}H(r_n)x_n$$

for $n=0,1,\ldots,N-1$ and

$$J_N(\vec{u}^*,x_0)=x_0^T\mathbf{E}\boldsymbol{\mathcal{D}}_*^NH(r_0)x_0,$$

where $\boldsymbol{\mathcal{D}}_*X(r)=\boldsymbol{\mathcal{D}}_{\boldsymbol{\mathcal{L}}X}X(r)$ and

$$\boldsymbol{\mathcal{L}}X(r)=[(B^T\bar{X}B+R)^+B^T\bar{X}A](r).$$

The open form of operator $\boldsymbol{\mathcal{D}}_*$ is as follows

$$\begin{aligned}
\boldsymbol{\mathcal{D}}_*X(r) &= Q(r)+[A^T\bar{X}A](r)\\
&\quad -[A^T\bar{X}B(B^T\bar{X}B+R)^+B^T\bar{X}A](r),
\end{aligned}$$

where: $X\in\mathcal{S}_{\mathbb{E}}^n$.

Proof. It suffices to use DPP together with the results of Lemma 2. ■

Now, consider the case of controls based on the observation of states and for-one-step-delayed observations of the Markov chain. To simplify the description of results, let us suppose that at the moment -1 the Markov chain is also defined. We assume $\mathbf{P}(r_{-1}=1)=1$ and $\mathbf{P}(r_0=i|r_{-1}=j)=p_i$, $i,j\in\mathbb{E}$. The cost criterion can be retyped in the equivalent form

$$\underline{J}_N(\vec{u},x_0)=\mathbf{E}\{\sum_{i=0}^{N-1}[x_i^T\bar{Q}(r_{i-1})x_i$$
$$+u_i^T\bar{R}(r_{i-1})u_i]+x_N^T\bar{H}(r_{N-1})x_N|x_0\} \tag{7}$$

and

$$\underline{J}_{n,N}(\vec{u},x^n,r^{n-1})=\mathbf{E}\{\sum_{i=n}^{N-1}[x_i^T\bar{Q}(r_{i-1})x_i$$
$$+u_i^T\bar{R}(r_{i-1})u_i]+x_N^T\bar{H}(r_{N-1})x_N|x^n,r^{n-1}\}.$$

Let $\Psi\in\mathcal{M}^n_{\mathbb{E}\times\mathbb{E}}$ and $X\in\mathcal{M}^n_{\mathbb{E}}$. Denote $\overline{\Psi^TX\Psi}(r)=\mathbf{E}[\Psi^T(r_0,r_1)X(r_1)\Psi(r_0,r_1)|r_0=r]$.

LEMMA 3 *Let us consider system (1) with cost criterion (7). Suppose*

$$u_i=-L(r_{i-1})x_i\ , \tag{8}$$

where: $L\in\mathcal{M}^{mn}_{\mathbb{E}}$, then for every x_0

$$\underline{J}_N(\vec{u},x_0)=x_0^T\mathcal{G}_L^N\bar{H}(r_{-1})x_0, \tag{9}$$

where: $\mathcal{G}_L:S^n_{\mathbb{E}}\to S^n_{\mathbb{E}}$ is defined by

$$\mathcal{G}_LX(r)=\mathcal{H}_LX(r)+\tilde{Q}_L(r)$$

where

$$\mathcal{H}_LX(r)=\overline{\Psi^TX\Psi_L}(r),$$

$\tilde{Q}_L(r)=\bar{Q}(r)+[L^T\bar{R}L](r)$ *and* $\Psi_L(r,s)=A(r)-B(r)L(s)$.

Proof. As in the proof of Lemma 2, the backward induction and the properties of the conditional expectation give

$$\underline{J}_{1,N}(\vec{u},x^1,r_0)=x_1^T\mathcal{G}_L^{N-1}\bar{H}(r_0)x_1.$$

We thus get

$$\underline{J}_{0,N}(\vec{u},x_0,r_{-1})$$
$$=x_0^T\mathbf{E}[Q(r_0)+(A(r_0)-B(r_0)L)^T\mathcal{G}_L^{N-1}\bar{H}(r_0)(A(r_0)-B(r_0)L)]x_0$$

which yields (9). ■

In the next proposition, the solution of the finite horizon optimal control problem for system (1) with cost criterion (2) and class of admissible controls (D1) is given.

PROPOSITION 2 *Let the system be described by (1) with the cost criterion given by (2). The solution of the finite horizon optimal control problem with a delay in the observation of Markov parameter r_n for one step is given by*

$$\begin{aligned} u_N^* &= 0 \\ u_n^* &= -\bar{\mathcal{L}}\mathcal{G}^{N-n-1}\bar{H}(r_{n-1})x_n \end{aligned}$$

for $n = 0, 1, \ldots, N-1$ and

$$\underline{J}_N(\vec{u}^*) = x_0^T \mathcal{G}_*^N \bar{H}(r_{-1})x_0,$$

where $\mathcal{G}_* X(r) = \mathcal{G}_{\bar{\mathcal{L}}X} X(r)$,

$$\bar{\mathcal{L}}X(r) = [(\overline{B^T X B} + \bar{R})^+ \overline{B^T X A}](r).$$

The open form of operator $\mathcal{G}_*$ is as follows

$$\begin{aligned} \mathcal{G}_* X(r) &= \bar{Q}(r) + \overline{A^T X A}(r) \\ &\quad -[\overline{A^T X B}(\overline{B^T X B} + \bar{R})^+ \overline{B^T X B}](r) \end{aligned}$$

Proof. This follows by the same arguments as in Proposition 1. We use DPP given by Lemma 1 with the obvious modification, the Markov property of parameter r_n and the properties of the conditional expectation. ∎

4. Stability of the system

In this section the stochastic and mean square stability for a closed-loop discrete time jump linear system are developed. First some definitions concerning the ordering and properties of positive operators in a finite dimensional Banach space are recalled.

Let $\mathcal{S}$ be a Banach space. A set $\mathcal{K} \subset \mathcal{S}$ is called *a cone* if the following conditions are satisfied (see Krasnosel'skij, 1964, Horn and Johnson, 1988): (i) the set $\mathcal{K}$ is closed; (ii) if $x, y \in \mathcal{K}$, then $\alpha x + \beta y \in \mathcal{K}$ for all $\alpha, \beta \in \Re$, $\alpha, \beta > 0$; (iii) of each pair of vectors $x, -x$ at least one does not belong to $\mathcal{K}$, provided that $x \neq 0$. By means of a cone $\mathcal{K}$ one can define a partial ordering relation $\preceq$ in Banach space $\mathcal{S}$. This is introduced in the following manner. Let $x, y \in \mathcal{S}$. We have $x \preceq y$ if $y - x \in \mathcal{K}$. Linear operator $\mathcal{A} : \mathcal{S} \to \mathcal{S}$ is called *positive* if it transforms cone $\mathcal{K}$ into itself. It is easy to check that the set $\mathcal{K}^n$ is a cone in $\mathcal{S}^n$ and the set $\mathcal{K}^n_{\mathbb{E}}$ is a cone in $\mathcal{S}^n_{\mathbb{E}}$. For $A \in \mathcal{K}^n$, we write $A \succ 0$ if A is positive definite. Similarly for $\boldsymbol{A} \in \mathcal{K}^n_{\mathbb{E}}$, we define $\boldsymbol{A} \succ \Theta$ if $A(i)$ is positive definite for every $i \in \mathbb{E}$.

Let $\mathcal{A} : \mathcal{M}^n_{\mathbb{E}} \to \mathcal{M}^n_{\mathbb{E}}$. The spectrum of $\mathcal{A}$ is denoted by $\sigma(\mathcal{A})$ and the spectral radius by $\rho(\mathcal{A})$. Operator $\mathcal{A}$ is called stable if $\rho(\mathcal{A}) < 1$. The space $\mathcal{M}^n_{\mathbb{E}}$ is linearly isomorphic with $\Re^{sn^2}$. We have: $\mathcal{A}$ is stable if and only if $\lim_{n\to\infty} \mathcal{A}^n X = 0$ for every $X \in \mathcal{M}^n_{\mathbb{E}}$.

Operator $\mathcal{A} : \mathcal{S} \to \mathcal{S}$ is called *monotone* in set $\mathcal{T} \subset \mathcal{S}$ if it follows from $x \preceq y$, $x, y \in \mathcal{T}$ that $\mathcal{A}x \preceq \mathcal{A}y$. Let $\mathcal{K}$ be a cone in $\mathcal{S}$. A linear operator is monotone if and only if it is positive. In further considerations, operators on $\mathcal{S}^n_{\mathbb{E}}$ will be used. In this space, all norms are equivalent. Let us mention that if the sequence $\{\mathcal{A}_n\}$ of positive operators is bounded and increasing with respect to relation $\preceq$, then there exists a positive operator $\mathcal{A}$ such that for every $X \in \mathcal{K}^n_{\mathbb{E}}$ we have $\lim_{n\to\infty} \mathcal{A}_n X = \mathcal{A}X$ and $\mathcal{A}_n X \preceq \mathcal{A}X$. The following lemma (see de Koning, 1982) for $\mathbb{E} = \{1\}$) concerning linear, positive or monotonic operators $\mathcal{A} : \mathcal{S}^n_{\mathbb{E}} \to \mathcal{S}^n_{\mathbb{E}}$ is stated:

LEMMA 4 *Let $X \in \mathcal{S}^n_{\mathbb{E}}$. We have*
(i) If operator $\mathcal{A}$ is monotonic and positive, then $\mathcal{A}^i$ for every $i \in \mathbb{N}$ is monotonic and positive.
(ii) If $\mathcal{A}$ is linear and positive, then $\|\mathcal{A}\| = \|\mathcal{A}\mathbb{I}\|_*$.
(iii) If $\lim_{n\to\infty} \mathcal{A}^n X = 0$ for $X \in \mathcal{K}^n_{\mathbb{E}}$ and $X \succ \Theta$, then $\mathcal{A}$ is stable.

Proof. From the definitions follows (i). The monotonicity of norm $\|\cdot\|_*$ and operator $\mathcal{A}$ gives (ii). To prove (iii) let $X \in \mathcal{K}^n_{\mathbb{E}}$. There is $\alpha > 0$ such that $\mathbb{I} \preceq \alpha X$ and by the monotonicity of $\mathcal{A}$ we have

$$\rho^i(\mathcal{A}) = \rho(\mathcal{A}^i) \le \|\mathcal{A}^i\| = \|\mathcal{A}^i\mathbb{I}\|_* \le \alpha\|\mathcal{A}^i X\|_* \to 0$$

when $i \to \infty$. Then $\rho(\mathcal{A}) < 1$ and $\mathcal{A}$ is stable. ∎

The main purpose of this consideration is the existence and the properties of the solution of the following Lyapunov-type equation

$$X = \mathcal{A}X + B \tag{10}$$

(see Krasnosel'skij, 1964, pp. 86-91, and de Koning, 1982).

LEMMA 5 *Let $\mathcal{A}$ be linear, positive and $B \in \mathcal{K}^n_{\mathbb{E}}$, then*
(i) If $\mathcal{A}$ is stable, then there exists a solution $\boldsymbol{X} \in \mathcal{K}^n_{\mathbb{E}}$.
(ii) There exists solution $\boldsymbol{X} \succeq \Theta$ and $\boldsymbol{B} \succ \Theta$, then $\mathcal{A}$ is stable and $\boldsymbol{X} \succ \Theta$.

Proof. Let us observe that if $\mathcal{A}$ is stable, linear and positive, then $\boldsymbol{X} = \sum_{k=0}^{\infty} \mathcal{A}^k \boldsymbol{B}$ is well defined. $\boldsymbol{X}$ fulfils (10) and $\boldsymbol{X} \succeq \Theta$. Let now (10) have a solution. Using induction and assuming $\boldsymbol{X} \succeq \Theta$ we have

$$\boldsymbol{X} = \sum_{k=0}^{n-1} \mathcal{A}^k \boldsymbol{B} + \mathcal{A}^n \boldsymbol{X} \succeq \sum_{k=0}^{n-1} \mathcal{A}^k \boldsymbol{B} \succeq \boldsymbol{B} \succ \Theta$$

for every $n \ge 1$. Hence we have $\lim_{n\to\infty} \mathcal{A}^n \boldsymbol{B} = \Theta$. Using Lemma 4 (iii), $\mathcal{A}$ is stable. ∎

Let us consider the stability of the closed loop control system

$$x_{k+1} = \Psi(r_{k-1}, r_k)x_k \tag{11}$$

for $k = 0, 1, 2, \ldots, N-1$. System (11) can be obtained from (1) when we assume controls (6) or (8). In the case of a system without a delay in the observation of the Markov chain (i.e. controls (6)), $\Psi(s,r) = \Psi_L(r) = \boldsymbol{A}(r) - \boldsymbol{B}(r)\boldsymbol{L}(r)$. In the case of system (1) with controls based on the observation of the Markov chain delayed for one step, $\Psi(s,r) = \Psi_L(s,r) = \boldsymbol{A}(r) - \boldsymbol{B}(r)\boldsymbol{L}(s)$.

DEFINITION 3 *(Ji and Chizeck, 1990) Closed loop system (11) is conditionally stochastically stable if for every initial state x_0 and r_{-1} there exists a finite number $M(x_0, r_{-1}) > 0$ such that*

$$\lim_{N\to\infty} \mathbf{E}\{\sum_{k=0}^{N} x_k^T x_k | x_0, r_{-1}\} < M(x_0, r_{-1}) \tag{12}$$

System (11) is stochastically stable if for every initial state x_0 there exists a finite number $M(x_0) > 0$ such that

$$\lim_{N\to\infty} \mathbf{E}\{\sum_{k=0}^{N} x_k^T x_k | x_0, r_{-1}\} < M(x_0) \tag{13}$$

DEFINITION 4 *System (11) is said to be conditionally ms-stable if for every x_0 and r_{-1}*

$$\lim_{k\to\infty} \mathbf{E}\{x_k^T x_k | x_1, r_0\} = 0 \tag{14}$$

and ms-stable if for every x_0

$$\lim_{k\to\infty} \mathbf{E}\{x_k^T x_k | x_0, r_{-1}\} = 0 \tag{15}$$

REMARK 1 *Definitions 3 and 4 are for a system with $\Psi(s,r)$ dependent on two successive states of the Markov chain. If $\Psi(s,r) = \Psi(r)$, condition x_0, r_{-1} given in (12) and (14) could be changed to x_0, r_0.*

We state without proof the following lemma:

LEMMA 6 *If system (11) is conditionally stochastically (ms-) stable, then it is stochastically (ms-) stable.*

Define for $\boldsymbol{X} \in \mathcal{M}_{\mathbb{E}}^n$

$$\begin{aligned} \boldsymbol{\mathcal{G}}\boldsymbol{X}(r) &= \mathbf{E}[\Psi^T(r_0, r_1)\boldsymbol{X}(r_1)\Psi(r_0, r_1) | r_0 = r] \\ &= \overline{\Psi^T \boldsymbol{X} \Psi}(r) \end{aligned} \tag{16}$$

i.e. $\boldsymbol{\mathcal{G}} : \mathcal{M}_{\mathbb{E}}^n \to \mathcal{M}_{\mathbb{E}}^n$. We have from (14) by (11) and (16)

$$\mathbf{E}[x_n^T x_n | x_1, r_0] = x_1^T \boldsymbol{\mathcal{G}}^n \mathbb{I}(r_0) x_1.$$

REMARK 2 *To calculate $\mathcal{G}\boldsymbol{X}$, one can use the stack and the stack inverse transformation (see Bellman, 1970, Horn and Johnson, 1988).*

We get

$$\begin{aligned} st(\mathcal{G}\boldsymbol{X}(r)) &= st(\overline{\Psi^T \boldsymbol{X} \Psi}(r)) = \\ &= \sum_{j \in \mathbb{E}} p_{rj} [\Psi^T(r,j) \otimes \Psi^T(r,j)] st(\boldsymbol{X}(j)) = \\ &= \sum_{j \in \mathbb{E}} p_{rj} \tilde{\Psi}(r,j) st(\boldsymbol{X}(j)) \end{aligned} \tag{17}$$

where $\otimes$ denotes the Kronecker product of matrices, st is the stack operation (for details see Lancaster and Tismenetsky, 1985) and $\tilde{\Psi}(r,j) = \Psi^T(r,j) \otimes \Psi^T(r,j)$. Expression (17) can be described shortly in matrix notation

$$st(\mathcal{G}X) = \tilde{\Phi}(p) st(X),$$

where

$$\tilde{\Phi}(p) = \begin{bmatrix} p_{11}\tilde{\Psi}(1,1) & p_{12}\tilde{\Psi}(1,2) & \dots & p_{1s}\tilde{\Psi}(1,s) \\ p_{21}\tilde{\Psi}(2,1) & p_{22}\tilde{\Psi}(2,2) & \dots & p_{2s}\tilde{\Psi}(2,s) \\ \dots & \dots & \dots & \dots \\ p_{s1}\tilde{\Psi}(s,1) & p_{s2}\tilde{(}s,2) & \dots & p_{ss}\tilde{\Psi}(s,s) \end{bmatrix}.$$

On the basis of the above definitions and Lemma 5 we get

LEMMA 7 *(i) Operator $\mathcal{G} : \mathcal{M}^n_{\mathbb{E}} \to \mathcal{M}^n_{\mathbb{E}}$ is linear and positive with respect to $\mathcal{K}^n_{\mathbb{E}}$.*

(ii) Closed loop system (11) is conditionally ms-stable if and only if operator $\mathcal{G}$ is stable.

(iii) Closed loop system (11) is conditionally ms-stable if and only if $\rho(\tilde{\Phi}(p)) < 1$.

Proof. To prove (i), let us observe that $\mathcal{G}$ is defined by the expected value which is linear. If $\boldsymbol{X} \in \mathcal{K}^n_{\mathbb{E}}$ then for all $i \in \mathbb{E}$, by definition, $\boldsymbol{X}(i)$ is non-negative definite and for all $j, i \in \mathbb{E}$ we have that $\Psi(j,i) = \Psi^T(j,i)\boldsymbol{X}(i)\Psi(j,i)$ is non-negative definite. The expected value is a convex combination of $\Psi(j,i)$, hence $\mathcal{G}\boldsymbol{X}(j)$ is non-negative definite and $\mathcal{G}\boldsymbol{X} \in \mathcal{K}^n_{\mathbb{E}}$. Statement (ii) follows from the definition of stability for the operator and close loop systems. Statement (iii) of the lemma follows from Remark 1. ■

For ms-stability we consider the behaviour of $V_n = \mathbf{E}[x_n^T x_n | x_0]$. Let us observe that

$$V_n = x_0^T \mathbf{E}[\Psi^T(r_{-1}, r_0) \mathcal{G}^n \mathbb{I}(r_0) \Psi(r_{-1}, r_0)] x_0.$$

When controls fulfil (D0), $\Psi(r_{-1}, r_0) = \Psi_L(r_0)$ and for controls fulfilling (D1), $\Psi(r_{-1}, r_0) = \boldsymbol{A}(r_0) - \boldsymbol{B}(r_0)\boldsymbol{L}$, where $\boldsymbol{L}$ does not depend on the states of the Markov chain.

Space $\mathcal{M}^n$ is linearly isomorphic with $\mathcal{M}^{n^2}$ and let $\tilde{\mathcal{G}} : \mathcal{M}^{n^2}_{\mathbb{E}} \to \mathcal{M}^{n^2}_{\mathbb{E}}$ correspond to $\mathcal{G} : \mathcal{M}^n_{\mathbb{E}} \to \mathcal{M}^n_{\mathbb{E}}$. For $\Psi \in \mathcal{M}^{n^2}_{\mathbb{E}}$

$$\tilde{\mathcal{G}}\Psi(r) = \mathbf{E}[\tilde{\Psi}(r)\Psi(r_1)|r_0 = r]$$

and we have

$$\tilde{V}_n = (p_1\tilde{\Psi}(1), p_2\tilde{\Psi}(2), \ldots, p_s\tilde{\Psi}(s))\tilde{\Phi}^n(p)\mathbb{I} \tag{18}$$

From the above considerations one can formulate the necessary and sufficient conditions for the ms-stability.

PROPOSITION 3 *System (11) is ms-stable if and only if*

$$\lim_{n\to\infty} \tilde{V}_n = 0.$$

From Proposition 3 we see that $\rho(\tilde{\Phi}(p)) < 1$ is sufficient for the ms-stability.

One can use the following results to establish the relation between the stochastic stability and ms-stability.

LEMMA 8 *(see Feng and Loparo, 1990). For any $F \in \Re^{q\times l}, \Phi \in \Re^{l\times l}, a \in \Re^l$, we have that $\lim_{n\to\infty} F\Phi^n a = 0$ implies $\sum_{k=0}^{\infty} F\Phi^k a < \infty$.*

PROPOSITION 4 *The (conditional) stochastic stability and ms-stability are equivalent.*

Proof. The proof follows from (18) and Lemma 8. ∎

The necessary and sufficient conditions for the stochastic stability can be formulated as follows:

PROPOSITION 5 *Closed loop system (11) is conditionally stochastically stable if and only if there exists a solution $\boldsymbol{M} \in \mathcal{K}^n_{\mathbb{E}}, \boldsymbol{M} \succ \Theta$ of the equation*

$$\boldsymbol{M}(j) - \sum_{i\in\mathbb{E}} p_{ji}\Psi^T(j,i)\boldsymbol{M}(i)\Psi(j,i) = \mathbb{I}(j), \;\; j \in \mathbb{E}. \tag{19}$$

Proof. By (11) we have

$$\sum_{k=n}^{N} x_k^T x_k = \sum_{k=n}^{N} x_n^T \Psi_{k,n}^T \Psi_{k,n} x_n$$

where $\Psi_{k,n} = \prod_{i=n}^{k}{}^{*}\Psi(r_{i-1}, r_i) = \Psi(r_{k-2}, r_{k-1})\Psi(r_{k-3}, r_{k-2})\ldots\Psi(r_{n-1}, r_n)$ for $k > n$ and $\Psi_{n,n} = \mathbb{I}$. Denote

$$\begin{aligned}\Phi(N-n, r) &= \mathbf{E}[\sum_{k=n}^{N} \Psi_{k,n}^T \Psi_{k,n}|r_n = r] \\ &= \mathbf{E}[\sum_{k=0}^{N-n} \Psi_{k,0}^T \Psi_{k,0}|r_0 = r].\end{aligned}$$

The properties of the conditional expectation and the fact that $\{r_k\}_{k=0}^{\infty}$ is a homogeneous Markov chain allow us to obtain the following relation

$$\Phi(N-n,r) = \boldsymbol{I} + \mathbf{E}[\sum_{k=n+1}^{N} \Psi_{k,n}^T \Psi_{k,n} | r_{n-1} = r] =$$
$$= \boldsymbol{I} + \mathbf{E}[\Psi^T(r_{n-1},r_n)\mathbf{E}[\sum_{k=n+1}^{N} \Psi_{k,n+1}^T, \Psi_{k,n+1}|r_n]\Psi(r_{n-1},r_n)|r_{n-1}=r]$$
$$= \boldsymbol{I} + \mathbf{E}[\Psi^T(r_{n-1},r_n)\Phi(N-n-1,r_n)\Psi(r_{n-1},r_n)|r_{n-1}=r]. \tag{20}$$

$\Phi(N-n,\cdot) \in \mathcal{K}_{\mathbb{E}}^n$ and it is nondecreasing because $\Psi_{k,n}^T\Psi_{k,n}$ is positive definite for every path of the Markov chain. From stochastic stability, $\Phi(N-n,\cdot)$ is bounded. There exists a unique limit

$$\boldsymbol{M}(j) = \lim_{k\to\infty} \Phi(k,j) \quad \text{for } j \in \mathbb{E}$$

and it fulfils the equation

$$\boldsymbol{M}(j) = \boldsymbol{I} + \mathbf{E}[\Psi^T(j,r_1)\boldsymbol{M}(r_1)\Psi(j,r_1)|r_0 = j]$$

for $j \in \mathbb{E}$. This proves the necessity of the condition given in the proposition.

Now the sufficiency will be proved. Let the system of equations (19) have solution $\boldsymbol{M} \in \mathcal{K}_{\mathbb{E}}^n$. Consider the stochastic Lyapunov function

$$F_k(x_k, r_{k-1}) = x_k^T \boldsymbol{M}(r_{k-1})x_k, \tag{21}$$

$k = 0, 1, \ldots$. We have from (11) and (21)

$$\begin{aligned}
&\mathbf{E}[F_{k+1}(x_{k+1},r_k)|x_k,r_{k-1}] - F_k(x_k,r_{k-1}) = \\
&= \mathbf{E}[x_k^T\Psi^T(r_{k-1},r_k)M(r_k)\Psi(r_{k-1},r_k)x_k|x_k,r_{k-1}] - x_k^T M(r_{k-1})x_k = \\
&= x_k^T\{\mathbf{E}[\Psi^T(r_{k-1},r_k)M(r_k)\Psi(r_{k-1},r_k)|x_k,r_{k-1}] - M(r_{k-1})\}x_k = \\
&= -x_k^T x_k.
\end{aligned} \tag{22}$$

Let $x \neq 0$. Denote

$$0 < \alpha = \min_{x,i} \frac{x^T x}{x^T \boldsymbol{M}(i) x} \tag{23}$$

From (22) and (23) we get

$$0 < \mathbf{E}[F_{k+1}(x_{k+1},r_k)|x_k,r_{k-1}] \leq (1-\alpha)F_k(x_k,r_{k-1}) \tag{24}$$

and we have

$$\begin{aligned}
&\mathbf{E}[F_{k+1}(x_{k+1},r_k)|x_1,r_0] = \mathbf{E}[\mathbf{E}[F_{k+1}(x_{k+1},r_k)|x_k,r_{k-1}]|x_1,r_0] \\
&\overset{(24)}{\leq} (1-\alpha)\mathbf{E}[F_k(x_k,r_{k-1})|x_1,r_0] \leq \ldots \leq (1-\alpha)^{k-1}\mathbf{E}[F_2(x_2,r_1)|x_1,r_0].
\end{aligned}$$

This implies the existence of $\lim_{n\to\infty} \mathbf{E}[\sum_{k=1}^{n} F_{k+1}(x_{k+1},r_k)|x_1,r_0]$. The conditional stochastic stability follows. ■

REMARK 3 *For the system without a delay in the observation of Markov parameter r_n, one can prove (Ji and Chizeck, 1990) that system (11) is conditionally stochastically stable if and only if there exists a solution $\boldsymbol{M} \in \mathcal{K}^n_{\mathbb{E}}$ of the equation*

$$\boldsymbol{M}(j) - \Psi_L^T(j) \sum_{i \in \mathbb{E}} p_{ji} \boldsymbol{M}(i) \Psi_L(j) = \boldsymbol{I} \quad \text{for every } j \in \mathbb{E}.$$

From (20) we have

$$\begin{aligned}
&\Phi(k, r) \\
&= \boldsymbol{I} + \sum_{\substack{i \in \mathbb{E} \\ i \neq j}} p_{ji} \Psi^T(j,i) \Phi(k-1,i) \Psi(j,i) - \bar{\Psi}^T(j,j) \Phi(k-1,j) \bar{\Psi}(j,j) = \\
&= \beta_{k-1}(j) + \bar{\Psi}^T(j,j) \Phi(k-1,j) \bar{\Psi}(j,j) = \\
&= \beta_{k-1}(j) + \bar{\Psi}(j,j) \beta_{k-2}(j) \bar{\Psi}(j,j) + (\bar{\Psi}^T(j,j))^2 \Phi(k-2,j) (\bar{\Psi}(j,j))^2 = \\
&= \sum_{l=1}^{k} (\bar{\Psi}^T(j,j))^{l-1} \beta_{k-l}(j) (\bar{\Psi}(j,j))^{l-1} + (\bar{\Psi}^T(j,j))^k \Phi(0,j) (\bar{\Psi}(j,j))^k \quad (25)
\end{aligned}$$

where: $\bar{\Psi}(j,j) = \sqrt{p_{jj}} \Psi(j,j)$ and

$$\beta_k(j) = \boldsymbol{I} + \sum_{\substack{i \in \mathbb{E} \\ i \neq j}} p_{ji} \Psi^T(j,i) \Phi(k,j) \Psi(j,i).$$

If system (11) is stable, then there exists $\lim_{k \to \infty} \Phi(k,r) = \boldsymbol{M}(r)$ for every $r \in \mathbb{E}$ and it is positive semidefinite. Put $\Phi(0,j) = \boldsymbol{I}$ and $\Phi(0,j) = \boldsymbol{O}$ in (25). We have then

$$\begin{aligned}
\boldsymbol{M}(r) &= \lim_{k \to \infty} \sum_{l-1}^{k} (\bar{\Psi}^T(j,j))^{l-1} \beta_{k-l}(j) (\bar{\Psi}(j,j))^{l-1} + (\bar{\Psi}^T(j,j))^k (\bar{\Psi}(j,j))^k \\
&= \lim_{k \to \infty} \sum_{l=1}^{k} (\bar{\Psi}^T(j,j))^{l-1} \beta_{k-1}(j) (\bar{\Psi}(j,j))^{l-1}
\end{aligned}$$

Hence $\lim_{k \to \infty} (\bar{\Psi}(j,j))^k = \boldsymbol{O}$ and $\bar{\Psi}(j,j)$ is stable for every $j \in \mathbb{E}$. On the other hand if system (11) is stable, then by (25) we have that $\boldsymbol{M}(r)$ can be approximated by recursively given sequence $\Phi(k,r)$ with $\Phi(0,r) = \boldsymbol{O}$.

Corollary. Stability $\bar{\Psi}(j,j)$ for every $j \in \mathbb{E}$ is necessary for the stability of (11). For stable system (11), $\boldsymbol{M} \in \mathcal{K}^n_{\mathbb{E}}$ is a limit of recursively definite function $\Phi(k, \cdot) \in \mathcal{K}^n_{\mathbb{E}}$.

REMARK 4 *For the system without a delay, the above corollary was formulated and proved by Ji and Chizeck (1990).*

On the basis of the above considerations of the stability of closed loop system (11) one can define the ms-stability of the open loop system (1). The definition will be different for a system with a delay in the observation of the Markov parameter and when the Markov parameter at moment n is observed.

DEFINITION 5 *System (1) is ms-stabilizable by controls fulfilling (D0) if there exists* $\boldsymbol{L}(r)$ *such that closed loop system (11) with* $\Psi(s,r) = \Psi_L(r)$ *is stable.*

The condition of ms-stabilizability will be formulated later.

5. Infinite horizon optimal control

Now we focus our attention on the infinite horizon optimal control problem for discrete-time linear system (1) and quadratic criteria (3), both with parameters dependent on the Markov chain. The considerations of the section are devoted to the system with delay for one step. Results for the system without delay follow similarly. Before that, some properties of operators $\boldsymbol{\mathcal{G}}_*$ and $\boldsymbol{\mathcal{G}}_L$ are proved. Similar properties will be formulated for $\boldsymbol{\mathcal{D}}_*$ and $\boldsymbol{\mathcal{D}}_L$.

PROPOSITION 6 *For every* $X \in \mathcal{K}^n_{\mathbb{E}}$ *we have* $0 \preceq \boldsymbol{\mathcal{G}}_*^N H \preceq \boldsymbol{\mathcal{G}}_L^N H$ *and* $\boldsymbol{\mathcal{G}}_*$ *is monotonic.*

Proof. From Proposition 2 we have for every x_1 and r_0

$$\mathbf{E}[J_N(\vec{u}^*)|x_1, r_0] \leq \mathbf{E}[J_N(\vec{u}_L)|x_1, r_0]$$

where $\vec{u}_L$ is given by (8) and from Lemma 3 we get

$$x_1^T \boldsymbol{\mathcal{G}}_*^{N-1} H(r_0) x_1 \leq x_1^T \boldsymbol{\mathcal{G}}_L^{N-1} H(r_0) x_1.$$

Let $H_1 \preceq H_2$ and $N = 2$, we have $\mathbf{E}[J_2(\vec{u}^*, H_1)|x_1, r_0] \leq \mathbf{E}[J_2(\vec{u}^*, H_2)|x_1, r_0]$. Hence $\boldsymbol{\mathcal{G}}_* H_1 \preceq \boldsymbol{\mathcal{G}}_* H_2$ and $\boldsymbol{\mathcal{G}}_*$ is monotonic. ∎

Up to this moment, we have been interested in the state equation only and not in the output, because our performance index was expressed in terms of states and inputs. Let us now define a fictitious output for our system. Let C be the square root of Q i.e. $Q = C^T C$. Define the output equation by

$$y_k = C x_k \tag{26}$$

For system (11) with output equation (26), the following notion of detectability is introduced.

DEFINITION 6 (A, C) *is ms-detectable if* $\mathbf{E}\{y_i^T y_i | x_0\} = 0$ *for every* $i \in \mathbb{N}$ *implies* $\mathbf{E}[x_i^T x_i | x_0] \to 0$ *as* $i \to \infty$.

LEMMA 9 *System (11) with output equation (26) is ms-detectable if and only if* $x_0^T \mathcal{G}^n C^T C(r_{-1}) x_0 = 0$ *for every* $i \in I\!N$ *implies* $\lim_{n \to \infty} x_0^T \mathcal{G}^n I\!I(r_{-1}) x_0 = 0$.

Proof. The lemma follows from the definition of ms-detectability and the definition of operator $\mathcal{G}$. ∎

Similarly as in de Koning (1982), one can show the following

LEMMA 10 *Let* $\bar{R}(s)$ *be positive definite for every* $s \in I\!E$, $\mathcal{G}_0^N \Theta > 0$ *or* $\bar{Q}(s)$ *is positive definite, then* $\mathcal{G}_L^N \Theta$ *is positive definite for every* $s \in I\!E$, *control* $\boldsymbol{L} \in \mathcal{M}_{I\!E}^{nm}$ *and* N.

Proof. From Lemma 3 we have $x_0^T \mathcal{G}_L^N \Theta x_0 = \mathbf{E}\{\sum_{i=0}^{N-1} x_i^T \bar{Q}(r_{i-1}) x_i + u_i^T \bar{R}(r_{i-1}) u_i\}$, for $u_i = -L x_i$, $i = 0, 1, \ldots, N-1$. If $x_0^T \mathcal{G}_L^N \Theta x_0 = 0$ and $R(s)$ is positive definite then $u_i = 0$ a.s. for $i = 0, \ldots, N-1$ and $x_0^T \mathcal{G}_L^N \Theta x_0 = x_0^T \mathcal{G}_0^N \Theta x_0 = 0$. If $\bar{Q}(r)$ is positive definite for $r \in I\!E$ then $\mathcal{G}_L^N \Theta = \sum_{i=0}^{N-1} \mathcal{H}_L^i (\bar{Q} + [L^T \bar{R} L]) \succeq \bar{Q}$. Hence $\mathcal{G}_L^N \Theta$ is also positive definite. ∎

One can state the following lemma

LEMMA 11 *When* $\bar{R}(s)$ *is positive definite,* (A, C) *ms-detectable or* $\bar{Q}(s)$ *positive definite, then* $(\Psi_L, (\bar{Q} + [L^T \bar{R} L])^{1/2})$ *is ms-detectable.*

The proof of the lemma follows from Lemma 10 and the definition of the ms-detectability.

The properties of operator $\mathcal{G}_*$ and $\mathcal{G}_L$ stated in Lemmas 9, 10, 11 and Proposition 6 show that the finite horizon optimal control problem is an approximation for the infinite horizon control problem (see de Koning, 1982).

THEOREM 1 *If system (1) is ms-stabilizable by a control dependent on the delayed observation of the Markov chain, then* $S(r) = \lim_{N \to \infty} \mathcal{G}_*^N \Theta$ *exists and* S *is the minimal solution in* $\mathcal{K}_{I\!E}^n$ *of the equation* $S = \mathcal{G}_* S$.

Proof. Since $\{\mathcal{G}_*^N \Theta\}$ is increasing and bounded, $S(r) = \lim_{N \to \infty} \mathcal{G}_*^N \Theta$ exists. We have $\mathcal{G}_*^{N+1} \Theta = \mathcal{G}_*(\mathcal{G}_*^N \Theta)$. It follows that $S = \mathcal{G}_* S$. If S_1 is another solution of this equation, then $G_*^i \Theta \preceq G_*^i S_1 = S_1$ and $S \preceq S_1$. ∎

THEOREM 2 *Let* $S = \lim_{N \to \infty} \mathcal{G}_*^N \Theta$ *exist and* $u_i = -\mathcal{L} S(r_{i-1}) x_i$ *for every* $i \in I\!N$. *We have*

$$J(\vec{u}) = x_0^T S x_0.$$

Considerations similar to those in the proof of Theorem 5.2 in de Koning (1982) prove the statement. We can also obtain the stability condition for the optimal control.

THEOREM 3 *Let* $S = \lim_{N\to\infty} \mathcal{G}_*^N \Theta$ *exist. If* $\bar{R}(r)$ *is positive definite for every* r, (A, C) *is ms-detectable or* $\bar{Q}(r)$ *is positive definite then system (11) with* $\Psi(r,s) = \Psi_{\mathcal{L}_S}(r,s)$ *is stable and* $S(r)$ *is the unique non-negative definite solution of the equation* $S = \mathcal{G}_* S$.

The equivalent conditions and theorem can be stated for the system without a delay of the observation of the Markov chain. To this end, operator $\mathcal{G}_*$ should be replaced by $\mathcal{D}_*$ and $\bar{R}, \bar{Q}$ by R, Q, respectively.

On the basis of the solution of the infinite horizon optimal control problem one can state the criterion of stabilizability.

THEOREM 4 *System (1) is ms-stabilizable by controls based on the delayed observation of the Markov chain if and only if sequence* $\{K_n(r)\}$, *obtained from the following recursive equation*

$$\begin{aligned} K_n &= \boldsymbol{I} + \overline{A^T \bar{K}_{n-1} A}(r) \\ &- \overline{A^T \bar{K}_{n-1} B}(\boldsymbol{I} + \overline{B^T \bar{K}_{n-1} B})^+ \overline{B^T \bar{K}_{n-1} A}(r) \end{aligned} \tag{27}$$

with initial condition $K_0(r) = 0$ *for every* $r \in \mathbb{E}$, *converges to a set of symmetric, non-negative definite matrices as* n *tends to infinity.*

Proof. If $R(r) = \boldsymbol{I}$ and $Q(r) = \boldsymbol{I}$ then the assumption of Theorem 3 are fulfilled. By Proposition 2, we have (27). Hence the statement follows from Theorem 3. ■

6. Conclusions

A problem of the optimal control of a class of a discrete time Markovian jump linear systems with perfect observation of the states of the Markov chain has been investigated. The quadratic cost is assumed. Two different models of the observation of the states of the Markov chain have been considered. The first one is without a delay in time, the second one with delay for one step. The differences between these observation schemes have been pointed out. The controls for the finite horizon of the control have been constructed. For the infinite horizon case the solution as well as stability conditions have been formulated. It is mentioned that the delay for one step model is related to the independent parameter case considered by de Koning (1982).

References

BELLMAN, R. (1970) *Introduction to Matrix Analysis*, McGraw-Hill, New York.

BLOM, H.A. (1991) Hybrid state estimation for systems with semi-markov switching coefficients. In: *Trans. of European Control Conference '91*, Grenoble, France, 1132–1137.

COSTA, O.L. and FRAGOSO, M.D. (1993) Stability results for discrete-time linear systems with markovian jumping parameters. *J. Math. Anal. Appl.*, 179, 154–178.

DE KONING, W.L. (1982) Infinite horizon optimal control of linear discrete time systems with stochastic parameters. *Automatica*, **18**, 443–453.

DE KONING, W.L. (1983) Detectability of linear systems with stochastic parameters. *Int. J. Control*, **38**, 1035–1046.

FENG, X. and LOPARO, K.A. (1990) Stability of linear markovian jump systems. In: *Proc. of the 29th Conference on Decision and Control, Honolulu, Hawaii*, December 1990, IEEE, 1408–1413.

FRAGOSO, M.D., COSTA, O. and DE SOUZA, C. (1993) *Some issues on generalized Riccati equations arising in stochastic control*, 1993. Submitted to SIAM J. Control & Optimization.

HORN, R.A. and JOHNSON, C.R. (1988) *Matrix Analysis.* Cambridge University Press, Cambridge.

JI, Y. and CHIZECK, H. (1990) Jump linear quadratic gaussian control: stady-state solution and testable conditions. *Control-Theory and Advanced Technology*, 6, 289–319.

KLAMKA, J. (1991) *Controllability of Dynamical Systems.* PWN - Polish Scientific Publishers and Kluwer Academic Publishers, Warsaw/Dordrecht/ Boston/London.

KLAMKA, J. and DONG, LE SI (1990) Stochastic controllability of discrete time systems with jump Markov disturbances. *Archiwum Automatyki i Telemechaniki*, **35**, 67–74.

KRASNOSEL'SKIJ, M. (1964) *Positive Solutions of Operator Equations.* P. Noordhoff Ltd, Groningen.

KUMAR, P.R. and VARAIYA, P. (1986) *Stochastic Systems: Estimation, Identification, and Adaptive Control.* Prentice-Hall Information and System Sciences Series, Prentice Hall, Inc., Englewood Cliffs.

LANCASTER, P. and TISMENETSKY, M. (1985) *The Theory of Matrices.* Academic Press, Orlando.

YANG, C. and BAR-SHALOM, Y. (1991) Stabilizing control of partially observed hybrid systems. In: *Trans. of European Control Conference '91*, Grenoble, France, 1152–1157.

ZABCZYK, J. (1975) On optimal stochastic control of discrete-time systems in Hilbert space. *SIAM J. Control*, **13**, 1217–1234.

ZABCZYK, J. (1981) Controllability of stochastic linear systems. *Systems and Control Letters*, **1**, 25–31.