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# Convergence and extensions of variational problems with non-coercive functionals

by

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**Abstract:** The paper is a transcript of the lecture given at the European Symposium on Well-Posedness in Optimization in Warsaw. It contains a complete theory of variational problems with integrands not depending on $x$, including existence and relaxation theorems, a complete description of solutions and the connection between variational convergences of functionals and convergence of value functions and solutions of associated variational problems with the main emphasis on functionals that lack coercivity.

**Keywords:** coercivity, relaxation, Γ-convergence, duality, vector measures.

## 1. Introduction

The magic word that links variational convergence of functionals and convergence of values and solutions of the associated variational problems is "coercivity". Recall that an (extended-real-valued) function $f$ on a topological (e.g. metric) space $X$ is called *coercive* if there is an $\alpha > \inf f$ such that the $\alpha$-sublevel set $\mathcal{L}_\alpha(f) = \{x : f(x) \le \alpha\}$ is relatively compact. A sequence $(f_n)$ is *uniformly coercive* if for any $\alpha \in \mathbb{R}$ there is an $n(\alpha)$ such that $\mathcal{L}_\alpha(f_n)$ is either empty or relatively compact for each $n \ge n(\alpha)$.

The two fundamental facts concerning coercivity and convergences are the following:

FACT 1: a coercive lower semicontinuous function attains its minimal value;
FACT 2: if a sequence $(f_n)$ Γ-converging to $f$ is uniformly coercive, then $\inf f_n$ converges to $\inf f$ and, in case when $|\inf f| < \infty$, any sequence $(x_n)$ such that $f_n(x_n) - \inf f_n \to 0$ contains a subsequence converging to a minimizer of $f$.

A natural question is what happens and what can be done if the coercivity property does not hold. As far as an individual function is concerned, a general recipe is to try to construct a coercive relaxation of the function, find a minimizer of the latter and then, with the help of this minimizer, try to recover a minimizing sequence for the original function.

Much less (if anything) is known about behavior of minimal values and minimizers of non-uniformly coercive sequences of functions. Simple examples can be given to show that in the absence of uniform coercivity the minimal values do not converge to the minimal value of the Γ-limit.

Consider for example the problems of minimizing

$$I_n(u(\cdot)) = \int_0^1 e^{-\frac{|u(t)|}{n}}\, dt$$

over the collection $S(x)$ of summable functions $u(t)$ whose integrals over $[0,1]$ are equal to $x$. It is an easy matter to verify that the functionals $I_n$ Γ-converge in $L_1$ to the function identically equal to one while the minimal value of every $I_n$ on every $S(x)$ is zero. On the other hand, the relaxation of every $I_n$ in the weak topology of $L_1$ is zero (see Theorem 1 below), and the natural question is whether and in which sense the functional identically equal to zero can be considered a sort of a limit of $I_n$.

The purpose of the lecture is to discuss the questions in the context of one of the simplest problem of calculus of variations.

## 2. Relaxations, extensions and Γ-limits

I shall begin, however, by recalling some basic definitions and concepts already mentioned in the Introduction.

A *relaxation* of a function $f$ is the greatest lower semicontinuous function majorized by $F$, that is to say, a lower semicontinuous envelope of $f$:

$$f(x) = \sup\{\beta(x) : \beta \leq f \ \&\ \beta \text{ continuous}\}.$$

The indefinite article has been used since there is always a certain freedom of choosing a space and/or topology with which the function is considered.

In case when the function is considered on a metric space, a convenient characterization of the relaxation can be given, namely $\overline{f}$ is the relaxation of $f$ if

- for any $x$ and for any sequence $(x_n)$ converging to $x$, $\liminf f(x_n) \geq \overline{f}(x)$;
- for any $x$ there is a sequence $(x_n)$ converging to $x$ such that $\limsup f(x_n) \leq \overline{f}(x)$.

We refer to Buttazzo (1989) and Dal Maso (1993) for details.

Along with relaxation we shall use a weaker concept of an extension of the function introduced in Ioffe-Tihomirov (1969). Namely, if $f$ is a function on a metric space $X$, then a function $g$ on (generally) another metric space $Y$ is an

$x \in X$ $g(\pi(x)) \le f(x)$ and for any $y \in Y$ either $g(y) = \infty$ or there is a sequence $(x_n)$ such that $\pi(x_n) \to y$ and $\limsup f(x_n) \le g(y)$.

It has to be emphasized that in both definitions the functions are assumed extended-real-valued and defined on the entire domain space. We shall adhere to this assumption.

The last definition to be recalled is that of Γ-convergence (or epi-convergence). We shall state the definition only for functions on metric spaces as we do not need more general settings. A sequence $(f_n)$ is said to Γ-converge to $f$ if

- for any $x$ and any sequence $(x_n) \to x$, $\liminf f_n(x_n) \ge \overline{f}(x)$ and;
- for any $x$ there is a sequence $(x_n)$ converging to $x$ such that $\limsup f_n(x_n) \le \overline{f}(x)$.

This concept is going back to works of Wijsman, Mosco and DeGiorgi of the 1960's and 1970's. We refer the reader to Attouch (1984), Dal Maso (1993) and (for the finite dimensional case) to Rockafellar and Wets (1997) for details and more information.

The following simple facts should be mentioned in connection with the definitions:

- every sequence has a Γ-converging subsequence;
- relaxation is the Γ-limit of the stationary sequence $f_n = f$;
- Γ-limit is always a lower semicontinuous function;
- extension is not necessarily lower semicontinuous.

## 3. The class of problems to be considered

These are problems of the form:

$$\text{minimize} \quad I_f(u(\cdot)) = \int_0^1 f(t, u(t))\,dt \tag{1}$$

over all summable $\mathbb{R}^d$-valued functions $u(\cdot)$ satisfying

$$\int_0^1 u(t)dt = x. \tag{2}$$

With all the simplicity of the formulation this class of problems contains optimal control problems with data depending linearly on the state variable (see Ioffe and Tihomirov, 1974, §9.3 for details):

$$\begin{array}{ll} \text{minimize} & \displaystyle\int_0^1 [(a(t)|u(t)) + b(t, u(t))]dt; \\ \text{s.t.} & \dot{x} = A(t)x + B(t, u), \quad u \in U(t); \\ & x(0) = x_0,\ x(1) = x_1. \end{array}$$

(Here $(\cdot|\cdot)$ stands for the inner product). Moreover, the results to be discussed can be extended to cases when integration is performed over a complete metric

We shall study the problem under fairly non-restrictive assumptions on $f$, namely:

($\mathbf{A}_1$) $f$ is a nonnegative extended-real-valued function on $[0,1] \times \mathbb{R}^d$;
($\mathbf{A}_2$) $f(t, u(t))$ is Lebesgue measurable if so is $u(t)$;
($\mathbf{A}_3$) there is a summable $\overline{u}(t)$ such that $f(t, \overline{u}(t))$ is summable.

The natural space to consider the problem is, of course, $L_1^d$, the Lebesgue space of all summable $\mathbb{R}^d$-valued functions on $[0,1]$. According to the well-known compactness criterium going back to de la Vallée–Poussin, the functional $I_f$ is coercive in the weak topology of $L_1^d$ (coercivity in the norm topology is of little interest) if and only if there is a function $\varphi(x)$ (on $\mathbb{R}^d$) growing to infinity superlinearly (that is, $(\varphi(x)/\|x\|) \to \infty$ when $\|x\| \to \infty$) such that $f(t,x) \geq \varphi(x)$ for all $x \in \mathbb{R}^d$ for almost every $t$. On the other hand, to guarantee that $I_f$ is lower semicontinuous with respect to the weak topology we have to require that $f$ be convex as a function of $u$.

If the integrand fails to have these two properties (superlinear growth and convexity), a coercive relaxation of the functional can be constructed in a different space, namely in the space $\mathcal{M}^d$ of all $\mathbb{R}^d$-valued Radon measures on $[0,1]$, if we consider every $u(\cdot) \in L_1^d$ as a density of an absolutely continuous measure and set for a $\nu \in \mathcal{M}^d$

$$J_f(\nu) = \begin{cases} I_f\left(\dfrac{d\nu}{dt}\right), & \text{if } \nu \text{ is absolutely continuous;} \\ \infty, & \text{otherwise.} \end{cases}$$

The relaxation theorem for $J_f$ proved by the end of the 1980s and associated mainly with the names of Ambrosio, Bouchitté, Buttazzo, De Giorgi and Valadier (see Buttazzo, 1989, for details) is stated as follows:

Let $\varphi^\infty$ stand for the recession function of a closed convex function $\varphi$ on $\mathbb{R}^d$:

$$\varphi(h) = \lim_{t \to \infty} t^{-1} \varphi(x + th)$$

(for $x \in \operatorname{dom} f$).

THEOREM 1 *Assume* ($\mathbf{A}_1$)–($\mathbf{A}_3$). *Then the relaxation of $J_f$ in the weak-star topology of $\mathcal{M}^d$ is*

$$\mathcal{J}_g(\nu) = J_g(\nu_a) + \int_0^1 g^\infty\left(t, \frac{d\nu_s(t)}{\delta|\nu_s|}\right) \delta|\nu_s|(t),$$

*where $g(t.u)$ is the pointwise supremum of functions $(a(t)|u) + b(t)$, such that $a(t)$ is continuous, $b(t)$ is measurable and $(a(t)|u) + b(t) \leq f(t,u)$ for all $u$ almost everywhere on $[0,1]$.*

Here $\nu_a$ and $\nu_s$ stand for the absolutely continuous and the singular parts of limiting behavior of $I_f$: rapid oscillation leading to convexification (as, say, with $u_m(t) = \mathrm{sign}(\sin mt)$) for the Bolza integrand $f(t,u) = (1-u^2)^2$) and blowing up trajectories in the absence of superlinear growth leading to singularities in the limiting measure (as say with $u_m(t) = \min\{m, t^{-1}\}$ in the case of the integrand $f(t,u) = t^2u^2$ suggested by Weierstrass)

The functional $\mathcal{J}_g$ is coercive (with respect to the weak-star topology) if and only if there are a positive $\alpha$ and a summable $\beta(t)$ such that $g(t,u) \geq \alpha\|u\| + \beta(t)$. Theorem 1, however, is valid without any a priori restrictions on the rate of growth of $f$.

## 4. Extension via duality

A certain inconvenience of the quoted relaxation theorem comes from the fact that it does not offer any constructive procedure to calculate the integrand $g$. Examples show (e.g. Buttazzo, 1989) that even in simple situations this requires substantial effort.

In this section I shall describe a theory developed in Ioffe and Tihomirov (1969, 1974), almost 20 years prior to the proof of the relaxation theorem which, however, remained largely unknown due to political situation in the former Soviet Union rather than for any scientific reason. The theory allows to obtain an easily calculable extension of $I_f$ in the same space $\mathcal{M}^d$ which, although not being lower semicontinuous, do have minimizers in the coercive case (and even under a somewhat weaker assumption). Moreover, the minimizing measures whose existence is provided by the theory have very simple structure, with the singular parts consisting of at most $d$ jumps (that is, they are SBV-functions in the modern terminology).

Consider the value function of (1),(2):

$$V(x) = \inf\Big\{I_f(u(\cdot)) : \int_0^1 u(t)\,dt = x\Big\}.$$

This is a convex function on $\mathbb{R}^d$, its Fenchel conjugate being

$$V^*(p) = \sup_x((p|x) - V(x)) = \int_0^1 f^*(t,p)\,dt,$$

where $f^*$ is the Fenchel conjugate of $f$ with respect to $u$, and consequently, its second conjugate is

$$\overline{V}(x) = \sup_p\Big((p|x) - \int_0^1 f^*(t,p)\,dt\Big).$$

In particular, if $x \in \mathrm{ri}(\mathrm{dom}\,V)$, then, of course, $V(x) = \overline{V}(x)$ and $\partial V(x) \neq \emptyset$, that is there is a $p_x$ such that

$$V(x) + V^*(p_x) = (p_x|x)$$

Definition 1 A point $t \in [0,1]$ is called *p-ordinary* if $f^*(t,p)$ is summable in a neighborhood of $t$ (of course we speak about a neighborhood in $[0,1]$!). Otherwise $t$ is called *p-extraordinary*.

Set

$$P(t) = \{p \in \mathbb{R}^d : t \text{ is } p-\text{ordinary}\}.$$

Then, $P(t)$ is a convex-valued lower semi-continuous mapping with nonempty values (indeed, by $(\mathbf{A}_3)$ $f^*(t,0)$ is summable, so $0 \in P(t)$ for every $t$). Set

$$P = \bigcap_t P(t).$$

Clearly, $P$ is nonempty and coincides with the domain of $V^*$.

Let $s(t,\cdot)$ be the support function of $P(t)$:

$$s(t,w) = \sup_{p \in P(t)} (p|w).$$

We define the collection $\mathcal{K}$ of measures $\nu \in \mathcal{M}^d$ with purely discrete singular parts containing at most $d$ jumps. That is, $\nu \in \mathcal{K}$ if and only if

$$\nu_s = \sum_{i=1}^{k} w_i \varepsilon_{\tau_i}, \quad k \leq d,$$

where $w_i \in \mathbb{R}^d$ and $\varepsilon_\tau$ is the unit mass at $\tau$.

Next we define an extension of $I_f$ to $\mathcal{M}^d$ by

$$\mathcal{I}_f(\nu) = \begin{cases} \int_0^1 f(t, d\nu_a(t)/dt)dt + \sum_{i=1}^{k} s(\tau_i, w_i), & \text{if } \nu \in \mathcal{K}; \\ \infty, & \text{if } \nu \notin \mathcal{K}. \end{cases}$$

The verification that $\mathcal{I}_f$ is indeed an extension of $I_f$ is not difficult. It is also clear that $J_f(\nu) \geq \mathcal{I}_f(\nu)$ for all $\nu \in \mathcal{M}^d$. It can be further shown that $\mathcal{I}_f$ is coercive if and only if $0 \in \operatorname{int} P$.

Consider the problem

$$\text{minimize} \quad \mathcal{I}_f(\nu), \text{ s.t. } \int_0^1 d\nu = x. \tag{3}$$

Theorem 2 *Assume* $(\mathbf{A}_1)$, $(\mathbf{A}_2)$. *Suppose* $x \in \operatorname{ri}(\operatorname{dom} V)$. *Then* $\nu$ *is a solution of (3) if and only if* $\int d\nu = x$ *and there is a* $p \in P$ *such that*

$$(a) \qquad f^*(t,p) + f\left(t, \frac{d\nu_a(t)}{dt}\right) = \left(p \middle| \frac{d\nu_a(t)}{dt}\right), \text{ a.e. on } [0,1]$$

*and*

$$(b) \qquad s(\tau_i, w_i) = (p|w_i), \quad i = 1, \ldots, k.$$

It follows in particular from (a) that $f(t, d\nu_a(t)/dt) = f^{**}(t, d\nu_a(t)/dt)$ almost everywhere. The second equality means that every $w_i$, if distinct from zero, belongs to the cone normal to $P(\tau_i)$ at $w_i$. Moreover, as $P \subset P(t)$, the equality $(p|x) = \sup_{q \in P}(q|x)$ must also hold, that is, $p$ is also normal to $P$ at every nonzero $w_i$.

The next theorem contains a condition, which guarantees the existence of a solution in (3). This condition turns out to be slightly weaker than coercivity: it actually says that the value function of (3) differs by a linear function from a coercive function. Recall that $f(t, u)$ is a normal integrand if it is lower semicontinuous with respect to $u$ and the epigraph of $g$,

$$\text{epi } g = \{(t, u, \alpha) \in [0, 1] \times \mathbb{R}^d \times \mathbb{R} : \alpha \geq g(t, u)\},$$

belongs to the $\sigma$-algebra generated by all products of Lebesgue measurable subsets of $[0, 1]$ and Borel subsets of $\mathbb{R}^d \times \mathbb{R}$.

THEOREM 3 *Let $f$ be a normal integrand. If* $\text{int}P \neq \emptyset$, *then (3) has a solution for every* $x \in \text{ri}(\text{dom}\, V)$.

The example below shows that the problem (3) may have solutions even without the assumption that the interior of $P$ is nonempty.

EXAMPLE (see Buttazzo, 1989). Let $f(t, u) = a(t)\sqrt{1 + u^2}$, $a(t) \geq 0$. Set

$$\overline{c} = \inf\{c > 0 : a(t) < c \text{ on a set of positive measure}\}$$

We have

$$f^*(t, p) = \begin{cases} -\sqrt{a^2(t) - p^2}, & \text{if } |p| \leq a(t) \\ \infty, & \text{otherwise.} \end{cases}$$

An easy calculation reveals that $P = [-\overline{c}, \overline{c}]$ and $\mathcal{I}_f$ is coercive if and only if $\overline{c} > 0$. If so, then by Theorem 2 a solution in (3) exists. Set

$$v_p(t) = \begin{cases} \dfrac{p}{\sqrt{a^2(t) - p^2}}, & \text{if } |p| \leq \overline{c}; \\ \infty, & \text{otherwise.} \end{cases} ; \quad y_p = \int_0^1 v_p(t)\, dt.$$

Then, $y_p$ is strictly increasing when $p$ changes between $-\overline{c}$ and $\overline{c}$ and the structure of the solution depends on whether $v_{\overline{c}}$ is summable. If not, then for any $x$ there is precisely one $p$ such that $y_p = x$ in which case by Theorem 2 the solution is absolutely continuous with density $v_p(t)$. Otherwise, such a solution exists for $x$ with $|x| \leq y_{\overline{c}}$ and in case when the absolute value of $x$ is greater than $y_{\overline{c}}$, we always have $p = \overline{c} \cdot \text{sign}\, x$ and in principle there may be many solutions with the common density $v_p(t)$ and a jump of the size $w = (|x| - y_{\overline{c}})\text{sign}x$ at any $\tau$ at which $P(\tau) = [-\overline{c}, \overline{c}]$.

If $\overline{c} = 0$ (non-coercive case), then $P = \{0\}$. Set $T_0 = \{t \in [0, 1] : P(t) = \{0\}\}$. Then, $T_0 \neq \emptyset$. Indeed, if we assume that $\text{int}P(t) \neq \emptyset$ for all $t$, then taking into account the obvious symmetry of every $P(t)$ with respect to zero and lower semicontinuity of $P(\cdot)$, we have to conclude that 0 is an interior point of $P$.

## 5. Sequences: uniformly coercive functionals

In what follows we shall consider the sequences of functionals

$$I_n(u(\cdot)) = I_{f_n}(u(\cdot)) = \int_0^1 f_n(t, u(t))\, dt$$

and associated variational problems (1), (2), first under the condition that the corresponding functionals $J_n(\nu)$ are uniformly coercive in the weak-star topology of $\mathcal{M}^d$. The structure of $\Gamma$-limits of such sequences in the weak-star topology was described in Bouchitté (1987). Under $(\mathbf{A}_1)$ and $(\mathbf{A}_2)$, the sequence is uniformly coercive in the weak-star topology of $\mathcal{M}^d$ if there are a $c > 0$ and a summable function $\rho_0(t)$ on $[0, 1]$ such that for every $n$

$$f_n(t, u) \geq c\|u\| + \rho_0(t), \ \forall\, u \in \mathbb{R}^d \text{ a.e. on } [0, 1]. \tag{4}$$

THEOREM 4 *Assume that all integrands $f_n$ satisfy $(\mathbf{A}_1)$, $(\mathbf{A}_2)$ and the following compatibility hypotheses:*

*$(\mathbf{A}_4)$ there is a $\overline{u}(\cdot) \in L_1^d$ and a summable function $\rho(t)$ such that for every $n$*

$$f_n(t, \overline{u}(t))dt \leq \rho(t) \ \ a.e..$$

*Suppose further that sequence $(J_n)$ is uniformly coercive in the weak-star topology of $\mathcal{M}^d$ and $\Gamma$-converges in the same topology to a functional $\mathcal{J}$. Then, there are a probability measure $\mu$ on [0,1] and a normal convex integrand $g(t, u)$ with $g^\infty$ being lower semicontinuous jointly in $(t, u)$ such that*

$$\mathcal{J}(\nu) = \int_0^1 g\left(t, \frac{d\nu_a(t)}{d\mu}\right) d\mu + \int_0^1 g^\infty\left(t, \frac{d\nu_s(t)}{d|\nu_s|}\right) d|\nu_s|(t).$$

Here $\nu_a$ and $\nu_s$ are absolutely continuous and the singular parts of $\nu$ with respect to $\mu$.

We recall that $g(t, u)$ is a *normal convex integrand* if it is a normal integrand and a convex function of $u$. It is also worth noting that the assumption that $J_n$ $\Gamma$-converge is not very restrictive as (thanks to the fact that bounded sets in $\mathcal{M}^d$ are metrizable in the weak-star topology) the restriction of the the functionals to any bounded set contains a $\Gamma$-converging subsequence.

There is one subtle difference between the relaxation theorem (Theorem 1) and Theorem 4: while the first offers, though non-constructive, description of the limiting integrand $g(t, u)$, the second is a pure existence theorem which does not give any indication of how the integrand of the $\Gamma$-limit can be found.

## 6. Sequences: the general case

We have already mentioned in the Introduction that the $\Gamma$-limit of the function-problems. We shall now describe another type of limiting behavior of the functionals $I_n$, which coincides with the $\Gamma$-convergence in the weak-star topology of the associated functionals $J_n$ if the sequence is uniformly coercive, and always generates convergence of the value functions of the problems. The proofs of the results will appear in the forthcoming paper by Ioffe and Freddi (2002).

We need some additional notation and definitions. The words "open interval" will be used to refer to subintervals of $[0,1]$ which are open in $[0,1]$, that is having one of the following forms: $[0,1]$, $[0,\alpha)$, $(\beta,1]$, $(\alpha,\beta)$, where $0<\alpha<\beta<1$. Set

$$
\begin{aligned}
V_n(x) &= \inf\{I_n(u(\cdot)) : \int_0^1 u(t)\,dt = x\};\\
I_n(\Delta, u(\cdot)) &= \int_\Delta f_n(t,u(t))\,dt;\\
V_n(\Delta, x) &= \inf\{I_n(u(\cdot)) : \int_\Delta u(t)\,dt = x\}.
\end{aligned}
$$

If $\pi$ is a partition of $[0,1]$ by points $0<\tau_1<\ldots<\tau_k<1$, then we say that the interval $\Delta$ *belongs* to $\pi$ if $\Delta$ is either $[0,\tau_i)$, or $(\tau_i,1]$, or $(\tau_i,\tau_j)$, $1\le i<j\le 1$. The *diameter* of $\pi$ is $\max_{0\le i\le k}(\tau_{i+1}-\tau_i)$, where we set $\tau_0=0$, $\tau_{k+1}=1$. A sequence $(\pi_m)$ of partitions *decreases* if every $\Delta\in\pi_m$ belongs to $\pi_{m+1}$. Finally, given a positive Radon measure $\mu$ on $[0,1]$, we say that a collection $\mathcal{D}$ of open intervals is $\mu$-dense if for every $\varepsilon>0$ and every open interval $\Delta$ there is a $\Delta'\in\mathcal{D}$ such that $\Delta'\subset\Delta$ and $\mu(\Delta\backslash\Delta')<\varepsilon$.

THEOREM 5 *Let $(f_n)$ be a sequence of integrands satisfying $(\mathbf{A}_1)$, $(\mathbf{A}_2)$ and $(\mathbf{A}_4)$. Then there are a probability measure $\mu$ on $[0,1]$, a normal convex integrand $g(t,u)$ on $[0,1]\times\mathbb{R}^d$, a lower semi-continuous convex-valued mapping $P(t)$ and a subsequence $n_j$ of indices such that for the functionals*

$$
\begin{aligned}
\mathcal{H}(\nu) &= \int_0^1 g\left(t,\frac{d\nu_a}{d\mu}\right)d\mu + \int_0^1 s\left(t,\frac{d\nu_s}{d|\nu_s|}\right)d|\nu_s|;\\
\mathcal{H}(\Delta,\nu) &= \int_\Delta g\left(t,\frac{d\nu_a}{d\mu}\right)d\mu + \int_\Delta s\left(t,\frac{d\nu_s}{d|\nu_s|}\right)d|\nu_s|;
\end{aligned}
$$

*and the value functions of the corresponding variational problems*

$$
\begin{aligned}
V(x) &= \inf\left\{\mathcal{H}(\nu) : \int_0^1 d\nu = x\right\};\\
V(\Delta,x) &= \inf\left\{\mathcal{H}(\Delta,\nu) : \int_\Delta d\nu = x\right\}
\end{aligned}
$$

*(a) the value functions $V_{n_j}(\Delta,\cdot)$ $\Gamma$-converge to $V^{**}(\Delta,\cdot)$ for every $\Delta$ of a $\mu$-dense collection of open subintervals of $[0,1]$;*

*(b) if a sequence $(\nu_j)\subset\mathcal{M}^d$ converges to $\nu$ in the weak-star topology, then*

$\liminf I_{n_j}(\nu_j)\ge\mathcal{H}(\nu)$;

*(c) for any $\nu \in \mathcal{M}^d$ there is a decreasing sequence $(\pi_m)$ of partitions of $[0,1]$, with diameters going to zero, by points which are not atoms of either $\mu$ or $\nu$ and a sequence $(\nu_j) \subset \mathcal{M}^d$ such that*

$$\limsup_{j\to\infty} J_{n_j}(\nu_j) \le \mathcal{H}(\nu);$$

*and for any $\Delta$ belonging to one of the partitions, the sequence $V_{n_j}(\Delta,\cdot)$ $\Gamma$-converge to $V^{**}(\Delta,\cdot)$ and*

$$\lim_{j\to\infty} \int_\Delta d\nu_j = \int_\Delta d\nu.$$

Here, as in the preceding section, $s(t,w)$ stands for the support function of $P(t)$ and as in Theorem 4, $\nu_a$ and $\nu_s$ are absolutely continuous and the singular parts of $\nu$ with respect to $\mu$.

This is the condition (c) that does not allow the type of convergence of $\nu_j$ considered in the theorem to reach up to the real $\Gamma$-convergence. However, if the original sequence of the functionals is uniformly coercive, in fact under a somewhat weaker condition in the spirit of Theorem 3, (c) reduces to the supremum inequality in the definition of the $\Gamma$-convergence. Moreover the following convergence and existence theorem holds true.

THEOREM 6 *Assume in addition that there are a $\bar{q} \in \mathbb{R}^d$, an $r > 0$ and a sequence $(\rho_n)$ of nonnegative functions with uniformly bounded integrals such that for every $n$*

$$f_n(t,u) \ge (q|u) - \rho_n(t) \quad a.e.,$$

*provided $\|q - \bar{q}\| \le r$. Then*

*(a) the conclusion of Theorem 5 holds with $\mathcal{H}$ being actually the $\Gamma$-limit of $J_{n_j}$ in the weak-star topology;*

*(b) for every $x \in \mathrm{ri}(\mathrm{dom}\, V)$ the problem*

$$\text{minimize} \quad \mathcal{H}(\nu), \ \text{s.t.} \int_0^1 d\nu = x$$

*has a solution belonging to $\mathcal{K}$ with jumps at points which are points of continuity of $\mu$.*

## 7. Constructions

I conclude by adding a brief description of the constructions of the objects whose existence is stated in Theorem 5.

**7.1.** Denote by $\mathcal{A}$ the collection of all open subsets of $[0,1]$ (open with respect to $[0,1]$!). Let us call an extended-real-valued function $S(p,E)$ on $\mathbb{R}^d \times \mathcal{A}$ a

(i) $S(\cdot, E)$ is convex lower semicontinuous for each $E \in \mathcal{A}$;

(ii) for every $p \in \mathcal{A}$, $S(p, \cdot)$ can be extended to a (possibly unbounded) positive Borel measure on $[0, 1]$. The following result (relating to so called $\overline{\Gamma}$-convergence) can be extracted from Dal Maso (1993).

THEOREM 7 *Let $(S_m)$ be a sequence of convex measures uniformly bounded from below. Then, there is a subsequence $(S_{n_j})$ converging to a certain convex measure $S$ in the following sense: whenever $E(t)$ is a strictly increasing family of open subsets of $[0, 1]$, the functions $S_{n_j}(\cdot, E(t))$ $\Gamma$-converge to $S(\cdot, E(t))$ for every $t$ with possible exception of countably many of them.*

At the first step we apply this theorem to $S_n(p, \Delta) = V_n^*(p, \Delta)$. Let $S(p, \Delta)$ be a limiting convex measure and $(n_j)$ the corresponding subsequence of indices.

**7.2.** Let $\sigma$ be a Borel measure on $[0, 1]$ bounded from below, e.g. nonnegative. We shall say (see Definition 1) that a $t \in [0, 1]$ is an *ordinary point* for $\sigma$ if $\sigma(\Delta) < \infty$ for some open interval $\Delta$ containing $t$. We denote the collection of all such points by $\mathrm{Od}(\sigma)$. This set is obviously open, that is-at most a countable union of disjoint open intervals $\Delta_i$ and this is an easy matter to show that there is a probability measure $\lambda$ on $\mathrm{Od}(\sigma)$ such that the restriction of $\sigma$ to $\mathrm{Od}(\sigma)$ is locally absolutely continuous with respect to $\lambda$. Such a $\lambda$ can, for instance, be defined as follows:

$$\lambda(E) = \sum_{i=1}^{\infty} 2^{-i} \frac{\sigma(E \cap \Delta_i)}{|\sigma|(\Delta_i)} \tag{5}$$

(the sum being taken over those $i$ for which $|\sigma|(\Delta_i) > 0$).

**7.3.** We now define $P(t)$ as the collection of all $p \in \mathbb{R}^d$ such that $t$ is an ordinary point for $S(p, \cdot)$. As all $f_n$ are nonnegative, all sets $P(t)$ contain zero, hence are nonempty. Convexity of $S(\cdot, \Delta)$ implies that $P(t)$ are convex sets.

Unlike $P(t)$ the measure $\mu$ is not uniquely defined. In specific cases, when an explicit expression for $S(p, \Delta)$ is available, $\mu$ can be obtained from it. A possible general scheme of obtaining a suitable $\mu$ is the following. Let $\mathcal{D}$ be the collection of open intervals with rational end points (including zero and one). For any $\Delta \in \mathcal{D}$ choose a dense countable subset of $\operatorname{dom} S(\cdot, \Delta)$ and let $\Pi$ be the union of all such points (over all $\Delta \in \mathcal{D}$). Let finally, $\lambda_p$ be the probability measure defined by (5) for $\sigma = S(p, \cdot)$. Then, $\mu$ is defined by

$$\mu(E) = (1/2) \sum_{p \in \Pi} (\lambda_p(E) + \mathrm{meas}(E)),$$

where $\mathrm{meas}(E)$ stands for the Lebesgue measure of $E$.

Finally the normal convex integrand $g(t, u)$ is defined in three steps as folto $\mu$ on $\mathrm{Od}(S(p,\cdot))$ for every $p \in \Pi$ and define for such $p$ the function $\varphi(t,p)$ as the density of $S(p,\cdot)$ with respect to $\mu$ if $t \in \mathrm{Od}\,(S(p,\cdot))$ and infinity otherwise. Then we set

$$\psi(t,p) = \inf\Big\{\sum_{i=1}^{d+1}\alpha_i\varphi(t,p_i) : p_i \in \Pi,\ \sum\alpha_i = 1,\ \sum\alpha_i p_i = p\Big\}.$$

and finally define $g$ as the Fenchel conjugate of $\psi$ with respect to the second argument:

$$g(t,u) = \sup_p((p|u) - \psi(t,p)).$$

We finally note that although $\mu$ is not uniquely defined, as soon as it is chosen, the integrand $g$ is fully determined.

## 7.4. Example

Consider the integrands (defined on $[0,1]\times R$)

$$f_n(t,x) = \begin{cases} 0, & \text{if } 0 \le t \le n^{-1}, \\ (a_n(t)/2)|x|^2, & \text{if } n^{-1} < t \le 1. \end{cases}$$

Assume that $a_n(t) > 0$ for all $n$ and $t$. Then

$$f_n^*(t,p) = \begin{cases} \delta_{\{0\}}, & \text{if } 0 \le t \le n^{-1}, \\ (2a_n(t))^{-1}|p|^2, & \text{if } n^{-1} < t \le 1, \end{cases}$$

where $\delta_{\{0\}}$ is the indicator of zero, that is the function equal zero at zero and infinity outside. Assuming that $b_n(t) = (a_n(t))^{-1}/2$ are summable and converge weakly in $L_1$ to some $b(t)$, we get

$$S_n(p,\Delta) = \begin{cases} \dfrac{|p|^2}{4}\displaystyle\int_\Delta b_n(t)\,dt, & \text{if } [0,n^{-1}]\cap\Delta = \emptyset \text{ or } p = 0, \\ \infty, & \text{if } [0,n^{-1}]\cap\Delta \ne \emptyset \text{ and } p \ne 0. \end{cases}$$

It is an easy matter to see that $S_n(\cdot,\Delta)$ $\Gamma$-converge to

$$S(p,\Delta) = \begin{cases} \dfrac{|p|^2}{4}\displaystyle\int_\Delta b(t)\,dt, & \text{if } 0 \notin \Delta \text{ or } p = 0, \\ \infty, & \text{if } 0 \in \Delta \text{ and } p \ne 0. \end{cases}$$

(Recall that $0 \notin \Delta$ means that either $\Delta = (\alpha,\beta)$ or $\Delta = (\alpha,1]$ with $\alpha > 0$ in either case.)

Therefore

$$P(t) = \begin{cases} \{0\}, & \text{if } t = 0,\end{cases}$$

Finally

$$s(t,w) = \begin{cases} 0, & \text{if } t = 0, \\ \delta_{\{0\}}, & \text{if } t \neq 0, \end{cases}$$

so that $\nu \in \operatorname{dom} \mathcal{H}$ only if the singular part of $\nu$ is a Dirac mesure at zero: $\nu_s = \lambda\varepsilon_0$ and therefore $d\nu = u(t)dt + \lambda\varepsilon_0$ for some $u(\cdot) \in L_1$. Thus

$$\mathcal{H}(\nu) = \int_0^1 b(t)|u(t)|^2\,dt.$$

In particular, $V(x) \equiv 0$ and for any $x \in R$ the measure solving the problem of minimizing $\mathcal{H}(\nu)$ subject to the condition $\int d\nu = x$ is $\bar{\nu} = x\varepsilon_0$.

As to the $\mathcal{H}_n$ and the corresponding variational problems, it is quite clear that all $V_n$ are identically equal to zero and the solutions of the problems

$$u_n(t) = \begin{cases} x/n, & \text{if } 0 \leq t \leq n^{-1}, \\ 0, & \text{if } n^{-1} < t \leq 1 \end{cases}$$

converge (weak-star in the space of measures) to $\bar{\nu}$.

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