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Control problems for systems described by hemivariational inequalities

by

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Abstract: The paper considers some control problems for the systems described by the evolution, as well as the stationary hemivariational inequalities (HVIs for short). First, basing on surjectivity theorems for pseudo-monotone operators we formulate some existence results for the solutions of the HVIs and investigate some properties of the solution set (like sensitivity; i.e. its dependence on data and operators). Next we quote some existence theorems for optimal solutions for various classes of optimal control like distributed control (e.g. Bolza problem), identification of parameters, or optimal shape design for systems described by HVIs. Finally, we discuss some common features in getting the existence of optimal solutions as well as some "well-posedness" problems.

Keywords: hemivariational inequalities, control problems (Bolza problem, identification of parameters, optimal shape design), well-posedness, sensitivity, Clarke subdifferential, multifunctions, pseudo-monotone operators, G and Γ convergence.

1. Introduction

Hemivariational inequalities are used to model such physical and engineering problems, in which nonconvex, nonmonotone, possibly multivalued laws appear; e.g. relation linking stress and strain or reactions and displacements in elasticity theory, flux and temperature in thermic problems, generalized forces and velocities in dynamic problems, and so on. The HVIs introduced in the 1980s by Panagiotopoulos can be considered as generalizations of partial differential equations (PDE's) and variational inequalities (VIs) (see e.g. Panagiotopoulos, 1993, Naniewicz and Panagiotopoulos, 1995).

Thus, similarly as in the case of PDE's and VIs (see e.g. Lions 1971, Tiba, 1990), it is quite natural to consider various types of control problems for systems described also by HVIs (see papers by Haslinger and Panagiotopoulos,

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1989, Miettinen and Haslinger, 1992, Haslinger and Panagiotopoulos, 1995, Denkowski and Migórski, 1998 A, B, Gasiński, 1998, Migórski and Ochal, 2000 a,b, Gasiński, 2000, Ochal, 2000,...).

The aim of this paper is to give a perspective for the existence results concerning the solutions (the nonemptyness of solution sets) of HVIs as well as the optimal solutions to various control problems for such systems. We find some common features as well as differences in the theory of stationary (let's say "elliptic") and evolution of first and second order (let's say "parabolic" and "hyperbolic", respectively) HVIs. For instance, the approach to the existence theory for HVIs (we consider) is based on surjectivity theorems for pseudomonotone operators, while in all considered optimal control problems the existence of optimal solutions is obtained by applying the direct method of calculus of variations. In the latter case the crucial problem (see Remark 5.1) is to find such topologies in the set of controls and in the space of solutions as to implement simultaneously two needed properties: the upper semicontinuity property (usc—for short) in the Kuratowski sense of the solution set as the function of control, and the lower semicontinuity of the cost functional (lsc—for short).

We also discuss some aspects of well-posedness for such kind of problems indicating some possibilities of getting results concerning the "upper semicontinuity" - dependence of the solution set or of optimal pairs (state - control) on the data or even on state relations (i.e. when the HVIs themselves are perturbed). In this part we use notions of G and PG convergence defined and developed for linear operators by Spagnolo (1967), De Giorgi and Spagnolo (1973), Colombini and Spagnolo (1977), and then generalized for monotone operators by Dal Maso, Defranceschi,... (see e.g. Chiado'Piat et al., 1990).

The paper is organized as follows. Before preliminaries of Section 3 we give in Section 2 a motivation (an example of controlled elastic beam with adhesive support and figures representing nonsmooth, nonmonotone, multivalued laws). Next, in Section 4 we formulate three types (elliptic, parabolic and hyperbolic) of HVIs and quote theorems assuring nonemptyness of their solution sets. In Section 5 we present some existence theorems for various kinds of optimal control for systems governed by HVIs mainly of hyperbolic type (similar results can be obtained for elliptic and parabolic cases). Finally, in Section 6 we formulate the concluding remarks concerning some well-posedness problems (e.g. the dependence on data or other perturbations of HVIs; i.e. we mention some results on asymptotic behaviour of the solution sets to parametrized HVIs as the parameter changes). The precise theorems on sensitivity of optimal solutions will be proven in the forthcoming paper.

2. A motivation

2.1. Beam in adhesive contact

Assume an elastic (obeing linear Hooke's law) beam, fixed at the ends $x =$

position and velocity at the initial time $t = 0$ be given functions $y_0(x)$ and $y_1(x)$, respectively.

Then, for small displacements $y(t,x)$ the beam can be modeled (see e.g. Banks et al., 1996, Panagiotopoulos and Pop, 1999) by the following mixed (initial-boundary value) problem for the hyperbolic PDE

$$\left\{ \begin{array}{l} \varrho \dfrac{\partial^2 y}{\partial t^2} + EI \dfrac{\partial^4 y}{\partial x^4} + c_D I \dfrac{\partial}{\partial t} \dfrac{\partial^4 y}{\partial x^4} = f_1 + f_2, \quad \text{for } (t,x) \in (0,T) \times (0,l), \\ y(0,x) = y_0(x), \quad \dfrac{\partial y}{\partial t}(0,x) = y_1(x) \quad \text{for all } x \in (0,l), \\ y(t,0) = y(t,l) = 0, \quad \dfrac{\partial^2 y}{\partial x^2}(t,0) = \dfrac{\partial^2 y}{\partial x^2}(t,l) = 0 \quad \text{for all } t \in (0,T), \end{array} \right. \tag{1}$$

coupled with the additional condition

$$-f_1(t,x) \in \partial j(y(t,x)) \quad \text{for } (t,x) \in (0,T) \times (0,l), \tag{2}$$

which describes the action of a gluing material on the beam, f_1 being the reaction force per unit length and ∂j denoting the Clarke subdifferential, see Section 3, of a locally Lipschitz function $j : \mathbb{R} \longrightarrow \mathbb{R}$); ϱ denotes linear mass density, while E, I, c_D stand, respectively, for the modulus of elasticity, the moment of inertia and the damping coefficient of the beam. The function f_2 is supposed to be more regular and completely known (e.g. gravity forces or forces generated by a controller).

Thus, relation (2) represents a physical law (here a link between reactions and displacements), which in general may have a nonmonotone, nondifferentiable, possibly multivalued character. It generalizes the case where j is differentiable ($\partial j = \nabla j$; i.e. j is a potential for the force field $-f_1$), or convex (for superpotential j its subdifferential ∂j is understood in the sense of convex analysis – see Moreau, 1968, Rockafellar, 1970).

By introducing the space $V = H^2(0,l) \cap H_0^1(0,l)$ with the inner product

$$\langle w, z \rangle = \int_0^l \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 z}{\partial x^2}\, dx.$$

and defining the operators $A, B : V \longrightarrow V^*$ (V^* being the topological dual of V) as

$$\langle Aw, z \rangle = \frac{c_D\, I}{\varrho} \langle w, z \rangle \quad \text{and} \quad \langle Bw, z \rangle = \frac{E\, I}{\varrho} \langle w, z \rangle,$$

we can reformulate the problem (1)(2) as the following differential inclusion (generalizing (PDE) as well as (VI)), also called hemivariational inequality:

(HVI)

$$\left\{ \begin{array}{l} \dfrac{\partial^2 y(t)}{\partial t^2} + A \dfrac{\partial y(t)}{\partial t} + By(t) + \dfrac{1}{\varrho}\, \partial j(y(t)) \ni \dfrac{1}{\varrho}\, f_2(t) \quad \text{for a.e. } t \in (0,T) \\ \ldots \end{array} \right.$$

The function f_2 appearing in (1) and in (HVI) represents the prescribed loading and it can have the form $f_2 = Cu$, the operator C being a controller acting on the control functions from an admissible set ($u \in \mathcal{U}_{ad} \subset \mathcal{U}$).

So the solution set for (HVI) depends on u and will be denoted $S_{(HVI)}(u)$.

Thus, we can formulate the control problem:

$$(\mathrm{CP})_{\mathrm{HVI}} \qquad \begin{cases} \text{Given a cost functional } F : \mathcal{U} \times \mathcal{Y} \longrightarrow \mathbb{R}, \\ \text{find } u^* \in \mathcal{U}_{ad} \text{ and } y^* \in S(u^*) \text{ such that} \\ F(u^*, y^*) \leq F(u, y) \quad \text{for all } u \in \mathcal{U}_{ad},\ y \in S(u), \end{cases}$$

when specifying the cost functional we can cover significant problems from engineering like e.g. the least deviation from a desired state at final time T, or the minimal energy of the control, and so on.

2.2. Examples of some nondifferentiable, nonmonotone, multivalued laws

We will now present three figures (see e.g. Naniewicz and Panagiotopoulos, 1995, Duvaut and Lions, 1976). The first two represent some adhesive laws, where the force f depends on the displacement y. The third figure represents the flux of heat $(\frac{\partial y}{\partial \vec{n}})$ as a function of the temperature y in the problem of regulating the temperature in some region Ω ($\vec{n} = \vec{n}(x)$ being the outward normal at point x of its boundary) to deviate as little as possible from the given interval $[h_1, h_2]$.

Fig. 1

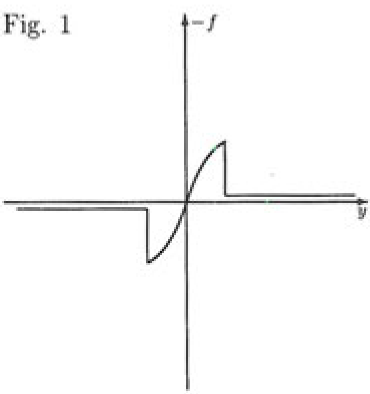


Fig. 2

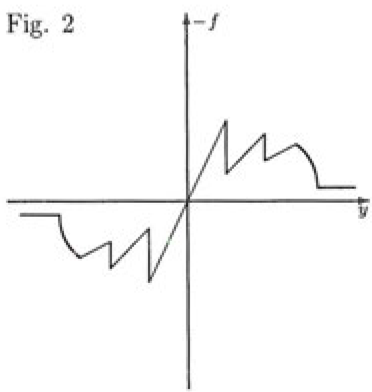
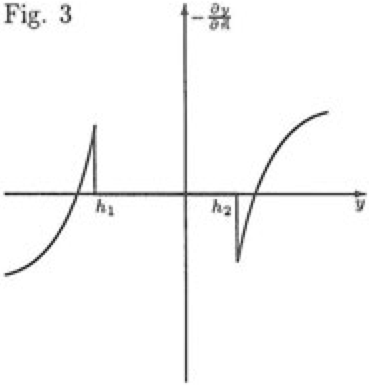


Fig. 3



3. Preliminaries

3.1. The Clarke subdifferential

Given a locally Lipschitz function $J : Z \longrightarrow \mathbb{R}$, where Z is a Banach space , we admit definitions (see Clarke, 1983):

(i) The generalized (in the sense of Clarke) directional derivative of J at point $u \in Z$ in the direction $v \in Z$, is defined by

$$J^0(u;v) = \limsup_{y \to u \ \lambda \searrow 0} \frac{J(y+\lambda v) - J(y)}{\lambda}.$$

(ii) The generalized gradient of J at u is the subset of the dual space Z^* given by

$$\partial J(u) = \{\zeta \in Z^* : \langle \zeta, v\rangle_{Z^* \times Z} \leq J^0(u;v) \quad \text{for all } v \in Z\}.$$

An important class of functionals is provided by the example below.

EXAMPLE 3.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary and set $Z = L^p(\Omega;\mathbb{R}^m)$ $(2 \leq p < +\infty$, m being a fixed natural number).*

We consider a function $j : (0,T) \times \Omega \times \mathbb{R}^m \longrightarrow \mathbb{R}$ which satisfies the following hypothesis

H(j): *$j : (0,T) \times \Omega \times \mathbb{R}^m \longrightarrow \mathbb{R}$ is a function such that for all $t \in (0,T)$ we have:*

(i) *$j(t,\cdot,v) : \Omega \longmapsto \mathbb{R}$ is measurable for all $v \in \mathbb{R}^m$ and $j(t,\cdot,0) \in L^1(\Omega)$;*

(ii) *$j(t,x,\cdot) : \mathbb{R}^m \longmapsto \mathbb{R}$ is locally Lipschitz for all $x \in \Omega$;*

(iii) *for each $x \in \Omega$ and $v \in \mathbb{R}^m$ if $\zeta \in \partial_v j(t,x,v)$, then $\|\zeta\|_{\mathbb{R}^m} \leq c(1 +$ $\|v\|^{\frac{2}{q}}$) for some $c > 0$*

Now we define the functional $J : (0,T) \times L^p(\Omega;\mathbb{R}^m) \longrightarrow \mathbb{R}$ *as*

$$J(t,v) = \int_\Omega j(t,x,v(x))\,dx. \tag{3}$$

From the theorem by Aubin and Clarke (1979), it follows that for every t the functional $J(t,\cdot)$ is well defined, Lipschitz continuous on every bounded subset of Z and for every $v \in Z = L^p(\Omega;\mathbb{R}^m)$ we have:

$$\zeta \in \partial J(t,v) \implies \zeta(x) \in \partial_v j(t,x,v(x)) \text{ for a.e. } x \in \Omega.$$

The inverse implication also holds provided j is regular (i.e. it has all the directional derivatives equal to the Clarke directional derivatives, see Naniewicz and Panagiotopoulos, 1995).

Hence and from the Hölder inequality we easily obtain that J, given by (3) with j such that **H(j)** holds, satisfies the following hypothesis:

H(J): $J : (0,T) \times Z \longrightarrow \mathbb{R}$ is a function such that

- **(i)** for each $v \in Z$, the map $t \longmapsto J(t,v)$ is measurable on $(0,T)$;
- **(ii)** for each $t \in (0,T)$ the function $v \longmapsto J(t,v)$ is locally Lipschitz on Z;
- **(iii)** the following growth condition holds: there exists a constant $\overline{c} > 0$ such that for any $v \in Z$ and $t \in (0,T)$ we have

$$\zeta \in \partial J(t,v) \implies \|\zeta\|_{Z^*} \le \overline{c}(1 + \|v\|_Z^{\frac{2}{q}}).$$

3.2. Multivalued operators

We start with basic definitions for multivalued operators and then we quote two main surjectivity results for the operator classes under consideration (see e.g. Browder and Hess, 1972, Naniewicz and Panagiotopoulos, 1995, Papageorgiou et al., 1999).

Let Y be a real reflexive Banach space and Y^* be its dual space and let $T : Y \longrightarrow 2^{Y^*}$ be a multivalued operator. By $\mathcal{R}(T) = \bigcup_{y\in Y} Ty$ we denote the range of T.

We say that T is:

(1) bounded if the set $T(C)$ is bounded in Y^* for any bounded subset $C \subseteq Y$,

(2) coercive if there exists a function $c : \mathbb{R}_+ \longrightarrow \mathbb{R}$ with $\lim_{r\to+\infty} c(r) = +\infty$ such that for all $y \in Y$ and $y^* \in Ty$, we have $\langle y, y^*\rangle \ge c(\|y\|)\|y\|$,

(3) upper semicontinuous if for any closed subset $C \subseteq Y^*$ the set $T^-(C) = \{y \in Y \ : \ Ty \cap C \ne \emptyset\}$ is closed in Y,

(4) monotone if for every $y, z \in T$, $y^* \in Ty$, $z^* \in Tz$, we have

(5) **maximal monotone** if
 a) T is monotone,
 b) for any $y \in Y$, $y^* \in Y^*$ such that $\langle y^* - z^*, y - z\rangle \geq 0$ for all $z \in Y$, $z^* \in Tz$, we have $y^* \in Ty$,

(6) **pseudomonotone** if the following conditions hold:
 a) the set Ty is nonempty, bounded, closed and convex for each $y \in Y$,
 b) T is upper semicontinuous from each finite-dimensional subspace of Y to Y^* furnished with the weak topology,
 c) if $\{y_n\} \subseteq Y, y_n \to y$ weakly in Y, $y_n^* \in Ty_n$, and $\limsup_{n\to+\infty}\langle y_n^*, y_n - y\rangle \leq 0$, then for each element $v \in Y$ there exists $y^*(v) \in Ty$ such that $\liminf_{n\to+\infty}\langle y_n^*, y_n - y\rangle \geq \langle y^*(v), y - v\rangle$,

(7) **generalized pseudomonotone** if the conditions $\{y_n\} \subseteq Y$, $y_n \longrightarrow y$ weakly in Y, $y_n^* \in Ty_n$, $y_n^* \longrightarrow y^*$ weakly in Y^* and $\limsup_{n\to+\infty}\langle y_n^*, y_n\rangle \leq \langle y^*, y\rangle$ imply $y^* \in Ty$ and $\lim_{n\to+\infty}\langle y_n^*, y_n\rangle = \langle y^*, y\rangle$.

 Now, let $L : Y \supseteq D(L) \to Y^*$ be linear, densely defined, maximal monotone operator.

(8) T is **L-generalized pseudomonotone** if the following conditions hold:
 a) for every $y \in Y$, Ty is a nonempty, convex and weakly compact subset of Y^*,
 b) T is upper semicontinuous from each finite-dimensional subspace of Y into Y^* equipped with the weak topology,
 c) if $\{y_n\} \subseteq D(L)$, $y_n \longrightarrow y$ weakly in Y, $y \in D(L)$, $Ly_n \longrightarrow Ly$ weakly in Y^*, $y_n^* \in Ty_n$, $y_n^* \longrightarrow y^*$ weakly in Y^* and $\limsup_{n\to+\infty}\langle y_n^*, y_n - y\rangle \leq \langle y^*, y\rangle$, then $y^* \in Ty$ and $\langle y_n^*, y_n\rangle \longrightarrow \langle y^*, y\rangle$.

The crucial point in the proofs of the existence of a solution to the hemivariational inequalities considered below are the following surjectivity results.

PROPOSITION 3.1 *If Y is a reflexive Banach space, and $T : Y \longrightarrow 2^{Y^*} \setminus \{\emptyset\}$ is a pseudomonotone coercive operator, then $\mathcal{R}(T) = Y^*$.*

PROPOSITION 3.2 *If Y is a reflexive, strictly convex Banach space, $L : Y \supseteq D(L) \to Y^*$ is a linear, densely defined, maximal monotone operator and $T : Y \longrightarrow 2^{Y^*} \setminus \{\emptyset\}$ is a bounded, coercive and L-generalized pseudomonotone operator, then $\mathcal{R}(L + T) = Y^*$.*

The proof of Proposition 3.1 can be found in Browder and Hess (1972), Theorem 3, p. 269 while the proof of Proposition 3.2 can be found in Papageorgiou et al. (1999), Theorem 2.1, p. 345.

3.3. Functional spaces

For the stationary HVIs we consider an evolution fivefold of spaces where V and Z are two reflexive, separable Banach spaces, H is a Hilbert space and H^*, Z^* and V^* denote dual spaces to H, X and V, respectively.

Assume that all embeddings are dense and continuous, and V embeds compactly in Z. As usual, we identify H with its dual.

We denote by $\langle\cdot,\cdot\rangle$ the duality of V and V^* and the pairing between Z and Z^* as well, by $\|\cdot\|_E$ the norm in the space E being respectively V, Z, Z^* or V^*, and by $|\cdot|_H$ the norm in H. Moreover, the symbol $w-E$ stands for the space E equipped with the weak topology.

EXAMPLE 3.2 *To have an example of such a situation, let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. We admit $V = W^{1,p}(\Omega;\mathbb{R}^m)$ (for the Neumann type boundary problems) or $V = W_0^{1,p}(\Omega;\mathbb{R}^m)$ (for the Dirichlet type problems), $Z = L^p(\Omega;\mathbb{R}^m)$ and $H = L^2(\Omega;\mathbb{R}^m)$ with some $2 \leq p < \infty$. From the Sobolev embedding theorem (see, e.g. Zeidler, 1990, p. 1026), we get that (V, Z, H, Z^*, V^*) is an evolution fivefold.*

For the case of evolution HVIs we need spaces of functions which depend also on time variable. Thus, given a fixed number $0 < T < +\infty$ for some p, $2 \leq p < \infty, 1/p + 1/q = 1$, we introduce the following function spaces:

$$\mathcal{V} = L^p(0,T;V),$$
$$\mathcal{Z} = L^p(0,T;Z),$$
$$\mathcal{H} = L^2(0,T;H),$$
$$\mathcal{Z}^* = L^q(0,T;Z^*),$$
$$\mathcal{V}^* = L^q(0,T;V^*),$$
$$\mathcal{W} = \{w \in \mathcal{V} : w' \in \mathcal{V}^*\},$$

(where the time derivative is understood in the sense of vector valued distributions). The latter is a separable, reflexive Banach space with the norm $\|w\|_{\mathcal{W}} = \|w\|_{\mathcal{V}} + \|w'\|_{\mathcal{V}^*}$ (see Zeidler, 1990, Proposition 23.7(c), p.411 and Proposition 23.23(i), pp. 422–423).

Clearly we have

$$\mathcal{W} \subset \mathcal{V} \subset \mathcal{Z} \subset \mathcal{H} \subset \mathcal{Z}^* \subset \mathcal{V}^*$$

with dense and continuous embeddings.

Since we assumed that $V \subset Z$ compactly, we have also that $\mathcal{W} \subset \mathcal{Z}$ compactly (see Lions, 1969, Theorem 5.1, p. 58). Moreover, the embedding $\mathcal{W} \subset C(0,T;H)$ is continuous (see Zeidler, 1990, Proposition 23.23(ii), p. 422). So every equivalence class in $\mathcal{W}$ has a unique representative in $C(0,T;H)$.

The pairing of $\mathcal{V}$ and $\mathcal{V}^*$ and also the duality between $\mathcal{Z}$ and $\mathcal{Z}^*$ are denoted by $\langle\langle f, g\rangle\rangle = \int_0^T \langle f(t), g(t)\rangle\, dt$.

As the spaces of solutions for the parabolic and hyperbolic HVIs below we admit, respectively:

The latter is also a separable, reflexive Banach space with the norm defined by

$$\|w\|_{\mathcal{Y}_h} = \|w\|_{\mathcal{V}} + \|w'\|_{\mathcal{W}}.$$

Moreover, each function $w \in \mathcal{Y}_h$ is an absolutely continuous function from $[0,T]$ to V and its distributional derivative w' possesses a representant which is continuous function from $[0,T]$ to H. Similarly, each element of $\mathcal{Y}_p$ after a change of values on a set of measure zero can be considered as an element of $C(0,T;H)$ (see e.g. Lions, 1971).

4. Hemivariational inequalities

4.1. Stationary (elliptic) hemivariational inequalities

Let V, Z, H, Z^*, V^* be as in Subsection 3.3 and assume we are given a (nonlinear) operator $A : V \longmapsto V^*$, a locally Lipschitz functional $J : Z \longmapsto \mathbb{R}$ and an element $f \in V^*$.

We consider the differential inclusion

$$(DI_e) \qquad \begin{cases} \text{Find } y \in V \text{ such that} \\ Ay + \partial J(y) \ni f \end{cases}$$

which, when explicited into the weak form, is equivalent to the following elliptic hemivariational inequality

$$(HVI_e)$$
$$\begin{cases} \text{Find } y \in V \text{ such that} \\ \langle Ay, v-y\rangle_{V^*\times V} + J^0(y; v-y) \geq \langle f, v-y\rangle_{V^*\times V} \quad \text{for all } v \in V, \end{cases}$$

This, in turn, can be written down as

$$(HVI_e^1) \quad \begin{cases} \text{Find } y \in V \text{ such that there is } \zeta \in \partial J(y) \subseteq Z^* \text{ and} \\ \langle Ay, v\rangle_{V^*\times V} + \langle \zeta, v\rangle_{V^*\times V} = \langle f, v\rangle_{V^*\times V} \quad \text{for all } v \in V, \end{cases}$$

In the case J is given by the integral formula

$$J(y) = \int_\Omega j(y(x))\,dx$$

with j satisfying hypothesis H(j) (omitting the dependence on t) the last HVI is equivalent to the following

$$(HVI_e^2)$$
$$\begin{cases} \text{Find } y \in V \text{ such that there is } \zeta(x) \in \partial j(y(x)) \text{ a.e. in } \Omega \\ \langle Ay, v\rangle_{V^*\times V} + \int_\Omega (\zeta(x), v(x))_{\mathbb{R}^m}\,dx = \langle f, v\rangle_{V^*\times V} \quad \text{for all } v \in V, \end{cases}$$

THEOREM 4.1 *If A is pseudomonotone, coercive and bounded, J satisfies hypothesis H(J) (omitting the dependence on t), then the solution set $S_{(HVI_e)}(f)$ contains at least one element.*

Proof. For the proof one has to show that the operator $\mathcal{T} = A + \partial J$ is pseudomonotone and coercive and next apply the surjectivity result of Proposition 3.1. ■

REMARK 4.1 *In the case A is a maximal monotone operator of the form $Ay = -div\ a(x, Dy(x))$, where $a \in \mathcal{M}_\Omega(\mathbb{R}^N)$ we can obtain a sensitivity result for solution set (depending also on A) $S_{(HVI_e)}(A, f)$ basing on the G-convergence for maximal monotone operators defined in Chiado'Piat et al. (1990).*

THEOREM 4.2 *Assume $a_n, a \in \mathcal{M}_\Omega(\mathbb{R}^N)$, $A_n y = -div\, a_n(x, Dy(x))$ and*
$A_n \longmapsto A$ G-converges; (or $K(w,s) - \lim GrA_n = GrA$)
∂J_n is usc; i.e. $K(s,w) - \limsup Gr\partial J_n \subseteq Gr\partial J$
$f_n \longmapsto f$ in $s - V^*$,
then $K(w) - \limsup S_{(HVI_e)}(A_n, f_n) \subseteq S_{(HVI_e)}(A, f)$

In the above, $K - \limsup Z_n$ (of a sequence of sets) is understood in the sense of Kuratowski (i.e. it is the set of all cluster points of all subsequences of points taken from Z_n), and Gr stands for graph. For the definitions of the class $\mathcal{M}_\Omega(\mathbb{R}^N)$ and the G-convergence see Chiado'Piat et al. (1990). The proof of the theorem follows directly from the definitions, see forthcoming paper.

4.2. First order evolution (parabolic) hemivariational inequalities

Suppose we are in the functional framework of spaces as in Subsection 3.3 and assume we are given an operator $A : (0,T) \times V \longmapsto V^*$, the elements $f \in \mathcal{V}^*$, $y_0 \in H$, and a functional $J : (0,T) \times Z \longmapsto \mathbb{R}$, which is locally Lipschitz with respect to the second variable for almost all $t \in (0,T)$.

Let us consider the following initial value problem for the parabolic differential inclusion:

$$(DI_p) \qquad \begin{cases} \text{Find } y \in \mathcal{Y}_p \text{ such that} \\ y'(t) + A(t)y(t) + \partial J(t, y(t)) \ni f(t) \quad \text{for a.e. } t \in (0,T) \\ y(0) = y_0, \end{cases}$$

which, due to the definition of generalized directional derivative, can also be written down (in the weak form) as

$$(HVI_p) \qquad \begin{cases} \text{Find } y \in \mathcal{Y}_p \text{ such that: for all } v \in V \text{ and a.e. } t \in (0,T) \\ \langle y'(t) + A(t)y(t) - f(t), v\rangle_{V^* \times V} + J^0(t, y(t); v) \geq 0 \end{cases}$$

With the use of a selection of the subdifferential ∂J the last HVI is often written down in the equivalent form:

(HVI_p^1)

$$\begin{cases} \text{Find } y \in \mathcal{Y}_p \text{ and } \zeta \in \mathcal{Z}^* \text{ such that: } \zeta(t) \in \partial J(t, y(t)) \text{ for a.e. } t \in (0,T), \\ \langle y'(t) + A(t)y(t) + \zeta(t) - f(t), v \rangle_{V^* \times V} = 0, \text{ for all } v \in V \text{ and a.e.} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in (0,T) \\ y(0) = y_0. \end{cases}$$

For an existence result we admit that J fulfils $\mathbf{H(J)}$ as in Preliminaries and the operator A satisfies the hypothesis:

<u>**H(A)**</u>: $A : (0,T) \times V \longmapsto V^*$ is an operator such that

- **(i)** for each $v \in V$, the map $t \longmapsto A(t,v) \in V^*$ is measurable on $(0,T)$;
- **(ii)** for each $t \in (0,T)$, the operator $v \longmapsto A(t,v) \in V^*$ is pseudomonotone;
- **(iii)** for almost every $t \in (0,T)$, the operator $A(t,\cdot)$ is bounded, i.e. there exist a nonnegative function $a_1 \in L^q(0,T)$ and a constant $b_1 > 0$ such that

$$\|A(t,v)\|_{V^*} \le a_1(t) + b_1 \|v\|_V^{p-1} \quad \text{for all } v \in V \text{ and a.e. } t \in (0,T);$$

- **(iv)** for almost every $t \in (0,T)$, the operator $A(t,\cdot)$ is coercive, i.e. there are constats $\beta_1 > 0$ and $\beta_2 \ge 0$ and a function $a \in L^1(0,T)$ such that for some $r < p$ we have
$\langle A(t,v), v \rangle_{V^* \times V} \ge \beta_1 \|v\|_V^p - \beta_2 \|v\|_V^r - a(t)$
for all $v \in V$ and a.e. $t \in (0,T)$.

THEOREM 4.3 *Under hypothesis $H(A)$, $H(J)$, for every $f \in \mathcal{V}^*$ and $y_0 \in H$ the solution set $S_{(HVI_p)}(f, y_0)$ contains at least one element.*

Proof. A detailed proof can be found in Migórski (2000). Here we indicate only an outline of the proof which is based on the surjectivity result of Proposition 3.2.

Consider the Nemyckii operator $\mathcal{A}$ corresponding to A (i.e. $(\mathcal{A}v)(t) = A(t)v(t)$), and the operator $\mathcal{N}$ defined for all $v \in \mathcal{V}$ by

$$\mathcal{N}v = \{w \in \mathcal{Z}^* : w(t) \in \partial J(t, v(t)) \text{ a.e. } t \in (0,T)\}.$$

Let modify them (by translating their domain by the initial condition), so that $y \in \mathcal{W}$ is the solution of (HVI_p) if and only if $w = y - y_0 \in \mathcal{W}$ solves the inclusion

$$\begin{cases} f \in w' + \mathcal{A}_1 w + \mathcal{N}_1 w \\ w(0) = 0, \end{cases}$$

where for all $v \in \mathcal{V}$ we have put $\mathcal{A}_1 v = \mathcal{A}(v + y_0)$ and

$$\mathcal{N}_1 v = \mathcal{N}(v + y_0) = \{w \in \mathcal{Z}^* : w(t) \in \partial J(t, (v + y_0)(t)) \text{ a.e. } t \in (0,T)\}$$

Here $v + y_1$ is understood as $(v + y_1)(\cdot) = v(\cdot) + y_1$.

Now observe that operator $L : \mathcal{V} \supseteq D(L) \longrightarrow \mathcal{V}^*$ defined by $Lv = v'$ with $D(L) = \{v \in \mathcal{W} : v(0) = 0\}$ is linear, densely defined and maximal monotone (see, e.g. Zeidler, 1990, Proposition 32.10, p. 855).

Next one can show that the operator $\mathcal{T} : \mathcal{V} \longrightarrow 2^{\mathcal{V}^*}$ defined by the formula $\mathcal{T}v = \mathcal{A}_1 v + \mathcal{N}_1 v$ is coercive, bounded and L-generalized pseudomonotone. Hence, due to Proposition 3.2 the operator $L + \mathcal{T}$ is surjective, which completes the proof. ■

REMARK 4.2 *A sensitivity result (like that of Theorem 4.2) for (HVI_p) with operators $Ay = -div\, a(t, x, Dy(t, \cdot))$, where functions a belong to a special class of single valued maximal monotone operators was obtained by Migórski (see Migórski, 2000) on the basis of PG convergence, which was earlier defined for linear operators by Colombini and Spagnolo (1977).*

4.3. Second order evolution (hyperbolic) hemivariational inequalities

Similarly as in the parabolic case we admit the functional setting of Subsection 3.3 and assume that apart from A and J as before we are given in addition an operator B satisfying the hypothesis:

H(B): $B : V \longrightarrow V^*$ is a bounded, linear, monotone and symmetric operator, i.e.

$$B \in \mathcal{L}(V, V^*), \quad \langle Bv, v\rangle \geq 0 \quad \text{for all } v \in V,$$
$$\langle Bv, w\rangle = \langle Bw, v\rangle \quad \text{for all } v, w \in V.$$

We admit also:

(H_0): $f \in \mathcal{V}^*, \;\; y_0 \in V, \;\; y_1 \in H.$

(H_1): If $p = 2$ then $\frac{\beta_1}{2} > \bar{c}\beta^2 T$, where β is an embedding constant of V into Z.

Let us consider the following initial value problem for the hyperbolic differential inclusion:

$$(DI_h) \qquad \begin{cases} \text{Find } y \in \mathcal{Y}_h \text{ such that: for a.e. } t \in (0, T) \\ y''(t) + A(t, y'(t)) + By(t) + \partial J(t, y(t)) \ni f(t) \\ y(0) = y_0, \quad y'(0) = y_1, \end{cases}$$

which is eqivalent to the following hyperbolic hemivariational inequality ($J^0(t, u; v)$ denoting the generalized directional derivative of $J(t, \cdot)$ at a point $u \in Z$ in the direction $v \in Z$):

$$(HVI_h) \quad \begin{cases} \text{Find } y \in \mathcal{Y}_h \text{ such that: for all } v \in V \text{ and a.e. } t \in (0, T) \\ \langle y''(t) + A(t, y'(t)) + By(t) - f(t), v\rangle_{V^* \times V} + J^0(t, y(t); v) \geq 0 \end{cases}$$

The latter is often written down as

$$(HVI_h^1)\begin{cases} \text{Find } y \in \mathcal{Y}_h \text{ and } \zeta \in \mathcal{Z}^* \text{ such that:} \\ \zeta(t) \in \partial J(t, y(t)) \quad \text{for a.e. } t \in (0,T) \\ y''(t) + A(t, y'(t)) + By(t) + \zeta(t) = f(t) \quad \text{for a.e. } t \in (0,T) \\ y(0) = y_0, \;\; y'(0) = y_1. \end{cases}$$

Let us notice that the initial conditions in the problems above have a sense since the embeddings $\mathcal{Y}_h \subset C(0,T;V)$ and $\mathcal{W} \subset C(0,T;H)$ are continuous.

With the help of the following lemma giving "a priori estimates" one can obtain the existence result below.

LEMMA 4.1 *Suppose that hypotheses $H(A)$, $H(B)$, $H(J)$ and (H_0) hold and y is a solution to (HVI_h). If $p > 2$, then there exists a constant $C > 0$ such that*

$$\|y\|_{C(0,T;V)} + \|y'\|_{\mathcal{W}} \le C(1 + \|y_0\|_V^{\frac{2}{q}} + |y_1|_H^{\frac{2}{q}} + \|f\|_{\mathcal{V}^*}^{\frac{2}{q}}). \tag{4}$$

Moreover, the estimate (4) still holds for $p = 2$ provided (H_1) is satisfied. If $Z = H$, then the estimate (4) holds for $p \ge 2$ without the assumption (H_1).

THEOREM 4.4 *If hypotheses $H(A)$, $H(B)$, $H(J)$, (H_0) and (H_1) hold, then the problem (HVI_h) has at least one solution (i.e. $S_{(HVI_h)}(f, y_0, y_1) \neq \emptyset$).*

Proof. For the complete proof of the theorem we refer to Gasiński (2000) and to Ochal (2001). We only mention here that (HVI_h) can be reduced to (HVI_p) by means of the operator $K : \mathcal{V} \longrightarrow C(0,T;V)$ defined by

$$Kv(t) = y_0 + \int_0^t v(s)\,ds \quad \text{for all } v \in \mathcal{V}.$$

Namely, let us notice that the problem (HVI_h) can be written as follows

$$(HVI_p^r) \qquad \begin{cases} \text{Find } z \in \mathcal{W} \text{ such that: for a.e. } t \in (0,T) \\ f(t) \in z'(t) + A(t, z(t)) + B(Kz(t)) + \partial J(t, Kz(t)) \\ z(0) = y_1. \end{cases}$$

It can be observed that z is a solution to (HVI_p^r) if and only if $y := Kz$ satisfies (HVI_h). Therefore it suffices apply to the reduced problem (HVI_p^r) the surjectivity result of Proposition 3.2 similarly as in the parabolic case. ■

Of course the solution set $S_{(HVI_h)}(f, y_0, y_1)$ depends also on A, B, J and sometimes we have to identify some of them in the appropriate classes of oper-

4.4. An example of A satisfying hypothesis of the existence theorem

EXAMPLE 4.1 (see Ochal, 2001). Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $Q = (0,T) \times \Omega$ and $V = W_0^{1,p}(\Omega)$ $(2 \le p < +\infty)$. We consider a family of functions

$$a_i : Q \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R} \quad \text{for } i = 1, \dots, N,$$
$$a_0 : Q \times \mathbb{R} \longrightarrow \mathbb{R}$$

satisfying the following hypothesis:

H(a): functions $a_i = a_i(t,x,\eta,\xi)$, $(i = 1,\dots,N)$, and $a_0 = a_0(t,x,\eta)$, are of the Carathéodory type (i.e. a_i, a_0 are measurable with respect to $(t,x) \in Q$ and continuous in other variables) and for a.e. $(t,x) \in Q$, for all $\eta \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$ we have

(i) there exist $c_1 > 0$ and $b \in L^q(Q)$ $\left(q = \frac{p}{p-1}\right)$ such that

$$|a_i(t,x,\eta,\xi)| \le c_1\Big(b(t,x) + |\eta|^{p-1} + \sum_{j=1}^{N} |\xi_j|^{p-1}\Big) \quad \text{for } i = 1,\dots,N,$$
$$|a_0(t,x,\eta)| \le c_1(b(t,x) + |\eta|^{p-1}),$$

(ii) $\sum_{i=1}^N (a_i(t,x,\eta,\xi) - a_i(t,x,\eta,\xi'))(\xi_i - \xi_i') \ge 0$ for all $\xi' \in \mathbb{R}^N$,

(iii) there exist $c_2 > 0$ and $k \in L^1(Q)$ such that

$$\sum_{i=1}^{N} a_i(t,x,\eta,\xi)\xi_i + a_0(t,x,\eta)\eta \ge c_2\Big(|\eta|^p + \sum_{i=1}^{N} |\xi_i|^p\Big) - k(t,x).$$

Now we define the operator $A : (0,T) \times V \longrightarrow V^*$ by the formula:

$$\langle A(t,v), w\rangle_{V^* \times V} = \int_\Omega \sum_{i=1}^{N} a_i(t,x,v,\mathrm{D}v)\mathrm{D}_i w\, dx + \int_\Omega a_0(t,x,v) w\, dx.$$

The operator A can be treated as the sum of two operators

$$A_1 : (0,T) \times V \longrightarrow V^*,$$
$$\langle A_1(t,v), w\rangle_{V^* \times V} = \int_\Omega \sum_{i=1}^{N} a_i(t,x,v,\mathrm{D}v)\mathrm{D}_i w\, dx$$

and

$$A_2 : (0,T) \times V \longrightarrow V^*, \quad \langle A_2(t,v), w\rangle_{V^* \times V} = \int_\Omega a_0(t,x,v) w\, dx.$$

According to Proposition 26.12 in Zeidler (1990, p. 572) the operator $A_1(t,\cdot)$ is monotone, coercive, continuous and bounded. From Corollary 26.14 in Zeidler the operator $A(t,\cdot): V \longrightarrow V^*$ is continuous. By Proposition 27.6(f) in Zeidler (1990, p. 586), the operator $A(t,\cdot)$ is pseudomonotone as a strongly continuous perturbation of the continuous monotone operator.

Let us notice that applying the hypotesis H(a)(i) and the Hölder inequality, we get

$$|\langle A(t,v),v\rangle_{V^*\times V}| \le \int_\Omega \Big(\sum_{i=1}^N |a_i(t,x,v,\mathrm{D}v)|\,|\mathrm{D}_i v| + |a_0(t,x,v)|\,|v|\Big)\,dx$$

$$\le \int_\Omega c_1\Big(b(t,x) + |v|^{p-1} + \sum_{j=1}^N |\mathrm{D}_j v|^{p-1}\Big)\Big(\sum_{i=1}^N |\mathrm{D}_i v| + |v|\Big)\,dx$$

$$\le c(\|b(t)\|_{L^q(\Omega)} + \|v\|^{p-1}_{W_0^{1,p}(\Omega)})\|v\|_V,$$

which implies that $\|A(t,v)\|_{V^*} \le a_1(t) + b_1\|v\|_V^{p-1}$ with $a_1(t) = c\|b(t)\|_{L^q(\Omega)}$, and $b_1 = c > 0$.

It follows from the assumption H(a)(iii) that

$$\langle A(t,v),v\rangle_{V^*\times V} = \int_\Omega \sum_{i=1}^N (a_i(t,x,v,\mathrm{D}v)\mathrm{D}_i v + a_0(t,x,v)v)\,dx$$

$$\ge \int_\Omega c_2\Big(|v|^p + \sum_{i=1}^N |\mathrm{D}_i v|^p\Big)\,dx - \int_\Omega k(t,x)v\,dx$$

$$\ge c_2\|v\|_V^p - \|k(t)\|_{L^q(\Omega)}\|v\|_{L^p(\Omega)} \ge \beta_1\|v\|_V^p - \beta_2\|v\|_V,$$

with $\beta_1 = c_2 > 0, \beta_2 = \|k(t)\|_{L^q(\Omega)} \ge 0$. Hence A is coercive (i.e. H(A)(iv) holds with $a \equiv 0$ and $r = 1 < p$) and finally the hypothesis H(A) is satisfied.

REMARK 4.3 *Other examples of operators satisfying* **H(A)** *are given by maximal monotone (also multivalued) operators in divergence form for the functions belonging to the class $\mathcal{M}_\Omega(\mathbb{R}^N)$ defined in Chiado'Piat et al. (1990) (see also Denkowski et al., 2001).*

5. Optimal control problems for systems governed by hemivariational inequalities

In this section we quote three classes of optimal control problems for the hyperbolic hemivariational inequalities (the case of elliptic as well as parabolic one can be treated similarly – see e.g. Migórski, 2000, Gasiński, 1998, Migórski and Ochal, 2000 a, Denkowski and Migórski (1998 a,b)).

1. The optimal control problem of distributed parameter system (here we consider only Bolza type problem, but the time optimal problem and maximum stay problem were considered as well by Ochal (2001)),
2. The optimal control in the superpotential (it may be treated as the iden-

3. The optimal shape design (OSD for short) problems (in contrary to the previous classes where controls were functions, the controls are geometrical domains changing in some admissible families of sets).

5.1. General remarks and a lower semicontinuity result

REMARK 5.1 *In all the optimal control cases mentioned above the main tool in getting some existence results is the direct method. It is based on two properties:*

(i) the closed graph (or usc in Kuratowski sense) property of the solution map $S:\mathcal{U}\ni u\longmapsto S(u)\subseteq\mathcal{Y}$, *where* $S(u)=S_{(HVI)}(u)$ *is the set of states* $y(u)$ *of the controlled system* (HVI) *under consideration,*

(ii) the lsc property of the corresponding cost functional.

In all the problems considered below the property (i) should be established separately in each case, while for the first two classes the property (ii) is based on the general theorem (quoted below), due to Balder, and for OSD it is based on the Serrin type theorem where the cost functional depends also on the domain of integration (see, e.g., Denkowski, 2001, 2000).

Let X be a separable Banach space and Y be a separable reflexive Banach space. Let $F:[0,T]\times X\times Y\to(-\infty,+\infty)$ be a given $\mathcal{L}(0,T)\times\mathcal{B}(X\times Y)$-measurable function (here $\mathcal{L}(0,T)$ denotes the family of all subsets of $[0,T]$ measurable in Lebesgue sense, and $\mathcal{B}(X\times Y)$ is the family of all Borel subsets of $X\times Y$). We define the functional $\mathcal{F}:L^1(0,T;X)\times L^1(0,T;Y)\to[-\infty,+\infty]$ as

$$\mathcal{F}(x,y)=\int_0^T F(t,x(t),y(t))\,dt.$$

We equip $L^1(0,T;X)$ with L^1-norm, and $L^1(0,T;Y)$ with the weak topology.

The following theorem (see Balder, 1987) presents sufficient conditions for strong–weak lower semicontinuity of the integral functional $\mathcal{F}$ on $L^1(0,T;X)\times L^1(0,T;Y)$.

THEOREM 5.1 *If the following three conditions hold:*

(i) $F(t,\cdot,\cdot)$ *is sequentially lower semicontinuous on* $X\times Y$ *for a.e.* $t\in(0,T)$,

(ii) $F(t,x,\cdot)$ *is convex on* Y *for every* $x\in X$ *and for a.e.* $t\in(0,T)$,

(iii) there exist $M>0$ *and* $\psi\in L^1(0,T;\mathbb{R})$ *such that*

$$F(t,x,y)\geq\psi(t)-M(\|x\|_X+\|y\|_Y)\ \text{for all}\ x\in X,\ y\in Y\ \text{and a.e.}\ t\in(0,T),$$

then the functional $\mathcal{F}$ *is sequentially lower semicontinuous in* $s-L^1(0,T;X)\times(w-L^1(0,T;Y))$*-topology.*

Moreover, the conditions (i)–(iii) are also necessary provided that $\mathcal{F}(\overline{x},\overline{y})<+\infty$

It should be pointed out that, in general, optimal control problems for hemivariational inequalities are formulated as double minimization, maximization or minimax problems since usually the hemivariational inequality does not possess a unique solution.

5.2. Distributed controls—Bolza problem

We begin with a system described by the following controlled second order evolution inclusion

$$(CHVI_h)\ \begin{cases} y''(t) + A(t, y'(t)) + By(t) + \partial J(t, y(t)) \ni f(t) + C(t)u(t) \\ \qquad\qquad \text{for a.e. } t \in (0, T) \\ y(0) = y_0, \quad y'(0) = y_1. \end{cases}$$

Here A, B, J, f, y_0, y_1 are as in Section 4, $y = y(u)$ denotes the solution state corresponding to a control $u \in \mathcal{U} = L^q(0, T; U)$, $2 \le p < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$, U is a space of control variables and C represents a controller.

We need the following hypothesis concerning the operator $C(\cdot)$ and the space U of controls.

H(C): $C \in L^\infty(0, T; \mathcal{L}(U, Z^*))$ and U is a separable reflexive Banach space.

Now the closed graph (or usc in Kuratowski sense) property of the solution map

$$S : \mathcal{U} \ni u \longmapsto S(u) \subseteq \mathcal{Y}_h,$$

where $S(u) = S_{(CHVI_h)}(f + Cu, y_0, y_1)$ is the set of states $y(u)$ of the controlled system $(CHVI_h)$ (f, y_0, y_1 being fixed), follows from the lemma below.

The solution set is a subset of the space $\mathcal{Y}_h = \{y \in \mathcal{V} : y' \in \mathcal{W}\}$. We say that $\{y, y_n\} \subseteq \mathcal{Y}_h$, $y_n \longrightarrow y$ weakly in $\mathcal{Y}_h$ if and only if $y_n \longrightarrow y$ weakly in $\mathcal{V}$ and $y_n' \longrightarrow y'$ weakly in $\mathcal{W}$.

LEMMA 5.1 *Assume the hypotheses $H(A)$, $H(B)$, $H(J)$, (H_0), (H_1) and $H(C)$ hold. Then the solution map $S : \mathcal{U} \ni u \longmapsto S(u) \in 2^{\mathcal{Y}_h} \setminus \{\emptyset\}$ has a closed graph in $(w - \mathcal{U}) \times (w - \mathcal{Y}_h)$-topology (so also in $(w - \mathcal{U}) \times s - L^1(0, T; H)$).*

Next, let us consider the control problem:

$$(\mathrm{CP})_1$$

$$\begin{cases} \Phi(y, u) = l(y(T), y'(T)) + \displaystyle\int_0^T F(t, y(t), y'(t), u(t))\, dt \longrightarrow \inf =: m \\ \text{where } y \in S(u) \text{ and } u(t) \in \mathbf{U}(t) \text{ a.e. in } (0, T), u(\cdot) \text{ is measurable.} \end{cases}$$

We admit the following hypotheses:

H(Φ): $l : H \times H \to \mathbb{R}$ is weakly lower semicontinuous; $F : [0, T] \times H \times H \times U \to$

(i) $F(t,\cdot,\cdot,\cdot)$ is sequentially lower semicontinuous on $H \times H \times U$ for a.e. $t \in (0,T)$,

(ii) $F(t,y,v,\cdot)$ is convex on U, for all $y \in H, v \in H$ and a.e. $t \in (0,T)$,

(iii) there exist $M > 0$ and $\psi \in L^1(0,T)$ such that for all $y \in H$, $v \in H$, $u \in U$ and a.e. $t \in (0,T)$, we have

$$F(t,y,v,u) \geq \psi(t) - M\left(|y|_H + |v|_H + \|u\|_U\right).$$

H(U): $\mathbf{U} : [0,T] \longrightarrow 2^U \setminus \{\emptyset\}$ is a multifunction such that for all $t \in [0,T]$, $\mathbf{U}(t)$ is a closed convex subset of U and $t \longmapsto |\mathbf{U}(t)| := \sup\{\|u\|_U : u \in \mathbf{U}(t)\}$ belongs to L^∞_+.

We recall that $S^q_{\mathbf{U}} = \{u \in \mathcal{U} = L^q(0,T;U) : u(t) \in \mathbf{U}(t) \text{ a.e. in } [0,T]\}$ is the set of all selectors of the multifunction $\mathbf{U}(\cdot)$. It is known that under the hypothesis $\mathbf{H(U)}$ the set $S^q_{\mathbf{U}}$ is nonempty.

By an admissible state-control pair (y,u) for $(\mathrm{CP})_1$ we understand a pair consisting of a state function $y \in S(u)$ (which solves $(CHVI_h)$) and a control function $u \in S^q_{\mathbf{U}}$. An admissible pair (y,u) is called an optimal solution to $(\mathrm{CP})_1$ if and only if $\Phi(y,u) = m$. We have the following

THEOREM 5.2 *If the hypotheses $H(A)$, $H(B)$, $H(J)$, (H_0), (H_1), $H(C)$, $H(\Phi)$ and $H(U)$ hold, then the problem $(\mathrm{CP})_1$ admits an optimal solution.*

Proof. It follows from the direct method due to Lemma 5.1 and Theorem 5.1. For the details see Ochal (2001).

We quote an example of a cost functional which satisfies $\mathrm{H}(\Phi)$.

EXAMPLE 5.1

$$\Phi(y,u) = \varrho_1|y(T) - y_d|_H^2 + \varrho_2|y'(T) - \overline{y_d}|_H^2 + \varrho_3\int_0^T |\mathcal{O}_1 y(t) - z_d(t)|_H^2\,dt$$
$$+\varrho_4\int_0^T |\mathcal{O}_2 y'(t) - \overline{z_d}(t)|_H^2\,dt + \varrho_5\int_0^T \langle Ru(t), u(t)\rangle_{U^*\times U}\,dt,$$

where $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{L}(H)$ are observation operators, $R \in \mathcal{L}(U,U^*)$ is a positive defined and symmetric operator on U, $y_d, \overline{y_d} \in H$, $z_d, \overline{z_d} \in \mathcal{H}$ are given elements (desired outputs) and $\varrho_i \geq 0$ $(i = 1,\ldots,5)$ are some constants (weights).

5.3. Problem with control in superpotential

The framework is quite general and covers, in particular, the parameter identification (inverse) problems for systems governed by hemivariational inequalities (see Panagiotopoulos and Haslinger, 1995).

The main theorem of this section generalizes the result of the paper of Miettinen and Haslinger (1992) who considered the stationary hemivariational inequal-

(1993), Naniewicz and Panagiotopoulos (1995) for some applications of the results for engineering structures.

The formulation of the problem is as follows

$$(\mathrm{CP})_2 \qquad \begin{cases} \text{Given a cost functional } F : \mathcal{U} \times \mathcal{Y} \longrightarrow \mathbb{R}, \\ \text{find } u^* \in \mathcal{U}_{ad} \text{ and } y^* \in S(u^*) \text{ such that} \\ F(u^*, y^*) \le F(u, y) \quad \text{for all } u \in \mathcal{U}_{ad},\ y \in S(u), \end{cases}$$

where $S(u) \subseteq \mathcal{Y} = \{y \in \mathcal{V} : y' \in \mathcal{W}\}$ denotes the set of solutions to the hemivariational inequality

$$\begin{cases} y''(t) + A(t, y'(t)) + By(t) + \partial J_u(t, y(t)) \ni f(t) \quad \text{for a.e. } t \in (0,T) \\ y(0) = y_0, \quad y'(0) = y_1. \end{cases}$$

Here $\mathcal{U}_{ad}$ denotes a class of admissible controls (parametrized superpotentials) in the control space $\mathcal{U}$ (while the cost functional may represent for instance a distance between the observed-measured and the calculated solution).

In order to formulate a theorem on the existence of solutions to $(CP)_2$ we need the following hypotheses:

H(U)$_1$: $\mathcal{U}_{ad}$ is a compact subset of a metric space of controls $\mathcal{U}$.

H(J)$_1$: for any $u \in \mathcal{U}$, $J_u : (0,T) \times Z \longrightarrow \mathbb{R}$ satisfies **H(J)** uniformly with respect to u and the following condition holds

$$\begin{cases} \text{if } u_n \longrightarrow u \text{ in } \mathcal{U}, \text{ then for a.e. } t \in (0,T) \\ K(Z \times (w - Z^*)) - \limsup\limits_{n \to +\infty} Gr\partial J_{u_n}(t, \cdot) \subset Gr\partial J_u(t, \cdot). \end{cases} \qquad (5)$$

REMARK 5.2 *We mention that the sufficient conditions for the above convergence of Clarke's generalized gradients have been found by Zolezzi (1994). Namely, if the sequence $\{J_u\}_{u \in U_{ad}}$ is Γ- (De Giorgi) or in other terms epi-(Attouch) convergent, locally equi-bounded and equi-lower semidifferentiable, then the relation (5) in* **H(J)$_1$** *holds (see Theorem 1 in Zolezzi, 1994, p. 384).*

Similarly as in the Bolza problem the crucial point in the proof of the existence result for $(\mathrm{CP})_2$ is to establish the closedness (in suitable topologies) of the graph of the solution map $S : \mathcal{U} \ni u \longmapsto S(u) \subseteq \mathcal{Y}_h$. Here we quote:

LEMMA 5.2 *If the hypotheses $H(A)$, $H(B)$, $H(J)_1$, (H_0), (H_1) and $H(U)_1$ hold, then the solution map $S : \mathcal{U} \ni u \longmapsto S(u) \in 2^{\mathcal{Y}_h} \setminus \{\emptyset\}$ has a closed graph in $s - \mathcal{U} \times (w - \mathcal{Y}_h)$-topology (so also in $s - \mathcal{U} \times s - L^1(0,T;H)$).*

Basing on this lemma and on Theorem 5.1 due to the direct method we obtain (for the details see Ochal, 2001):

THEOREM 5.3 *If the hypotheses $H(A)$, $H(B)$, $H(J)_1$, (H_0), (H_1), $H(U)_1$ hold and the cost functional F is lower semicontinuous in $\mathcal{U} \times (w - \mathcal{Y}_h)$-topology,*

5.4. Optimal shape design problems (OSDs)

In distinction from the previously considered control problems where the state relation was considered on a fixed domain $\Omega \subset \mathbb{R}^N$ and the controls were functions ($u \in \mathcal{U}_{ad}$), in this special class (OSD) of optimal control the controls are geometrical domains ($u = \Omega$), so the state relation should be considered on the changing set.

Let $\mathcal{O}_{ad}$ denote a class of admissible shapes (e.g. open and bounded subsets of $\mathbb{R}^N$). For a given $\Omega \in \mathcal{O}_{ad}$ consider a state relation E on Ω (E stands for a PDE or VI or HVI). By setting $S_E(\Omega)$ = the solution set of E, we define

$$\mathcal{D} \stackrel{df}{=} \bigcup_{\Omega \in \mathcal{O}_{ad}} (\{\Omega\} \times S_E(\Omega))$$

Then, given a cost functional $\mathcal{F}: \mathcal{D} \ni (\Omega, y) \longmapsto \overline{\mathbb{R}}$ we formulate the (OSD) problem as follows:

$$(OSD)_E \qquad \begin{cases} \text{Find } \Omega^* \in \mathcal{O}_{ad} \text{ and } y^* \in S_E(\Omega^*) \text{ such that} \\ \mathcal{F}(\Omega^*, y^*) = \min\limits_{\Omega \in \mathcal{O}_{ad}} \min\limits_{y \in S_E(\Omega)} \mathcal{F}(\Omega, u). \end{cases}$$

The problem $(OSD)_E$ was solved (using the so called "mapping method") in the case $E = HVI_c$ by Denkowski and Migórski (2000 A, B) in the case $E = HVI_p$ and $E = HVI_h$ by Gasiński (1998, 2000).

For the convenience of the reader we quote here some basic facts from the mapping method. This method (originated by Micheletti, and then developed by Murat and Simon) provides us with both:

1. The set of admissible controls $\mathcal{O}_{ad}$ (they are images of a fixed regular domain G by transformations belonging to an appropriately defined family of applications in $\mathbb{R}^N$),
2. The topology in $\mathcal{O}_{ad}$ permitting to get the usc property for the solution map $\Omega \longmapsto S_E(\Omega)$, as well as the lsc property of the cost functional $\mathcal{F}$.

Thus, let G be an open bounded subset of $\mathbb{R}^N$ with the boundary ∂G of class $W^{k,\infty}$ ($k \geq 1$) such that $\operatorname{int} \overline{G} = G$.

For $k \geq 1$ we introduce spaces of applications in $\mathbb{R}^N$ (here $I : \mathbb{R}^N \longmapsto \mathbb{R}^N$ denotes the identity mapping)

$$W^{k,\infty}(\mathbb{R}^N; \mathbb{R}^N) \stackrel{df}{=} \{v : D^\alpha v \in L^\infty(\mathbb{R}^N; \mathbb{R}^N) \ \ \forall \alpha : 0 \leq |\alpha| \leq k\},$$
$$\mathcal{V}^{k,\infty} \stackrel{df}{=} \{\tau : \tau - I \in W^{k,\infty}(\mathbb{R}^N; \mathbb{R}^N)\},$$
$$\mathcal{F}^{k,\infty} \stackrel{df}{=} \{\tau : \tau \text{ is bijective and } \tau, \tau^{-1} \in \mathcal{V}^{k,\infty}\}.$$

The derivative D^α in the definition of $W^{k,\infty}$ is understood in the distributional sense. So, the space $\mathcal{F}^{k,\infty}$ consists of "essentialy bounded perturbations" (with

The class of all considered shapes will be denoted by

$$\mathcal{O}^{k,\infty} \stackrel{df}{=} \{\Omega : \Omega = \tau(G),\ \tau \in \mathcal{F}^{k,\infty}\}.$$

It can be proved (see Murat and Simon, 1976) that the sets $\Omega \in \mathcal{O}^{k,\infty}$ are bounded and their boundaries $\partial\Omega$ belong to the class $W^{k,\infty}$.

In the space $W^{k,\infty}(\mathbb{R}^N;\mathbb{R}^N)$ we define the norm setting

$$\|v\|_{k,\infty} \stackrel{df}{=} ess\,sup_{x\in\mathbb{R}^N}\Big(\sum_{0\le|\alpha|\le k} |D^\alpha v(x)|^2_{\mathbb{R}^N}\Big)^{\frac{1}{2}}.$$

Let us observe that the function $\delta_{k,\infty} : \mathcal{O}^{k,\infty} \times \mathcal{O}^{k,\infty} \longmapsto \mathbb{R}$, defined as

$$\delta_{k,\infty}(\Omega_1,\Omega_2) \stackrel{df}{=} \inf_{\substack{\tau\in\mathcal{F}^{k,\infty}\\ \tau(\Omega_1)=\Omega_2}} \left(\|\tau - I\|_{k,\infty} + \|\tau^{-1} - I\|_{k,\infty}\right),\ \forall \Omega_1,\Omega_2 \in \mathcal{O}^{k,\infty}.$$

is not a metric as it does not satisfy the triangle inequality (see Murat and Simon, 1976). However, after a modification we can get a metric according to:

THEOREM 5.4 *Let $k \ge 1$. Then:*
(a) *There exists a constant $\mu_k > 0$ such that the application $d_{k,\infty} : \mathcal{O}^{k,\infty} \times \mathcal{O}^{k,\infty} \longmapsto \mathbb{R}$, given by*

$$d_{k,\infty}(\Omega_1,\Omega_2) = \sqrt{\max\{\delta_{k,\infty}(\Omega_1,\Omega_2),\mu_k\}}, \quad for\ \Omega_1,\Omega_2 \in \mathcal{O}^{k,\infty},$$

is a metric in $\mathcal{O}^{k,\infty}$.
(b) *The metric space $\left(\mathcal{O}^{k,\infty}, d_{k,\infty}\right)$ is complete.*
(c) *For $k \ge 2$ the embedding $\mathcal{O}^{k,\infty} \subseteq \mathcal{O}^{k-1,\infty}$ is compact; i.e. for every bounded (in the metric $d_{k,\infty}$) sequence $\{\Omega_n\}_{n\ge1} \subseteq \mathcal{O}^{k,\infty}$ there exists a subsequence $\{\Omega_{n_k}\}_{k\ge1}$ which is convergent in the metric $d_{k-1,\infty}$.*

Proof. See Murat and Simon (1976), Proposition 2.3, Théorème 2.2 and Théorème 2.4. ∎

A characterization of the convergence in $\mathcal{O}^{k,\infty}$ is given by the remark below.

REMARK 5.3 *Assume $\{\Omega_n\}_{n\ge1} \subseteq \mathcal{O}^{k,\infty}$, $\Omega \in \mathcal{O}^{k,\infty}$ and $\{\tau_n\}_{n\ge1} \subseteq \mathcal{F}^{k,\infty}$, $\tau \in \mathcal{F}^{k,\infty}$ are such that $\tau_n(G) = \Omega_n$ and $\tau(G) = \Omega$. Then $\Omega_n \longrightarrow \Omega$ in $\mathcal{O}^{k,\infty}$ if and only if $\tau_n \longrightarrow \tau$ and $\tau_n^{-1} \longrightarrow \tau^{-1}$ in $W^{k,\infty}(\mathbb{R}^N;\mathbb{R}^N)$.*

Now, passing to HVIs we quote an existence result for $(OSD)_{HVI_h}$ obtained by L. Gasiński (see Gasiński, 2000). In the particular case ($p = q = 2$) we consider the spaces and operators:

$$H(\Omega) = L^2(\Omega),$$
$$V(\Omega) = H^1(\Omega) = \{v:\ v \in L^2(\Omega),\ D^\alpha v \in L^2(\Omega)\ \text{ dla } 0 \le |\alpha| \le 1\},$$
$$\mathcal{H}(\Omega) = L^2(0,T;H(\Omega)),$$
$$\mathcal{V}(\Omega) = L^2(0,T;V(\Omega)),$$
$$\mathcal{W}(\Omega) = \{y:\ y \in \mathcal{V}(\Omega),\ y' \in \mathcal{V}^*(\Omega)\},$$

First we solve the problem

(HVI_h)

$$\begin{cases} \text{Find } y \in \mathcal{Y}_h(\Omega), \text{ such that there is } \zeta \in \mathcal{H}(\Omega) \text{ satisfying} \\ \langle y''(t), v\rangle_{V^*(\Omega)\times V(\Omega)} + a_\Omega(y'(t), v) + b_\Omega(y(t), v) + (\zeta(t), v)_{H(\Omega)} \\ \qquad = \langle f(t), v\rangle_{V^*(\Omega)\times V(\Omega)} \quad \forall v \in V(\Omega) \text{ and for a.e. } t \in (0,T), \\ y(0) = y_0, \quad y'(0) = y_1 \quad \text{in } \Omega, \\ \zeta(t,x) \in \partial j(g(y(t,x), y'(t,x))) \quad \text{for a.e. } (t,x) \in (0,T)\times\Omega, \end{cases}$$

where f, y_0, y_1 are given functions, j and g fulfill, respectively, hypothesis **H(j)** of Section 3 and the following one

H(g) $g : \mathbb{R}\times\mathbb{R} \longmapsto \mathbb{R}$ is continuous and satisfies the condition

$$|g(\xi,\zeta)| \le \alpha_1|\xi| + \alpha_2|\zeta|, \quad \text{for every } \xi, \zeta \in \mathbb{R},$$

with some constants $\alpha_1, \alpha_2 \ge 0$.

We assume the bilinear functions $a_\Omega, b_\Omega : V \times V \longmapsto \mathbb{R}$ satisfy the hypothesis below.

$\mathbf{H(a)_\Omega}$ $a_\Omega : V(\Omega)\times V(\Omega) \longmapsto \mathbb{R}$ is a bilinear function given for $v, w \in V(\Omega)$ by

$$a_\Omega(v,w) \stackrel{df}{=} \int_\Omega \left[(\overline{A}(x)\nabla v(x), \nabla w(x))_{\mathbb{R}^N} + \overline{a}(x)v(x)w(x)\right] dx,$$

where

(i) $\overline{A} \in \left[C(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\right]^{N^2}$ denotes a coercive matrix with a coercivity constant $\beta_1 > \beta$; (see hypothesis **H(A)** and **($\mathbf{H_1}$)** in Section 4);

(ii) $\overline{a} \in C(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ is a function such that for a constant $\overline{\overline{a}} > \beta$, we have $\overline{a}(x) \ge \overline{\overline{a}}$ a.e. in $\mathbb{R}^N$.

$\mathbf{H(b)_\Omega}$ $b_\Omega : V(\Omega)\times V(\Omega) \longmapsto \mathbb{R}$ is a bilinear function given by

$$b_\Omega(v,w) \stackrel{df}{=} \int_\Omega \left[(\overline{B}(x)\nabla v(x), \nabla w(x))_{\mathbb{R}^N} + \overline{b}(x)v(x)w(x)\right] dx,$$

where

(i) $\overline{B} \in \left[C(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\right]^{N^2}$ denotes a symmetric and nonnegative matrix

(ii) $\overline{b} \in C(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ is a function such that $\overline{b}(x) \ge 0$ a.e. in $\mathbb{R}^N$.

H'(f, y_0, y_1) $f \in \mathcal{H}(\mathbb{R}^N)$, $y_0 \in V(\mathbb{R}^N)$, $y_1 \in H(\mathbb{R}^N)$.

We have the following existence result:

THEOREM 5.5 *Let $p = q = 2$ and let Ω be an open and bounded subset of $\mathbb{R}^N$. If the hypotheses* **H(j)**, **H(g)**, $\mathbf{H(a)_\Omega}$, $\mathbf{H(b)_\Omega}$ *and* **H'(f, y_0, y_1)** *are fulfilled, then $S_{(HVI_h)}(\Omega) \neq \emptyset$, (i.e. the problem $(HVI)_h$) has at least one solution.*

Proof. For the proof it suffices to define operators respectively by the formulae (for every $v, w \in V(\Omega)$),

$$\langle A(t,v), w\rangle_{V^*(\Omega)\times V(\Omega)} \stackrel{df}{=} a_\Omega(v,w), \quad \langle Bv, w\rangle_{V^*(\Omega)\times V(\Omega)} \stackrel{df}{=} b_\Omega(v,w)$$

and observe that they satisfy hypotheses **H(A)** and **H(B)**, so the existence Theorem 4.4 can be applied. ■

Next, we admit hypothesis:

H(G,$\mathcal{O}_{ad}$) (i) G is an open and bounded subset of $\mathbb{R}^N$ with the boundary of the class $W^{1,\infty}$ and such that $int\,\overline{G} = G$;

(ii) $\mathcal{O}_{ad}$ is a closed and bounded subset of $\mathcal{O}^{k,\infty}$, where $k \geq 3$.

H($\mathcal{F}$) The cost functional $\mathcal{F} : \mathcal{D} \longmapsto \mathbb{R}$ is sequentially lsc with respect to the following convergence in $\mathcal{D}$:

$(\Omega_n, y_n) \longrightarrow (\Omega_0, y_0)$ iff $\Omega_n \longrightarrow \Omega_0$ in $\mathcal{O}^{k-1,\infty}$ and $\underline{y_n} \longrightarrow \underline{y_0}$, $\underline{y_n'} \longrightarrow \underline{y_0'}$ in $\mathcal{H}(\mathbb{R}^N)$.

Above, by $\underline{y}$ we denoted the function $y \in \mathcal{Y}(\Omega)$ extended by zero outside Ω ; i.e.

$$\underline{y}(t,x) \stackrel{df}{=} \begin{cases} y(t,x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Finally, we can formulate

THEOREM 5.6 *Assume $p = q = 2$ and the hypotheses $H(j)$, $H(g)$, $H'(f, y_0, y_1)$, $H(G,\mathcal{O}_{ad})$, $H(\mathcal{F})$ hold. Then, if the assumptions $H(a)_\Omega$ and $H(b)_\Omega$ are satisfied for every $\Omega \in \mathcal{O}_{ad}$, the problem $(OSD)_{HVI_h}$ admits at least one solution.*

Proof. It goes by the direct method, for details see Gasiński (2000).

6. Concluding remarks

- The problems with HVIs as the state equations (e.g. (CP)) are not well posed in the sense of Hadamard, since in general the solution set (if not empty) contains more than one element.
- For the unicity of solutions some additional hypothesis are needed like, for instance, strict convexity of superpotentials and maximal monotonicity of the involved operators (see, e.g., Miettinen and Haslinger, 1992).
- As far as it concerns sensitivity and robustness of such systems some information on the asymptotic behaviour of the solution set can be obtained from the Kuratowski-usc property of the solution map but only for HVIs with operators belonging to special classes for which we can assure G or PG convergence (see Remark 4.1, Theorem 4.2, Remark 4.2,...). Namely, we can infer that any accumulation point of a sequence of solutions to HVI

- Similarly, for the set of optimal solutions one can expect (in analogy with control problems for PDE's or differential inclusions, see e.g. Denkowski and Mortola, 1993) that the convergence (in Kuratowski sense) of solution sets of perturbed systems and some complementary Γ convergence of cost functionals will imply the convergence of optimal solutions of the perturbed systems to an optimal solution of the limit problem (see forthcoming paper).
- There are many open problems of the above mentioned type for HVIs with multivalued operators A. For instance, under what conditions PG (see Migórski, 2000) or even G convergence (see e.g. Denkowski et al., 2001) will imply a sensitivity result for hyperbolic HVIs. Open problems of another type concern the relaxation of HVIs in a similar way as of PDE's (see e.g. Smołka, 2000, Denkowski, 2000).

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