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Construction of a k – immunization strategy with the highest convexity

by

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Abstract: Assuming that interest rate shocks are proportional to their values plus one, we prove in Theorem 1 the existence of and construct a portfolio Z^* with the highest convexity in the class of portfolios that solve the immunization problem to meet the liability to pay C dollars K years from now. Z^* appears to be a barbell strategy with two zero-coupon bonds with the shortest and the longest maturities. This intuitively clear result has been obtained here in a rigorous way by means of the K-T conditions. In addition, we show that our result is strictly related to the problem of maximization of the unanticipated rate of return on a portfolio solving the above immunization problem (Theorem 2). Two more results concerning the unanticipated return after K years are provided with proofs. An example illustrating the role of convexity in maximization of the unanticipated return is included. Despite the fact that there exists a pretty vast literature on bond portfolio strategies, the present paper offers a new methodological approach to this area (see Ingersoll, Skelton, Weil, 1978).

Keywords: convexity, immunization, $K - T$ conditions, unanticipated return

1. Introduction

By the basic immunization problem we mean here the problem of finding the least expensive bond portfolio that ensures meeting a single liability of C dollars to be paid K years from now, irrespective of unknown shocks h_t in spot rates $y_t(y_t \rightarrow y_t + h_t)$. Here K may be any number between the shortest t_0 and the longest (t_n) bond maturity. It is well known (see the pioneering work of Macaulay, 1938, Redington, 1952, and Fisher, 1971), that any bond portfolio with duration of K years and an appropriate investment value, solves this problem in the simplest case when the YTM curve is flat $(y_t = y)$ and $h_t = h$, where t varies over all instants $t_0, t_1, \ldots, t_n$ when currently available bonds pay coupons (for all such t, spot rates y_t are derived from current bond prices).

In Zaremba (1995, Theorems 4 and 5) we solved this problem assuming that shifts h_t in spot rates y_t were expected to be proportional to their values plus 1, i.e.,

$$\frac{h_t}{1+y_t} = \frac{h_{t_0}}{1+y_{t_0}} \quad t = t_0, t_1, \ldots, t_n. \tag{1}$$

We proved that any bond portfolio Z that (i) dominates, Zaremba (1995, Definition 4), a real or imaginary zero–coupon bond 0 maturing K years from now with the only payment of C dollars and such that (ii) the investment value of Z is not smaller than that of the bond 0, solves the basic immunization problem and vice versa. We also demonstrated that bond portfolios which dominate the bond 0 are exactly those whose durations are equal to K, Zaremba (1995, Theorem 6).

In this paper we prove the existence and then construct (Theorem 1) a portfolio Z^* with the highest convexity in the class of portfolios that solve the basic immunization problem. Z^* appears to be a barbell strategy based upon two zero–coupon bonds with the shortest and the longest maturities. This result is interesting not only in itself but should be of interest to all those whose goal is to maximize the unanticipated rate of return on a portfolio solving the basic immunization problem due to unknown proportional shifts in spot rates that take place instantly after the bond portfolio is acquired.

In fact, the actual unanticipated return on a portfolio Z solving the basic immunization problem differs only very little from that given by the formula (11). It means that the maximum return given by (11), which we derive in Theorem 2, is very close to the actual maximum unanticipated return due to proportional shifts in spot rates. In Theorem 3 we suppose that no other changes in the spot rates will take place in the period of K years, to give a formula for the resulting return on Z^* after K years.

2. Duration and convexity for bond portfolios

We will recall now some notions and facts introduced and shown in Zaremba (1995). By a bond A one understands a financial instrument paying coupons of C_t dollars t years from now, $t = t_0, t_1, \ldots, t_n$, with $t_0(t_n)$ being the maturity of a shortest (longest) bond; for a concrete bond, some or even most of C_t's are equal to zero. Treasury bills with maturities less than 1 year are treated as (zero–coupon) bonds. Thus, a bond A can be identified with a sequence of coupons

$$C_{t_0}, C_{t_1}, \ldots, C_{t_n}$$

The investment value of a bond A is therefore equal to

$$P_A = P_A(y_{t_0}, y_{t_1}, \ldots, y_{t_n}) = \sum C_t(1+y_t)^{-t}. \tag{2}$$

Since a bond portfolio $Z = (z_1, x_2, \ldots, x_r)$ is a collection of x_i copies of bonds O_i, the investment value of Z is defined to be the sum of the investment value of all bonds O_i present in portfolio Z. It is also well known that the duration of a bond O_m is given by the formula

$$D(O_m) = \sum t x_t^m, \quad x_t^m = C_t^m (1 + y_t)^{-t} / P_m, \tag{3}$$

where P_m is the investment value of the bond O_m generating the coupons

$$C_{t_0}^m, C_{t_1}^m, \ldots, C_{t_n}^m,$$

so that x_m^t are the weights of coupons C_t^m. Thus, duration is the average of the dates on which cash flows are promised, where those dates having the larger current values of the cash flows receive the greater weight. Accordingly, the duration of a portfolio $Z = (x_1, x_2, \ldots, x_r)$ is understood in the same fashion, that is,

$$D(Z) = \sum_{t=t_0}^{t=t_n} t \cdot \sum_{m=1}^{r} C_t^m (1 + y_t)^{-t} / P^*, \quad P^* = \sum_{m=1}^{r} P_m. \tag{4}$$

However, it is well known (for a rigorous proof see Zaremba, 1995, Lemma 1) that $D(Z)$ is a convex combination of durations $D(O_m)$, that is,

$$D(Z) = \sum_{m=1}^{r} x_m D(O_m),\ x_m = \frac{P_m}{P^*}, \quad P^* = \sum_{m=1}^{r} P_m. \tag{5}$$

It is also well known, Elton, Gruber (1995), p. 548, that the convexity of a bond A is defined to be the number

$$V(A) = \frac{1}{2} \sum_{t=t_0}^{t=t_n} t(t+1) x_t^A,\ x_t^A = \frac{C_t (1 + y_t)^{-t}}{P_A}. \tag{6}$$

In the same way one understands the convexity of a bond portfolio $Z = (x_1, x_2, \ldots, x_r)$, that is to say,

$$V(Z) = \frac{1}{2} \sum_{t=t_0}^{t=t_n} t(t+1) x_t^Z, \tag{7}$$

where x_t^Z stands for the weight of all coupons to be received from Z at time t. Like in case of duration, $V(Z)$ is a convex combination of convexities of bonds O_m (see, for instance, Zaremba, 1995, Theorem 2), that is,

$$V(P) = \sum_{m=1}^{r} x_m V(O_m), \quad x_m = \frac{P_m}{P^*}, \tag{8}$$

where, as usual, P_m stands for the investment value of the bond O_m, while P^* is the investment value of Z (see (4)).

Now, suppose that spot rates y_t have been changed proportionally so that (1) holds. Denoting by $\bar{P}_Z$ the new investment value of a bond portfolio Z, the unanticipated rate of return on Z (see, Elton, Gruber, 1995, p. 544), due to the proportional shifts in the spot rates, is expressed as in Zaremba (1995, Theorem 3).

$$\frac{dP_Z}{P_Z} = \frac{\bar{P}_Z - P_Z}{P_Z} = -D(Z)\frac{h_{t_0}}{1+y_{t_0}} + V(Z)\left(\frac{h_{t_0}}{1+y_{t_0}}\right)^2 + o, \tag{9}$$

where o, being a third order term with respect to h_{t_0}, may be neglected. Note that on many investment (financial) markets one has $-.01 \leq h_t \leq .01$, which makes the term o really small (see, Zaremba, Example 1)

3. Basic immunization problem

Assume one has to discharge the financial liability to pay C dollars K years from now, $t_0 \leq K \leq t_n$, where as usual, $t_0(t_n)$ is the maturity of the shortest (longest) bond available on a given financial market. The basic immunization problem can be stated as follows: How to meet this liability with the help of a bond portfolio, expecting proportional shocks in spot rates, with the least amount of money spent for this purpose? Analogously as in Zaremba (1995), let us set

$$C_K^* = [(1+y_k)^{-K}]C. \tag{10}$$

Obviously, C_K^* can be thought of as the investment value of a real or imaginary zero-coupon bond o paying C dollars at time K. Following Zaremba (1995) we shall say that a bond portfolio Z is a K - immunization portfolio if Z solves the basic immunization problem defined above. It was proved in Zaremba (1995) that any portfolio Z with $P_Z \geq C_K^*$ and $D(Z) = K$ is a K - immunization bond portfolio. Moreover, if $P_Z < C_K^*$ or $P_Z = C_K^*$ and simultaneously $D(Z) \neq K$ then Z is not a K - immunization bond portfolio.

The last result obtained in Zaremba (1995) can be formulated as follows. Suppose that spot rates y_t have been changed proportionally instantly after an investor has acquired a K - immunization portfolio. Then the investor obtains the unanticipated rate of return on Z, R_Z^u, which is approximately equal to (the third order term with respect to h_{t_0} has been neglected)

$$R_Z^u = \left[V(Z) - \frac{1}{2}K(K+1)\right]\left(\frac{h_{t_0}}{1+y_{t_0}}\right)^2 + \left(\frac{h_{t_0}}{1+y_{t_0}}\right)^{-K} - 1. \tag{11}$$

Using this equality and Theorem 1, we derive several corollaries in Section 5.

4. Maximum convexity in the class of K – immunization strategies

The main goal of this paper is to construct a portfolio Z^* with the highest convexity $V(Z^*)$ in the class of K – immunization portfolios. Following Elton, Gruber (1995, p. 552), a bond portfolio $Z = (x_1, x_2, \ldots, x_n)$ consisting of two types of bonds only, say O_i, O_j (in other words, $x_k = 0$ except for $k = i$ and $k = j$) is said to be a barbell strategy for the basic immunization problem if $D(Z) = K$, while $D(O_i)$ and $D(O_j)$ are "very different". On the other hand, Z is said to be a focused strategy if $D(O_k)$ are centered around the duration of the liability, that is, the number K, $k = 1, 2, \ldots, r$.

THEOREM 4.1 *A K – immunization portfolio Z^* with the highest convexity exists and appears to be a barbell strategy built up with the shortest and the longest zero–coupon bonds, say, O_1, O_2, whose maturities are equal to t_0, t_n, respectively. The investment values of these bonds O_1, O_2, are equal to*

$$P_{O_1} = \frac{t_n - K}{t_n - t_0} C_K^* \text{ and } P_{O_2} = \frac{K - t_0}{t_n - t_0} C_K^* \tag{12}$$

respectively (they sum up to C_K^*).

Proof. The maximization of $V(Z)$ in the class of portfolios Z satisfying $D(Z) = K$ and $P_Z = C_K^*$ (see (10)) leads to the following optimization problem

$$\max \frac{1}{2} \sum t(t+1) x_t^Z; \; \sum_{t=t_0}^{t=t^n} t x_t^Z = K, \; \sum_{t=t_0}^{t=t^n} x_t^Z = 1, \; x_t^Z \geq 0 \tag{13}$$

where

$$x_t^Z = \frac{\sum_{m=1}^{r} C_t^m (1 + y_t)^{-t}}{P_Z}, \; t = t_0, t_1, \ldots, t_n. \tag{14}$$

Since all the functions occurring in (13) are linear with respect to x_1^Z, the Kuhn – Tucker conditions are necessary and sufficient for the optimality of a given portfolio Z. We thus have

$$\frac{\partial L}{\partial x_t^Z} = 0, \; \sum_{t=t_0}^{t=t^n} \lambda_t x_t^Z = 0, \; \lambda_t = 0, \; x_t^Z \geq 0, \tag{15}$$

where

$$L = \frac{1}{2} \sum_{t=t_0}^{t=t^n} (t^2 + t) x_t^Z + \sum_{t=t_0}^{t=t^n} (\lambda_t z_t^Z + \mu_1 t x_t^Z + \mu_2 x_t^Z). \tag{16}$$

At first, we show that $x_t^Z \neq 0$ for at most two instances t_i, t_j, which means the remaining x_t^Z's vanish. Assuming on the contrary, that $x_t^Z \neq 0$ for t_i, t_j,

t_l, with t_i, t_j, t_l being pairwise distinct, one arrives at a contradiction. In fact, because by virtue of (14) λ_{t_i}, λ_{t_j}, λ_{t_l}, should be equal to zero, the necessary condition

$$0 = \frac{\partial L}{\partial x_t^Z} = \frac{1}{2}(t^2 + t) + \lambda_t + \mu_1 + \mu_2 \tag{17}$$

would then mean three distinct roots of the quadratic equation $\frac{1}{2}(t^2+t)+\mu_1 t + \mu_2 = 0!$, which is impossible. Now we shall guess (intuitively it is quite clear) that the portfolio Z^* is given by

$$x_{t_0}^{Z^*} = \frac{t_n - K}{t_n - t_0}, \;\; x_t^{Z^*} = 0, \;\; t \neq 0, \;\; t \neq t_n, \;\; x_{t_n}^{Z^*} = \frac{K - t_0}{t_n - t_0} \tag{18}$$

and next show that the necessary and sufficient conditions (14) hold for Z^*. In fact, by setting

$$\lambda_t = \frac{1}{2}(t_n - t)(t - t_0), \;\; \mu_1 = -\frac{1}{2}(t_n + t_0 + 1), \;\; \mu_2 = \frac{1}{2}t_n t_0 \tag{19}$$

one sees that (16) holds. The remaining three conditions given in (14) are obviously satisfied because $x_t^Z \geq 0$, $\lambda_t \geq 0$ and $\lambda x_t^Z = 0$. It all means that Z^* is the only solution to the optimization problem (13); see Remark 1 to follow. Since Z^* pays two coupons only at $t = t_0$ and $t = t_n$, the simplest way to define it is to set $Z^* = (x_1^*, x_2^*, 0, \ldots, 0)$ with x_1^* copies of the shortest bond O_1 paying

$$\frac{C_K^*(1 + y_{t_0})^{t_0}(t_n - K)}{(t_n - t_0)}$$

dollars at $t = t_0$ and x_2^* copies of bond O_2 paying

$$\frac{(1 + y_{t_n})^{t_n}(K - t_0)}{(t_n - t_0)}$$

dollars at $t = t_n$. It is now clear that the investment values of all bonds O_1 present in Z^* and O_2 are given by (12), as required. The proof is complete. ■

REMARK 4.1 *The reader may wonder why another barbell strategy $\bar{Z}$ built up with two zero–coupon bonds O_i, O_j with maturities $t_i \neq t_0$ or $t_j \neq t_n (t_i < t_j)$ cannot be a solution to the optimization problem (13). The answer is simple: this is so because then $\lambda_t = \frac{1}{2}(t_j - t)(t - t_i)$ would not be nonnegative for all t. In fact, if $t_i \neq t_0$ then $\lambda_{t_0} < 0$ and similarly, if $t_j \neq t_n$ then $\lambda_{t_n} < 0$. The proof of the fact that λ_t must be of the form $\lambda_t = \frac{1}{2}(t_j - t)(t - t_i)$ is given in the Appendix.*

5. Unanticipated rate of return

In this Section we derive a few corollaries resulting from the formula (11) and Theorem 1. An example is also included.

THEOREM 5.1 *The unanticipated rate of return on the K – immunization bond portfolio Z^* due to proportional shifts in spot rates is approximately equal to*

$$R^u_{Z^*} = \frac{1}{2}[K(t_n + t_0 - K) - t_0 t_n]\left(\frac{h_{t_0}}{1+y_{t_0}}\right)^2 + \left(1 + \frac{h_{t_0}}{1+y_{t_0}}\right)^{-K} - 1 \quad (20)$$

Proof. Based on (11) it is enough to demonstrate that

$$\max\left[V(Z) - \frac{1}{2}K(K+1)\right] = \frac{1}{2}[K(t_n + t_0 - K) - t_0 t_n]. \quad (21)$$

Taking into account the formulae for $x_t^{Z^*}$ given in (14) and using the definition of convexity for a bond portfolio (see (7)) we have

$$V(Z) = \frac{1}{2}\left[t_0(t_0+1)\frac{t_n - K}{t_n - t_0} + (t_n+1)\frac{K - t_0}{t_n - t_0}\right].$$

By virtue of Theorem 1, the maximum in (21) is attained at $Z = Z^*$. Therefore, it is sufficient to prove that $V(Z^*) - \frac{1}{2}K(K+1)$ equals the right hand side of (21). Towards this end, observe that

$$\begin{aligned}
&2(t_n - t_0)\left[V(Z^*) - \frac{1}{2}K(K+1)\right] = \\
&K[t_n^2 + t_n - t_n^2 - t_0 - (K+1)(t_n - t_0)] + (t_n t_0^2 - t_n^2 t_0) \\
&= (t_n - t_0)[K(t_n + t_o - K) - t_n - t_0].
\end{aligned}$$

The proof is complete. ■

THEOREM 5.2 *The unanticipated rate of return on a K – immunization bond portfolio Z after K years due to proportional shifts in spot rates (if no other changes in the spot rates occur in the meantime) equals*

$$R_Z^K = \left[V(Z) - \frac{1}{2}K(K+1)\right]\left(\frac{h_{t_0}}{1+y_{t_0}}\right)^2 (1+y_k+h_k)^K + [(1+y_K)^K - 1], \quad (22)$$

where $R_0^K = [(1+y_K)^K - 1]$ is the rate of return at the maturity on an imaginary or real zero–coupon bond O maturing at time $t = K$.

Proof. Since the investment value of the bond O equals $C(1+y_k)^{-K} = C_K^*$ where C is the face value of O, we have

$$R_0^K = \frac{C - C(1+y_k)^{-K}}{C(1+y_K)^{-K}} = (1+y_K)^K - 1. \quad (23)$$

To prove the first part of the theorem, let us observe that

$$R_Z^K = \frac{\tilde{P}_Z(1+y_K+h_K)^K - P_Z}{P_Z},$$

where $\tilde{P}_Z$, the investment value of Z instantly after the change in the spot rates, equals

$$\tilde{P}_Z = \tilde{P}_0 + P_Z[V(Z) - V(O)]\left(\frac{h_{t_0}}{1+y_{t_0}}\right)^2. \tag{24}$$

To demonstrate (23) let us suppose without loss of generality that $P_Z = P_0 = C_K^*$. Besides, $D(Z) = K = D(0)$. We already know from the very end of Section 2, (see also Zaremba, 1995, Lemma 3) that in such an instant $V(Z) > V(O) = \frac{1}{2}K(K+1)$ and, by Equation (9),

$$\frac{dP_Z}{P_Z} - \frac{dP_O}{P_O} = [V(Z) - V(O)]\left(\frac{h_{t_0}}{1+y_{t_0}}\right)^2 > 0.$$

Since $dP_Z - dP_0 = \tilde{P}_Z - \tilde{P}_0$, we infer that

$$\tilde{P}_Z - P_0 = P_Z[V(Z) - V(O)]\left(\frac{h_{t_0}}{1+y_{t_0}}\right)^2, \tag{25}$$

what proves (23). Let us note that after K years the investment value of Z will be equal to $\tilde{P}_Z(1+y_K+h_K)^K$, while the investment value of the bond O will be equal to $\tilde{P}_0(1+y_K+h_K)^K = C$, the latter meaning $\tilde{P}_0 = C(1+y_K+h_K)^{-K}$. Finally,

$$R_Z^K = \frac{\left\{C(1+y_K+h_K)^{-K} + C(1+y_K)^{-K}[V(Z)-V(O)]\left(\frac{h_{t_0}}{1+y_{t_0}}\right)^2\right\}}{C(1+y_K)^{-K}} \cdot$$
$$\cdot (1+y_K+h_K)^K - C(1+y_K)^{-K}$$

what completes the proof of (22). ■

COROLLARY 5.1 *The unanticipated rate of return after K years on the portfolio Z^* with the highest convexity due to proportional shifts in spot rates (if no other changes in the spot rates occur in the meantime) is approximately equal to*

$$R_{Z^*}^K = \frac{1}{2}[K(t_n + t_0 - K) - t_0 t_n]\left(\frac{h_{t_0}}{1+y_{t_0}}\right)^2 (1+y_K+h_K)^K + R_0^K, \tag{26}$$

where $R_{Z^*}^K = (1+y_K)^K - 1$ is the rate of return at maturity on an imaginary or real zero – coupon bond O maturing at $t = K$.

The proof follows from (21) and (22).

EXAMPLE 5.1 *Let us suppose that* $K = 4$, $t_0 = \frac{8}{52}$ *(the shortest bond is an 8 – week treasury bill),* $t_n = 10$ *(the longest zero–coupon bond matures after 10 years) and* $y_{t_0} = .34$. *Setting* $h_{t_0} = -.04$ *and assuming that spot rates changed proportionally, we know that the unanticipated rate of return on the K – immunization portfolio* Z^* *due to proportional shifts in spot rates is given by the formula (20).*

Therefore, $R^u_{Z^*} = 11.53846(.02985074)^2 + (1.1288748 - 1) = .139156356$, which is 13.9156%. Note that the first term depending solely on the convexity represents 7.9779% of the entire $R^u_{Z^*}$ (so much can be gained by choosing the portfolio Z^* with the highest convexity).

Finally, suppose an investor is unfamiliar with Theorem 2 and applies a "naive" strategy by purchasing a K – immunization focused portfolio $\tilde{Z} = (O_1, O_2)$ with O_1 and O_2 being zero–coupon bonds with maturities of 2 and 5 years, respectively. By making use of (11), we obtain

$$R^u_{\tilde{Z}} = (10.5 - 10)(.02985074)^2 + (1.01288748 - 1) = .12932,$$

that is: 12.932%. It is now obvious that the last choice of $\tilde{Z}$ was very unfortunate because the first term depending solely on convexity is now over 23 times less than it was in case of $Z * 11.53846(10.5 - 10)^{-1} > 23$.

References

ELTON, E., GRUBER, M. (1995) *Modern Portfolio Theory and Investment Analysis.* Wiley.

FISHER, L., WEIL, R. (1971) Coping with the risk of interest rat fluctuations returns to bondholders from naive and optimal strategies. *J. Business*, **44**, 408–431.

INGERSOLL, J.E., SKELTON, J., WEIL, R. (1978) Duration forty years later. *J. Financial & Quantitative Analysis*, **13**, 627–650.

MACAULAY, F. (1983) *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields and Stock Prices in the United States since 1856.* National bureau of Economic Research, New York.

PRISMAN, E.Z. (1986) Immunization as a maximin strategy. *J. Banking & Finance*, **10**, 491–509.

PRISMAN, E.Z., SHORES, M.R. (1988) Duration measures for specific term structure estimations and applications to bond portfolio immunization. *J. Banking & Finance*, **12**, 493-504.

REDINGTON, F. (1952) Review of the principles of life – office valuations. *J. Inst. Actuaries*, **3**, 286–315.

ZAREMBA, L.S. (1995) *Solution of immunization problem in case of proportional spot rates shifts.* Working paper, WP-3-1995, Systems Research Inst., Polish Academy of Sciences.

Appendix

We will demonstrate here that the coefficients λ_t, appearing in the Kuhn-Tucker conditions are of the form

$$\frac{1}{2}(t_j - t)(t - t_i) \tag{27}$$

Proof. It was shown in the proof of Theorem 1 that just two of the numbers x_t^Z are different from zero, say $x_{t_i}^Z$, $x_{t_j}^Z$, resulting in $\lambda_{t_i} = 0$, $\lambda_{ij} = 0$ due to the equality $\lambda_t x_t^Z = 0$ valid for all $t \in \{t_0, t_1, \ldots, t_n\}$. It follows from (17) that

$$0 = \frac{1}{2}(t_i^2 + t_i) + \mu_1 t_i + \mu_2 = \frac{1}{2}(t_j^2 + t_j) + \mu_1 t_j + \mu_2 \tag{28}$$

and consequently

$$\mu_2 = -\frac{1}{2}(t_i^2 + t_i) - \mu_1 t_i + \mu_2 = -\frac{1}{2}(t_j^2 + t_j) + \mu_1 t_j, \tag{29}$$

the latter resulting in

$$\mu_1 = -\frac{1}{2}(t_j^2 - t_i + 1). \tag{30}$$

Having proven this, we infer from (29) that $\mu_2 = -\frac{1}{2}(t_i^2 + t_i) + \frac{1}{2}t_i(t_j + t_i + 1) = t_i t_j$.

Finally, it follows from (17) and the obtained formulae for μ_1, μ_2 that

$$\lambda_t = -\frac{1}{2}(t^2 + t) - \mu_1 t - \mu_2 = \frac{1}{2}(t_j - t)(t - t_i). \tag{31}$$

■