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A comparative study of the fuzzy linear model and the DEA in evaluation of efficiency of the DMUs

by

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Abstract: Evaluation of efficiency for every DMU (Decision Making Unit) in a company is a very important issue. Thus, the studies of evaluation of efficiency are being actively carried out on the basis of production functions. Until now, loglinear production function (Cobb-Douglas model) has been used for evaluation. This loglinear model evaluates DMUs by measuring the average. Recently, DEA (Data Envelopement Analysis) has been applied as the available method involving, for example, the CCR (Charnes-Cooper-Rhodes) and the BCC (Banker-Charnes-Cooper) models. However, the DEA models do not have the lower limit on the production set, but only the upper limit. Since, however, we consider that the real problems have the production set extending from the lower limit to the upper limit, we propose the possibility production function obtained by introducing the fuzziness into the loglinear production function. As we try to evaluate the efficiency by this possibility production function we can obtain two efficiency ratings: for the upper and lower limits. Though both DEA and fuzzy loglinear model include all the DMU data, in the DEA approach we obtain the lower limit on inputs for the given output, while in the fuzzy loglinear approach we obtain the possibility maximum output for the given inputs. By making full use of the difference between the two approaches, we try to compare the DEA and the fuzzy loglinear model in the evaluation of efficiency of the DMUs. In terms of the two efficiency ratings, fuzzy loglinear model can yield more exact ranking for every DMU than DEA. Generally, when a DMU has efficiency less that 1 by fuzzy loglinear analysis, it means that there is a possibility of obtaining larger output for the given inputs.

Keywords: DEA, BCC model, fuzzy loglinear model, possibility production set

1. Introduction

Evaluation of efficiency for every DMU (Decision Making Unit) in a company actively carried out on the basis of production functions. Until now, loglinear production function (Cobb-Douglas model) has been used in the evaluation methods, Sato (1975). This loglinear model evaluates DMUs by measuring the average. Recently, DEA (Data Envelopement Analysis) has been applied as the available method involving, for example, the CCR (Charnes-Cooper-Rhodes) and the BCC (Banker-Charnes-Cooper) models, Charnes, Cooper, Rhodes (1978). However, the DEA models do not have the lower limit of the production set, but only the upper limit. Since, however, we consider that the real problems have the production set extending from the lower limit to the upper limit, we propose the possibility production function obtained by introducing the fuzziness into the loglinear production function, Watada, Morimoto (1992), Uemura, Kobayashi, Hiro (1996). As we try to evaluate the efficiency by this possibility production function we can obtain two efficiency ratings: for the upper and lower limits, Uemura, Kobayashi, Hiro (1996). Though both DEA and fuzzy loglinear model include all the DMU data, in the DEA approach we obtain the lower limit on inputs for the given output, while in the fuzzy loglinear approach we obtain the possibility maximum output for the given inputs. By making full use of the difference between the two approaches, we try to compare the DEA and the fuzzy loglinear model in the evaluation of efficiency of the DMUs. In terms of the two efficiency ratings, fuzzy loglinear model can yield more exact ranking for every DMU than DEA. Generally, when a DMU has efficiency less that 1 by fuzzy loglinear analysis, it means that there is a possibility of obtaining larger output for the given inputs. We show this difference characteristics for the two approaches in Fig. 1. By making full use of the difference of the two approaches, we try to compare DEA and fuzzy loglinear model in the evaluation of efficiency for DMUs.

2. Evaluation of efficiency with the loglinear model

The loglinear regression analysis provides the basis for evaluation by addressing the balance far from the loglinear regression line. The Cobb-Douglas model, Charnes, Cooper, Rhodes (1978) is formulated as follows:

$$Q = X_0 \cdot A_1^{x_1} \cdot A_2^{x_2} \cdots A_n^{x_n} \tag{1}$$

where Q is single output of DMU, and A_i $(i = 1, \dots, n)$ are multiple inputs of the DMU.

By taking the logarithm of (1), we transform this model into the loglinear regression model:

$$q = x_0 + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \tag{2}$$

where $x_0 = \ln X_0$, and $a_i = \ln A_i$ $(i = 1, \dots, n)$.

Now, let us denote the data of DMU_j $(j = 1, \dots, m)$ as $(q_j, A_{1j}, A_{2j}, \dots, A_{nj})$

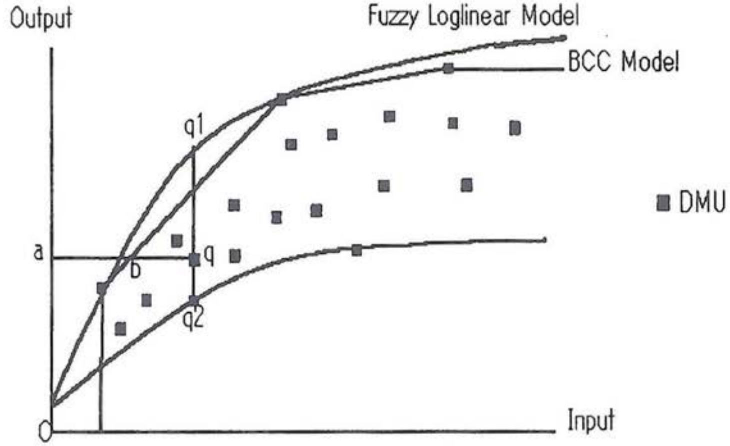

Figure 1. The image of evaluation according to the two models

The parameters a_i are identified by the least squares method. This loglinear model evaluates DMUs by measuring the average.

3. Evaluation of efficiency by the fuzzy loglinear model

Now introduce fuzziness into the loglinear model. The values of x_i in the model (1) are extended using the $L-L$ fuzzy parameter $\tilde{x}_i = (x_i, d_i)_L$. Because the object of our study is the construction of the evaluation of the efficiency for DMUs, it is natural for the possibility production function to have the upper curve and the lower curve, so as to include all the inputs of DMUs corresponding to the same source point of the output axis. Therefore, we set X_0 as a non-fuzzy value. Using the notation of Section 2 so as to avoid unnecessary complications, we define the possibility production function as follows:

$$Q = X_0 \cdot A_1^{\tilde{x}_1} \cdot A_2^{\tilde{x}_2} \cdots A_n^{\tilde{x}_n} \tag{3}$$

By taking the logarithm of (3), we transform this model into the fuzzy loglinear regression model:

$$q = x_0 + a_1\tilde{x}_1 + a_2\tilde{x}_2 + \cdots + a_n\tilde{x}_n \tag{4}$$

By the extension principle, the resulting model is obtained as follows:

$$\begin{aligned} &x_0 + a_1\tilde{x}_1 + a_2\tilde{x}_2 + \cdots + a_n\tilde{x}_n \\ &= \left(x_0 + \sum^{n} a_{ij}x_i, \sum^{n} a_{ij}d_i \right) \end{aligned} \tag{5}$$

Using the concept of identification of fuzzy parameters in fuzzy regression analysis, Tanaka (1991), the fuzzy parameters in (4) are identified via the following mathematical programming problem:

$$\begin{aligned} &\min \sum_{i=1}^{n} a_{ij} d_i \\ &\text{s.t. } x_0 + \sum_{i=1}^{n} a_{ij} x_i + |L^{-1}(h)| \cdot \sum_{i=1}^{n} a_{ij} d_i \geq q_j \\ &\quad\;\; x_0 + \sum_{i=1}^{n} a_{ij} x_i + |L^{-1}(h)| \cdot \sum_{i=1}^{n} a_{ij} d_i \leq q_j \\ &\quad\;\; x_i \geq 0, \;\; i = 1, \ldots, n \end{aligned} \tag{6}$$

where h is the level of possibility proper for a given decision maker, and $L(\cdot)$ is the form of the $L-L$ fuzzy number. Though it is usual that x_i is not constrained in fuzzy regression analysis, in the identification of fuzzy parameters in (6) x_i is bigger than 0, because of the construction of the possibility production set.

It is generally accepted to use the possibility interval of output for $h = 0$ in fuzzy regression analysis. However, because the object of our study is to obtain the efficiency ratings, these ratings are obtained from the ratios of output to the upper and lower limit. According to the following theorems, the ratios of output to the upper limit and lower limit are constant for the given h level. Therefore, in this paper the h level is fixed at 0.95.

THEOREM 3.1 *(Theorem 9.4, Tanaka (1991)) The optimal solution $A^*_{h'}$ of the fuzzy regression analysis for h' level is obtained from A^*_h for h level as follows:*

$$A^*_{h'} = (\alpha^*_H, c^*_h |L^{-1}(h)| \setminus |L^{-1}(h')|)_L \tag{7}$$

where $A^*_h = (\alpha^*_h, c^*_h)_L$.

THEOREM 3.2 *By Theorem 9.4 in Charnes, Cooper, Rhodes (1978), the possibility interval of the identified output is constant for the given h level.*

Proof. The h level set in A^*_h is $[\alpha^*_h \pm c^*_h |L^{-1}(h)|]$. By Theorem 1, the h' level set in $A^*_{H'}$ is obtained as follows:

$$[\alpha^*_h \pm c^*_h |L^{-1}(h')| |L^{-1}(h)| \setminus |L^{-1}(h')|] = [\alpha^*_h \pm c^*_h |L^{-1}(h)|] \tag{8}$$

THEOREM 3.3 *(Existence of efficiency and non-efficiency of DMUs) For every j, $Q_j \leq \widehat{QU}_j$, and $\exists ku$, $Q_{ku} = \widehat{QU}_{ku}$. Furthermore, for every j, $Q_j \geq \widehat{QL}_j$, and $\exists kl$, $Q_{kl} = \widehat{QL}_{kl}$, where $[\widehat{QL}_j, \widehat{QU}_j]$ is the estimated output interval defined*

Definition 3.1 *(The efficiency rating for DMU_j according to the possibility production function.) The efficiency rating for DMU_j equals $\frac{Q_j}{\widehat{QU_j}}$ and the non-efficiency rating for DMU_j is $\frac{\widehat{QU_j}}{Q_j}$.*

Lemma 3.1 *(Improvement of the non-efficient DMU.) In order to improve the non-efficient DMU_j, we can take $A_{ij}^{xu_i}$ instead of A_{ij}, where xu_i is the upper limit of the possibility interval for the fuzzy value of $\tilde{x}_i$ in fuzzy loglinear model (4).*

4. DEA based on the possibility production set

The first formulation of DEA was fractional mathematical programming meant for direct estimation of efficiency of the DMUs, Sato (1975). However, along with this direct approach, DEA has also been formulated on the basis of construction of the possibility production set, Tone (1993). In this section we explain the DEA approach based on the possibility production set.

Assume that we have the input set $X_j = (x_{j1}, \ldots, x_{jm})$ and the output set $Y_j = (y_{j1}, \ldots, y_{js})$ for the DMU_j $(j = 1, \ldots, n)$. We set the input vector as $\vec{X} = [X_1, \ldots, X_n]$ and the output vector as $\vec{Y} = [Y_1, \ldots, Y_n]$. The input and output vectors are supposed to take positive values. Using the data set of DMUs, $(\vec{X}, \vec{Y})$, we define the possibility production set (x, y) as the set satisfying the following constraints:

$$
\begin{aligned}
x &\geq \vec{X}\lambda \\
y &\leq \vec{Y}\lambda \\
\lambda &\geq 0 \\
L &\leq e^T\lambda \leq U
\end{aligned}
\tag{9}
$$

where $x \in R^m$, $y \in R^s$, $\lambda \in R^n$, and $e^T = (1, \ldots, 1)$.

By (9), we constrain the possibility production set. Therefore, various DEA-generated models resulting form it have been proposed by the variation of L and U. Specially, when $L = 0$, $U = \infty$, this model is the CCR (Charnes-Cooper-Rhodes) model, and when $L = U = 1$, it is the BCC (Banker-Charnes-Cooper) model. In this paper we focus on the BCC model, as we compare the DEA and the fuzzy loglinear model in terms of the possibility production set. The BCC model is formulated as follows:

BCC model

$$
\begin{aligned}
&\min \Theta - e(e^T s^+ + e^T s^-) \\
&\text{s.t. } \Theta x_0 - s^+ = \vec{X}\vec{\lambda} \\
&y + s^- = \vec{Y}\vec{\lambda}
\end{aligned}
\tag{10}
$$

	Inputs		Inputs	Fuzzy loglinear model		BCC Model
	A_1	A_2	Q	Efficiency	Inefficiency	D-Efficiency
Market1	3414	5280	378760	0.7724	0.8968	0.9304
Market2	21475	26030	2541518	0.8668	0.7468	1.0000
Market3	15184	12860	1538742	1.0000	0.6677	1.0000
Market4	3477	3800	269772	0.7026	1.0000	0.6963
Market5	4849	4700	401516	0.8018	0.8685	0.7828
Market6	12603	12340	1147413	0.8156	0.6453	0.8245
Market7	10646	10720	1032815	0.8616	0.7804	0.8648
Market8	7952	9710	1015017	1.0000	0.6757	1.0000
Market9	7469	7180	611233	0.6023	0.8852	0.7631
Market10	2162	2890	198807	0.7401	0.9608	0.8444
Market11	1268	1310	124524	1.0000	0.7357	1.0000
Market12	3095	2770	239558	0.8237	0.8651	0.8121

Table 1. Evaluation of efficiency for some supermarkets according to the two models

$$e^T \vec{\lambda} = 1$$
$$\vec{\lambda}, s^+, s^- \geq 0$$

where $\Theta = \vec{Y_0} \cdot \vec{\lambda}$, and s^+, s^- are the slack vectors. We call $\Theta^* = 1$, which is the solution to (10), D-efficiency.

5. An example of application

We applied the BCC model and the fuzzy loglinear model to evaluation of efficiency for some big supermarkets in Japan. Let us take two inputs: employment, A_1, and total sales area, A_2, and one output: total sales Q.

First, the fuzzy loglinear production function is identified with the respective data (see Table 1) as follows:

$$Q = 56.657 \cdot A_1^{(0.313, 0.000)_L} \cdot A_2^{(0.739, 0.021)_L} \tag{11}$$

The data and the efficiency ratings obtained from the BCC and the fuzzy loglinear models are shown in Table 1. Having evaluated the efficiency with the two models, we may obtain the following four characteristic situations:

1. If we have D-efficiency in the BCC model and the efficiency rating in the fuzzy loglinear model equals one, we can judge that the given DMU is perfect.
2. We always have D-efficiency in the BCC model when the efficiency rating

3. When a DMU features D-efficiency in the BCC, but less than 1 in the fuzzy loglinear model, there is a possibility for the DMU to achieve a bigger output for the current input.
4. When we make full use of the inefficiency ratings if two DMUs have the same efficiency rating in the BCC model, the fuzzy loglinear model can provide a more exact ranking for the DMUs.

6. Conclusions

Because we consider that a more realistic production function should have both upper and lower limits, we propose the fuzzy loglinear model by introducing the fuzziness into the loglinear model. Each fuzzy parameter in the fuzzy loglinear production function is identified very easily by the use of the concept of fuzzy regression analysis. As we try to evaluate efficiency by this fuzzy loglinear production function, the fuzzy loglinear regression analysis can yield two efficiency ratings: for the upper and lower limits. After having applied the BCC model and the fuzzy loglinear model to the evaluation of efficiency for some big supermarkets in Japan, we show that there is a possibility that a DMU who obtains D-efficiency in the BCC, but less than 1 in the fuzzy loglinear model may get a bigger output for the given inputs. Furthermore, by making full use of inefficiency rating, we can show that the fuzzy loglinear model provides more exact ranking for DMUs than the BCC model.

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