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Changes of the set of efficient solutions by extending the number of objectives and its evaluation

by

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Abstract: In this paper the vector optimization problem (P) with continuous and convex objective functions on a compact convex feasible set is considered. We form a new vector optimisation problem $(\widetilde{P})$ from (P) by adding an objective function to the problem (P). The necessary and sufficient conditions for the sets of efficient solutions of these two problems to be equal are given. In the case where the set of efficient solutions of the problem $(\widetilde{P})$ contains that of (P), we also suggest how the difference between the sets of efficient solutions of the problems $(\widetilde{P})$ and (P) might be evaluated. Examples are given to illustrate our results.

Keywords: vector optimization problem, efficient solutions, nonessential objective function.

1. Introduction and notations

In this paper, we study the following vector optimization problem

$$P = (X, \mathbf{F}^n, R), \tag{1}$$

where

1) $X \subset \mathbf{R}^k$ is the feasible set;

2) $\mathbf{F}^n(\mathbf{x}) = [f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})]^T : X \to \mathbf{R}^n (n > 1)$ is a vector-valued function, each is called an objective function;

3) $f_i (i \in \{1, \ldots, n\})$ is the objective space and

$$Y = \mathbf{F}^n(X) = \{\mathbf{y} \in \mathbf{R}^n \mid \mathbf{F}^n(\mathbf{x}) = \mathbf{y}, \mathbf{x} \in X\}$$

is the feasible objective set;

4) R is the binary relation on $\mathbf{R}^n$:

$$\mathbf{y}^1 = [y_1^1, \ldots, y_n^1], \mathbf{y}^2 = [y_1^2, \ldots, y_n^2]^T \in \mathbf{R}^n$$

$$(\mathbf{y}^1, \mathbf{y}^2) \in R \Leftrightarrow \mathbf{y}^1 \leq \mathbf{y}^2 \Leftrightarrow \forall i \in \{1, \ldots, n\} :$$

$$y_i^1 \leq y_i^2 \wedge \exists i \in \{1, \ldots, n\} : y_i^1 < y_i^2.$$

The solution of the problem (1) consists finding all solutions that are efficient in the sense of the following definition.

DEFINITION 1 *A vector $\mathbf{x}^0 \in X$ is said to be an efficient solution of the problem (1) iff there exists no $\mathbf{x} \in X$ such that $\mathbf{F}^n(\mathbf{x}) \leq \mathbf{F}^n(\mathbf{x}^0)$.*

The set of efficient solutions of the problem (1) is denoted by $E(X, \mathbf{F}^n)$. Throughout this paper we assume that:

A.1. X is a nonempty, compact and convex set.

A.2. Each $f_i (i \in \{1, \ldots, n\})$ is a continuous and convex function.

We form a new vector optimisation problem $(\widetilde{P})$ from (P) by adding an objective function f_{n+1} to the problem (P). If the set of efficient solutions of the problem $(\widetilde{P})$ equals that of (P), then the objective function f_{n+1} is called nonessential. The concept of nonessential objective function was proposed by Gal and Leberling (Gal and Leberling, 1977; Gal, 1980). They investigated, however, only a linear vector optimization problem. In this paper, the concept of nonessential objective functions is generalised. If the problem $(\widetilde{P})$ satisfies the assumptions A.1, A.2, we show necessary and sufficient conditions for the objective function f_{n+1} to be nonessential (Section 2). In Section 3, the evaluation of the difference between the sets of efficient solutions of the problems $(\widetilde{P})$ and (P) is proposed and the sufficient conditions for that difference to be contained in the boundary of X are given (Corollary 2).

Before going further, let us introduce some notations. For a set A in $\mathbf{R}^k$, clA, $intA$, bdA are respectively the closure, the interior and the boundary of A. The Euclidean norm of a vector $\mathbf{x} \in \mathbf{R}^k$ is denoted by $\|\mathbf{x}\| = (\sum_{i=1}^k x_i^2)^{\frac{1}{2}}$. The Euclidean distance function between a point $\mathbf{x}^0$ and a set A is denoted by $d(\mathbf{x}^0, A) = \inf_{\mathbf{x} \in A} \|\mathbf{x}^0 - \mathbf{x}\|$. The symbol $B(\mathbf{x}^0, \delta)$ denotes an open ball with a centre $\mathbf{x}^0$ and a radius $\delta > 0$, i.e. $B(\mathbf{x}^0, \delta) = \{\mathbf{x} \in \mathbf{R}^k \mid \|\mathbf{x}^0 - \mathbf{x}\| < \delta\}$.

2. Nonessential functions

Let $f_{n+1} : X \to \mathbf{R}$ be a continuous convex function and let $E(X, \mathbf{F}^{n+1})$ denote the set of efficient solutions of the problem

$$\widetilde{P} = (X, \mathbf{F}^{n+1}, R), \tag{2}$$

where $\mathbf{F}^{n+1}(\mathbf{x}) = [f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}), f_{n+1}(\mathbf{x})]^T$.

With these notations we introduce the definition of the nonessential objective function.

DEFINITION 2 *The objective function* f_{n+1} *is said to be nonessential if*

$$E(X, \mathbf{F}^{n+1}) = E(X, \mathbf{F}^n).$$

Let $E(X, f_{n+1})$ denote the set of solutions of the following single objective optimisation problem

$$Min\{f_{n+1}(\mathbf{x}) : \mathbf{x} \in X\}.$$

In other words

$$E(X, f_{n+1}) = \{\mathbf{x}^0 \in X \mid \forall \mathbf{x} \in X : f_{n+1}(\mathbf{x}^0) \leq f_{n+1}(\mathbf{x})\}.$$

THEOREM 1 *Let the problem (1) satisfy the assumptions A.1, A.2. Then the objective function* f_{n+1} *is nonessential if and only if the following three conditions hold:*

i) $E(X, \mathbf{F}^n) \subset E(X, \mathbf{F}^{n+1})$;
ii) $E(X, \mathbf{F}^n) \cap E(X, f_{n+1}) \neq \emptyset$;
iii) $\forall \mathbf{x} \in X \setminus E(X, \mathbf{F}^n) \exists \mathbf{x}' \in \mathbf{R}^k : \mathbf{F}^{n+1}(\mathbf{x}') \leq \mathbf{F}^{n+1}(\mathbf{x})$.

Proof. The condition i) is immediate from Definition 2. To prove condition ii) is suffices to note that under the assumptions A.1, A.2 the set $E(X, \mathbf{F}^{n+1}) \cap E(X, f_{n+1})$ is nonempty (Galas et al., 1987). Hence, by the definition of nonessential objective function, also the set $E(X, \mathbf{F}^n) \cap E(X, f_{n+1})$ is nonempty. For the condition iii), suppose to the contrary that there is $\mathbf{x}^0 \in X \setminus E(X, \mathbf{F}^n)$ with the property

$$\neg \exists \mathbf{x}' \in \mathbf{R}^k : \mathbf{F}^{n+1}(\mathbf{x}') \leq \mathbf{F}^{n+1}(\mathbf{x}^0).$$

In particular

$$\neg \exists \mathbf{x}' \in X : \mathbf{F}^{n+1}(\mathbf{x}') \leq \mathbf{F}^{n+1}(\mathbf{x}^0).$$

Consequently, $\mathbf{x}^0 \in E(X, \mathbf{F}^{n+1})$. This is a contradiction because, by the assumption $E(X, \mathbf{F}^n) = E(X, \mathbf{F}^{n+1})$, $\mathbf{x}^0 \in X \setminus E(X, \mathbf{F}^{n+1})$.

"$\Leftarrow$" In view of condition i) we shall show only that $E(X, \mathbf{F}^{n+1}) \subset E(X, \mathbf{F}^n)$. Set $\mathbf{x} \in X \setminus E(X, \mathbf{F}^n)$. Suppose that $\mathbf{x} \in intX$. Then

$$\exists \delta > 0 : B(\mathbf{x}, \delta) \subset X.$$

By condition ii) there exists $\mathbf{x}^1 \in \mathbf{R}^k$ such that

$$\forall i \in \{1, \ldots, n+1\} : f_i(\mathbf{x}^1) \leq f_i(\mathbf{x}) \wedge \exists i \in \{1, \ldots, n+1\} : f_i(\mathbf{x}^1) < f_i(\mathbf{x}) \quad (3)$$

Let $\mathbf{x}^2 = (1 - \lambda)\mathbf{x} + \lambda \mathbf{x}^1$, where $0 < \lambda < 1$ is selected such that $\mathbf{x}^2 \in B(\mathbf{x}, \delta)$. The convexity of $f_i (i \in \{1, \ldots, n+1\})$ and (3) imply that $\mathbf{F}^{n+1}(\mathbf{x}^2) \leq \mathbf{F}^{n+1}(\mathbf{x})$. Hence, $\mathbf{x} \notin E(X, \mathbf{F}^{n+1})$ and in this case $E(X, \mathbf{F}^{n+1}) \subset E(X, \mathbf{F}^n)$. Suppose now that $\mathbf{x} \in bdX$. For $\mathbf{x}^0 \in E(X, \mathbf{F}^n) \cap E(X, f_{n+1})$ only one of the following cases holds:

a) $\mathbf{F}^n(\mathbf{x}^0) \leq \mathbf{F}^n(\mathbf{x})$,
b) neither $\mathbf{F}^n(\mathbf{x}^0) \leq \mathbf{F}^n(\mathbf{x})$ nor $\mathbf{F}^n(\mathbf{x}) \leq \mathbf{F}^n(\mathbf{x}^0)$.

In case a), since $\mathbf{x}^0 \in E(X, f_{n+1})$, we have $\mathbf{F}^{n+1}(\mathbf{x}^0) \leq \mathbf{F}^{n+1}(\mathbf{x})$. This means that $\mathbf{x} \notin E(X, \mathbf{F}^{n+1})$ and $E(X, \mathbf{F}^{n+1}) \subset E(X, \mathbf{F}^n)$.

In case b) we may assume without loss of generality that

$$f_i(\mathbf{x}^0) < f_i(\mathbf{x}), \quad i \in \{1, \ldots, l\}, \tag{4}$$

$$f_j(\mathbf{x}^0) \geq f_j(\mathbf{x}), \quad j \in \{l+1, \ldots, n\}, \tag{5}$$

$$f_{n+1}(\mathbf{x}^0) \leq f_{n+1}(\mathbf{x}), \tag{6}$$

where at least one of the inequalities (5) is strict. If there exists $\mathbf{x}^3 = (1-\lambda)\mathbf{x} + \lambda\mathbf{x}^0$, for some $0 < \lambda < 1$, such that

$$f_j(\mathbf{x}^3) \leq f_j(\mathbf{x}), \quad j \in \{l+1, \ldots, n\},$$

then by (4), (6) and the convexity of f_i, f_{n+1} we have

$$f_i(\mathbf{x}^3) < f_i(\mathbf{x}), \quad i \in \{1, \ldots, l\},$$

$$f_{n+1}(\mathbf{x}^3) \leq f_{n+1}(\mathbf{x}).$$

Moreover, since X is convex, $\mathbf{x}^3 \in X$. Thus $\mathbf{x} \notin E(X, \mathbf{F}^{n+1})$ and $E(X, \mathbf{F}^{n+1}) \subset E(X, \mathbf{F}^n)$. Assume now that there is an index $j \in \{l+1, \ldots, n\}$ such that

$$f_j(\mathbf{x}^3) > f_j(\mathbf{x}),$$

for all $\mathbf{x}^3 = (1-\lambda)\mathbf{x} + \lambda\mathbf{x}^0$, where $0 < \lambda < 1$. Since $\mathbf{x} \notin E(X, \mathbf{F}^n)$, there exists $\mathbf{x}^4 \in X$ with the property

$$\mathbf{F}^n(\mathbf{x}^4) \leq \mathbf{F}^n(\mathbf{x}). \tag{7}$$

If $f_{n+1}(\mathbf{x}^4) \leq f_{n+1}(\mathbf{x})$, then it is clear that $\mathbf{x} \notin E(X, \mathbf{F}^{n+1})$. If

$$f_{n+1}(\mathbf{x}^4) > f_{n+1}(\mathbf{x}), \tag{8}$$

then we may build the two-dimensional simplex $\Delta(\mathbf{x}, \mathbf{x}^0, \mathbf{x}^4)$, because the objective functions are convex and conditions (4)–(6), (7), (8) hold. Applying Darboux's Theorem (Engelking and Sieklucki, 1986) to the set $\Delta(\mathbf{x}, \mathbf{x}^0, \mathbf{x}^4)$ and each objective function, by condition iii), we conclude that there exists $\mathbf{x}^5 \in \Delta(\mathbf{x}, \mathbf{x}^0, \mathbf{x}^4)$ such that $\mathbf{F}^{n+1}(\mathbf{x}^5) \leq \mathbf{F}^{n+1}(\mathbf{x})$. Hence, since $\Delta(\mathbf{x}, \mathbf{x}^0, \mathbf{x}^4) \subset X$, we have $\mathbf{x} \notin E(X, \mathbf{F}^n)$. Consequently $E(X, \mathbf{F}^{n+1}) \subset E(X, \mathbf{F}^n)$. ■

It is interesting to observe that, in the case where $X \subset \mathbf{X}$, one of the conditions in Theorem 1 can be dropped.

LEMMA 1 *With the assumptions A.1, A.2, if $X \subset \mathbf{X}$, then the conditions ii) and iii) in Theorem 1 are equivalent.*

Proof. The proof is easy to obtain by applying the definition of an efficient solution and the properties of a convex function. ■

Generally, if $X \subset \mathbf{R}^k (k > 1)$, the lemma above is not true. The following example shows this fact.

EXAMPLE 1 The set X, the vector-valued function $\mathbf{F}^n(\mathbf{x})$ and the objective function $f_{n+1}(\mathbf{x})$ are given by

$$X = \{[x_1, x_2]^T \in \mathbf{R}^2 \mid x_1 + 4x_2 \leq 24, -x_1 - x_2$$
$$\leq -6, -x_2 \leq -1, 3x_1 + 2x_2 \leq 32\},$$
$$\mathbf{F}^2(\mathbf{x}) = [-x_2, -4x_1 - x_2]^T,$$
$$f_3(\mathbf{x}) = 5x_1 + 3x_2.$$

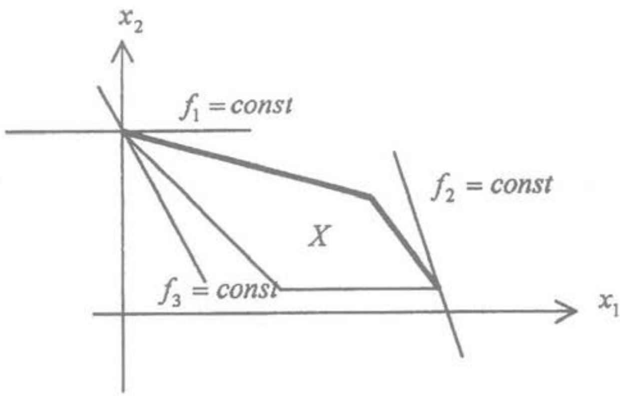

Figure 1. Condition ii) of Theorem 1 does not imply condition iii).

In this problem

$$E(X, \mathbf{F}^2) \cap E(X, f_3) = \{[0, 6]^T\},$$

but there is, for example, $\mathbf{x}^0 = [6, 1]^T \in X \setminus E(X, \mathbf{F}^2)$ such that

$$\neg \exists \mathbf{x} \in \mathbf{R}^2 : \mathbf{F}^3(\mathbf{x}) \leq \mathbf{F}^3(\mathbf{x}^0).$$

In Fig. 1, the set $E(X, \mathbf{F}^2)$ (bold line) and the set $E(X, \mathbf{F}^3) = X$, are illustrated. Of course, function f_3 is not nonessential.

Now let us change functions f_1 and f_3 into

$$f_1(\mathbf{x}) = -x_1 - 2x_2,$$
$$f_3(\mathbf{x}) = x_1 - 4x_2$$

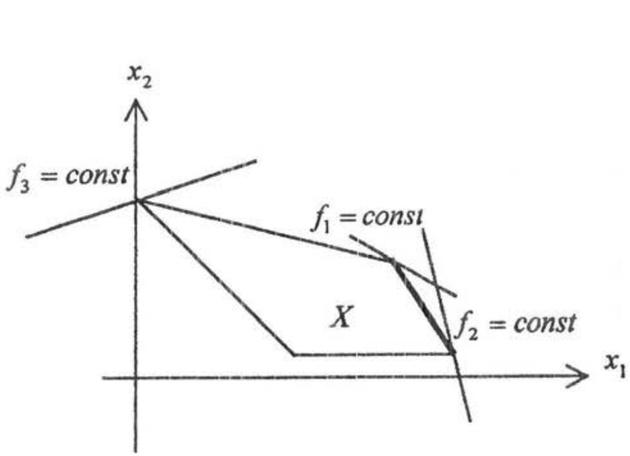

Figure 2. The condition iii) of Theorem 1 does not imply the condition ii).

In this case condition iii) holds, that is

$$\forall \mathbf{x} \in X \setminus E(X, \mathbf{F}^2) \exists \mathbf{x}' \in \mathbf{R}^2 : \mathbf{F}^3(\mathbf{x}') \leq \mathbf{F}^3(\mathbf{x}),$$

although $E(X, \mathbf{F}^2) \cap E(X, f_3) = \emptyset$. The bold line in Fig. 2 represents the set $E(X, \mathbf{F}^2)$. The set $E(X, \mathbf{F}^3)$ is the north-east edge of the feasible set. Again, function f_3 is not nonessential.

It should be noted that the condition i) of Theorem 1 is not obvious. But if the vector-valued function $\mathbf{F}^n(\mathbf{x})$ is one-to-one on the set $E(X, \mathbf{F}^n)$, then this condition holds (Gutenbaum and Inkielman, 1998).

Theorem 1 can fail if the objective functions are not convex.

EXAMPLE 2 The set X, the vector-valued function $\mathbf{F}^n$ and the objective function f_{n+1} are given by

$$X = \{x \in \mathbf{R} \mid -x \leq 0, x \leq 6\},$$
$$\mathbf{F}^2(x) = \left[x^2, \frac{1}{2}(x-4)^2 + 2\right],$$
$$f_3(x) = -\frac{2}{9}(x-3)^2 + 5.$$

Fig. 3 shows the set X and the objective functions. It is easy to see that

$$E(X, \mathbf{F}^2) = \{x \in \mathbf{R} \mid 0 \leq x \leq 4\},$$
$$E(X, \mathbf{F}^3) = X,$$
$$E(X, \mathbf{F}^2) \cap E(X, f_3) = \{0\}$$

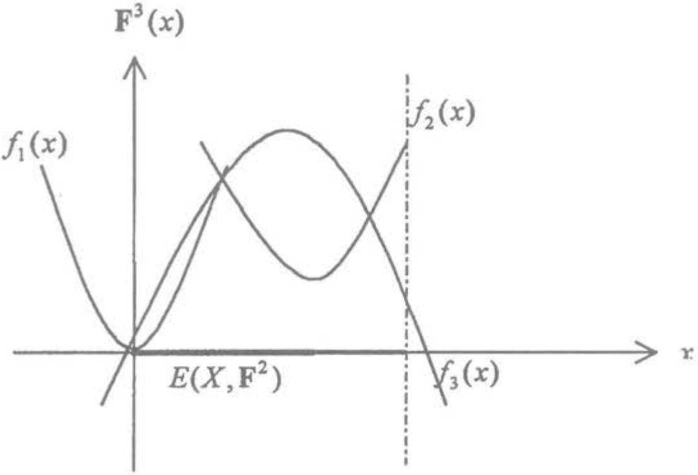

Figure 3. The conditions of Theorem 1 hold, but the function f_3 is not nonessential.

Hence, by Lemma 1, conditions of Theorem 1 hold. But the function f_3 is not nonessential. Obviously f_3 is not convex.

3. The evaluation of extension of the set of efficient solutions

Now we mention some consequences of Theorem 1 and Lemma 1.

COROLLARY 1 *With the assumptions A.1, A.2 the following implication holds*

$$E(X,\mathbf{F}^{n+1}) \setminus E(X,\mathbf{F}^n) = \emptyset \Rightarrow \inf_{\mathbf{x}\in E(X,\mathbf{F}^n)} \{d(\mathbf{x}, E(X, f_{n+1}))\} = 0.$$

Proof. The proof is immediate from Theorem 1 and the definition of Euclidean distance function. ■

It is worth noticing that with the assumptions A.1, A.2 the condition $inf_{\mathbf{x}\in E(X,\mathbf{F}^n)}\{d(\mathbf{x}, E(X, f_{n+1}))\} = 0$ is necessary, but not sufficient for the set $E(X,\mathbf{F}^{n+1}) \setminus E(X,\mathbf{F}^n)$ to be empty. As an example, in $\mathbf{R}$, let

$$X = \{x \in \mathbf{R} \mid 0 \le x \le 6\}$$

and

$$E(X,\mathbf{F}^n) = \{x \in X \mid 1 < x < 3\}, \quad E(X,\mathbf{F}^{n+1}) = \{3\}.$$

It can be seen that $E(X,\mathbf{F}^{n+1}) \setminus E(X,\mathbf{F}^n) = \{3\} \neq \emptyset$, but

$$\inf_{\mathbf{x}\in E(X,\mathbf{F}^n)} \{d(\mathbf{x}, E(X, f_{n+1}))\} = 0.$$

However, the following proposition holds

PROPOSITION 1 *Let the problem (1) satisfy the assumptions A.1, A.2 and let the set $E(X, \mathbf{F}^n)$ be closed. If $X \subset \mathbf{R}$ or the condition iii) of Theorem 1 holds, then*

$$E(X, \mathbf{F}^{n+1}) \setminus E(X, \mathbf{F}^n) = \emptyset \Leftrightarrow \inf_{\mathbf{x} \in E(X, \mathbf{F}^n)} \{d(\mathbf{x}, E(X, f_{n+1}))\} = 0.$$

Proof. In view of Corollary 1 we shall show only the implication "$\Leftarrow$ ". For this, observe that the set $E(X, \mathbf{F}^n)$ is compact, because it is closed and contained in the feasible set, which is compact. Furthermore the Euclidean distance function is continuous. Hence if $\inf_{\mathbf{x} \in E(X,\mathbf{F}^n)}\{d(\mathbf{x}, E(X, f_{n+1})\} = 0$, then there exists $\mathbf{x}^0 \in E(X, \mathbf{F}^n)$ such that $d(\mathbf{x}^0, E(X, f_{n+1})) = 0$. Moreover, since f_{n+1} is continuous, the set $E(X, f_{n+1})$ is closed. Therefore $\mathbf{x}^0 \in E(X, f_{n+1})$ and the set $E(X, \mathbf{F}^n) \cap E(X, f_{n+1})$ is nonempty. Finally, due to Theorem 1 (and Lemma 1 if necessary), $E(X, \mathbf{F}^n) \subseteq E(X, \mathbf{F}^{n+1})$ so the set $E(X, \mathbf{F}^{n+1}) \setminus E(X, \mathbf{F}^n)$ is empty. ∎

From Proposition 1 we conclude that, if the assumptions of that proposition hold, then we may use the number $\inf_{\mathbf{x} \in E(X,\mathbf{F}^n)}\{d(\mathbf{x}, E(X, f_{n+1}))\}$ as the evaluation of the difference between the sets of efficient solutions of the problems $(\widetilde{P})$ and (P).

In the case where $X \subset \mathbf{R}$ the number $\inf_{\mathbf{x} \in E(X,\mathbf{F}^n)}\{d(\mathbf{x}, E(X, f_{n+1}))\}$ has a very interesting property.

REMARK 1 *With the assumptions A.1, A.2, if $X \subset \mathbf{R}$, the set $cl(E(X, \mathbf{F}^{n+1}) \setminus E(X, \mathbf{F}^n))$ is a closed interval and its length equals*

$$\inf_{x \in E(X,\mathbf{F}^n)} \{d(x, E(X, f_{n+1}))\}.$$

It is rather obvious that if we drop the assumption $X \subset \mathbf{R}$, then the remark above becomes false. Nevertheless we give an example of this fact.

EXAMPLE 3 *The set X, the vector-valued function $\mathbf{F}^n(\mathbf{x})$ and the objective function $f_{n+1}(\mathbf{x})$ are given by*

$$X = \{[x_1, x_2]^T \in \mathbf{R}^2 \mid -x_2 + g(x_1) \le 0, \ \textit{where}$$

$$g(x_1) = \begin{cases} 0,5x_1^2 - 4x_1 + 12 & \textit{for } x_1 < 4 \\ 4 & \textit{for } x_1 \ge 4 \end{cases},$$

$$x_2 - 6 \le 0, -x_2 + x_1 - 3 \le 0\}$$

$$\mathbf{F}^2(x_1, x_2) = [(x_1 - 4)^2 + x_2^2 - 1, (x_1 - 7)^2 + x_2^2 - 2]^T$$

$$f_3(x_1, x_2) = (x_1 - 10)^2 + (x_2 - 6)^2 - 3$$

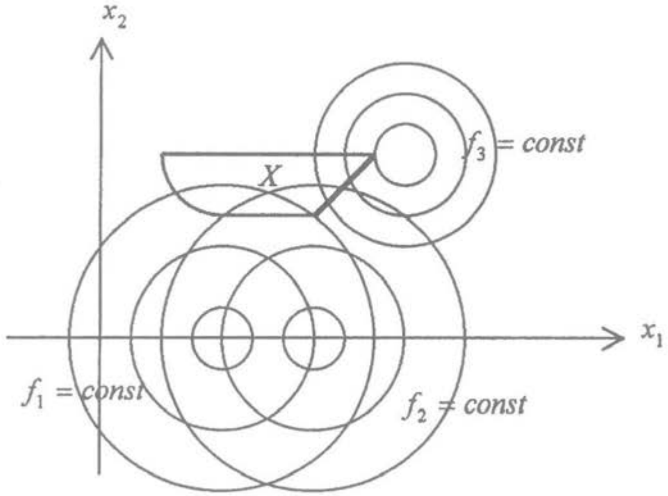

Figure 4. The set $cl(E(X,\mathbf{F}^3) \setminus E(X,\mathbf{F}^2))$ is a line segment.

From Fig. 4, it can be seen that

$$E(X,\mathbf{F}^2) = \{[x_1,x_2] \in X \mid 4 \le x_1 \le 7, x_2 = 4\},$$

$$E(X,\mathbf{F}^3) = \{[x_1,x_2] \in X \mid x_2 = x_1 - 3, 7 \le x_1 \le 9\} \cup E(X,\mathbf{F}^2).$$

Hence, the objective function f_3 is not nonessential. The set $cl(E(X,\mathbf{F}^3) \setminus E(X,\mathbf{F}^2))$ is a line segment (bold line) and its length equals

$$\inf_{\mathbf{x} \in E(X,\mathbf{F}^2)} \{d(\mathbf{x}, E(X,f_3))\} = 2\sqrt{2}.$$

Now let us change the objective function f_3 into

$$f_3(x_1,x_2) = (x_1-1)^2 + (x_2-6)^2 - 3.$$

In this case the objective function f_3 is still not nonessential, because

$$E(X,\mathbf{F}^3) = \{[x_1,x_2] \in X \mid x_2 = 0,5x_1^2 - 4x_1 + 12, 2 \le x_1 \le 4\}$$
$$\cup\, E(X,\mathbf{F}^2).$$

But now, the set $cl(E(X,\mathbf{F}^3) \setminus E(X,\mathbf{F}^2))$ is a curve segment ($x_2 = 0,5x_1^2 - 4x_1 + 12$) and its length is greater than $\inf_{\mathbf{x} \in E(X,\mathbf{F}^2)}\{d(\mathbf{x}, E(X,f_3))\} = 2\sqrt{2}$. The bold line in Fig. 5 represents this set.

We finish this section by presenting sufficient conditions for the difference between the sets of efficient solutions of problems $(\widetilde{P})$ and (P) to be contained in the boundary of X.

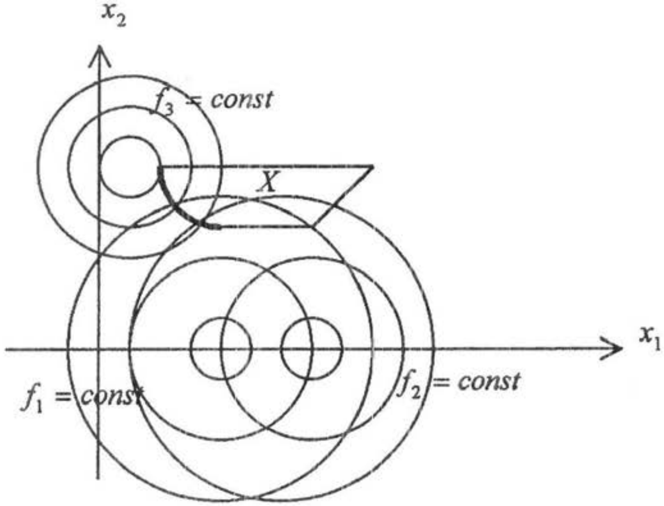

Figure 5. The set $cl(E(X, \mathbf{F}^3) \setminus E(X, \mathbf{F}^2))$ is a curve segment $(x_2 = 0, 5x_1^2 - 4x_1 + 12)$.

COROLLARY 2 *With the assumptions A.1, A.2, if the condition iii) of Theorem 1 holds, then* $E(X, \mathbf{F}^{n+1}) \setminus E(X, \mathbf{F}^n) \subset bdX$.

Proof. The case where $E(X, \mathbf{F}^n) \subseteq E(X, \mathbf{F}^{n+1})$ is trivial. If $E(X, \mathbf{F}^{n+1}) \setminus E(X, \mathbf{F}^n) \neq \emptyset$, then the proof is obtained from the proof of Theorem 1. ∎

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