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Boundary observability, controllability and stabilizability of linear distributed systems with a time reversible dynamics

by

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Abstract: The paper provides a short introduction into the theory of boundary exact controllability and uniform stabilization of linear distributed systems, based on the multiplier method.
Keywords: observability, wave equation, Petrovsky system.

1. Introduction

These notes correspond to a series of lectures given in September 1996 in the *Istituto per le Applicazioni del Calcolo "Mauro Picone"* of the *Consiglio Nazionale delle Ricerche.* Its purpose is to give a short introduction into the theory of boundary exact controllability and uniform stabilizability of linear distributed systems, based on the multiplier method. Many more results are given in Lions (1988a,b), Komornik (1994), and in their references.

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The plan of the paper sections is the following:

2. Observability. The multiplier method

2.1. The one-dimensional wave equation

Let Ω be a bounded domain in $\mathbf{R}^n$ with boundary Γ and consider the problem

$$\begin{cases} u'' - \Delta u = 0 \quad \text{in} \quad \Omega \times \mathbf{R}, \\ u = 0 \quad \text{on} \quad \Gamma \times \mathbf{R}, \\ u(0) = u_0 \quad \text{and} \quad u'(0) = u_1 \quad \text{in} \quad \Omega. \end{cases} \tag{1}$$

Among many other things, for $n = 2$ this modelizes the small transversal vibrations of an elastic membrane of the form Ω, stretched along its boundary.

We recall the following facts (see, e.g., Lions and Magenes, 1968-70):

- given $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ arbitrarily, (1) has a unique (so-called *weak*) solution
  $$u \in C(\mathbf{R}; H_0^1(\Omega)) \cap C^1(\mathbf{R}; L^2(\Omega));$$
- the energy of the solution, defined by the formula
  $$E(t) = \frac{1}{2} \int_\Omega u'(t)^2 + |\nabla u(t)|^2 \, dx, \tag{2}$$
  is in fact independent of $t \in \mathbf{R}$. We shall therefore denote it simply by E;
- if Ω is of class C^2 and if the initial data (u_0, u_1) belong to
  $$(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega),$$
  then the corresponding (so-called *strong*) solutions are smoother:
  $$u \in C(\mathbf{R}; H^2(\Omega)) \cap C^1(\mathbf{R}; H^1(\Omega)) \cap C^2(\mathbf{R}; L^2(\Omega)). \tag{3}$$

Now consider the following question. Suppose we are only able to observe the solution in some small neigbourhood of the boundary Γ. Is this observation sufficient to distinguish solutions corresponding to different initial data? In the one-dimensional case it is easy to obtain an affirmative answer by a direct computation. For simplicity we only consider an interval of length π.

PROPOSITION 2.1 *Let $\Omega = (0, \pi)$. Then all strong solutions of the problem (1) satisfy the following equality:*

$$\int_0^\pi u_x(0,t)^2 + u_x(\pi,t)^2 \, dt = 4E. \tag{4}$$

It follows from this proposition that if two strong solutions v and w of (1) (corresponding to initial data (v_0, v_1) and (w_0, w_1)) coincide in some neighbourhood $(0, \varepsilon) \cup (\pi - \varepsilon, \pi)$ of the extremities of Ω for $0 < t < \pi$, then in fact v and w are the same solutions. Indeed, applying the proposition with $u := v - w$ (which also solves (1) with the initial data $(u_0, u_1) = (v_0 - w_0, v_1 - w_1)$, the left-hand side of (4) vanishes by our assumption. Hence $E = 0$ and in particular $\nabla u(t) = 0$ in Ω for all $t \in \mathbf{R}$. Since $u(t) \in H_0^1(\Omega)$, applying the Poincaré inequality we conclude that $u(t) = 0$ in Ω for all $t \in \mathbf{R}$. Hence $v \equiv w$.

Proof of the proposition Using the Fourier method, the solutions of (1) are given by the formula

$$u(x,t) = \sum_{k=1}^{\infty} (\alpha_k \cos kt + \beta_k \sin kt) \sin kx$$

with suitable real coefficients α_k and β_k. We have

$$\begin{aligned}
\int_0^\pi u_x(0,t)^2 + u_x(\pi,t)^2 \, dt &= \int_0^\pi \left(\sum_{k=1}^{\infty} k(\alpha_k \cos kt + \beta_k \sin kt) \right)^2 \\
&+ \left(\sum_{k=1}^{\infty} (-1)^k k(\alpha_k \cos kt + \beta_k \sin kt) \right)^2 dt \\
&= \int_0^\pi 2 \left(\sum_{k \text{ is odd}} k(\alpha_k \cos kt + \beta_k \sin kt) \right)^2 \\
&+ 2 \left(\sum_{k \text{ is even}} k(\alpha_k \cos kt + \beta_k \sin kt) \right)^2 dt \\
&= \int_0^\pi \sum_{k=1}^{\infty} 2k^2 \alpha_k^2 \cos^2 kt + 2k^2 \beta_k^2 \sin^2 kt \, dt \\
&= \pi \sum_{k=1}^{\infty} k^2 (\alpha_k^2 + \beta_k^2)
\end{aligned}$$

because during the integration all mixed products disappear.

Furthermore,

$$4E(0) = 2 \int_0^\pi \left(\sum_{k=1}^{\infty} k\beta_k \sin kx \right)^2 + \left(\sum_{k=1}^{\infty} k\alpha_k \cos kx \right)^2 dx$$

$$= 2\int_0^\pi \sum_{k=1}^\infty k^2\beta_k^2 \sin^2 kx + k^2\alpha_k^2 \cos^2 kx \, dx$$

$$= \pi \sum_{k=1}^\infty k^2(\alpha_k^2 + \beta_k^2)$$

and the proof is completed. ∎

Unfortunately, the proof of Proposition 2.1 does not extend to general domains in $\mathbf{R}^n$. In the next subsection we shall apply another, more powerful method.

2.2. The wave equation in several space dimensions

The main result of this subsection is the following, in which we denote by ν the outward unit normal vector to Γ.

THEOREM 2.1 *Assume that Ω is of class C^2 and let $B(x_0, R)$ be the smallest ball containing Ω. Then for every number $T > 2R$ there exist two constants $c_1,\ c_2 > 0$ such that*

$$c_1 E \le \int_0^T \int_\Gamma |\partial_\nu u|^2 \, d\Gamma \, dt \le c_2 E \tag{5}$$

for all strong solutions of (1).

REMARK 2.1

1. The second inequality in (5) is often called a *direct* or *admissibility* inequality. It is due to Lasiecka and Triggiani (1983) and to Lions (1983). It allows us to define $\partial_\nu u$ as an element of $L^2(0,T;L^2(\Gamma))$ for all weak solutions by an easy density argument. Observe that this does not follow from the regularity in the definition of the weak solutions and from the usual trace theorems as those in Lions and Magenes (1968-70). Therefore it is often called a *hidden regularity* result. Note that using this definition the inequalities (5) remain valid for all *weak* solutions.
2. This hidden regularity result will allow us to define the solutions of some dual problem for rather irregular boundary data. (This explains the word "admissibility": some nonsmooth boundary data are admissible for the dual problem to be well-posed.) This will be important in the solution of a corresponding controllability problem in the next section.
3. The first inequality in (5) is often called an *inverse* or *observability* inequality. It was first proved by Ho (1986) under a stronger hypothesis on T and then by Lions (1988a) under the present condition $T > 2R$.
   Applying the same argument as in the preceding subsection, this implies the following observability result. Assume that two weak solutions of (1) coincide in $\Gamma_\varepsilon \times (0,T)$ for some $\varepsilon > 0$ and $T > 2R$ where Γ_ε is the ε-neighbourhood of Γ in Ω:
   $$\Gamma_\varepsilon = \{x \in \Omega \mid \text{dist}\,(x,\Gamma) < \varepsilon\}.$$
   Then in fact $v \equiv w$ in $\Omega \times \mathbf{R}$.

4. Using the finite propagation property for the wave equation, it is not difficult to show that the first inequality of (5) *cannot* hold for arbitrarily small T. If $\Omega = B(x_0, R)$, then a short elementary proof in remark 3.6 of Komornik (1994) shows that we cannot take $T < 2R$. Joó (1991) proved that we cannot take $T = 2R$ either if $n \geq 2$; this contrasts with the one-dimensional case. For a general domain the determination of the critical value of T is a difficult problem: see Bardos, Lebeau and Rauch (1992) and Tataru (1996). It turns out that the critical value is the length of the longest line segment lying entirely in Ω.

The proof of Theorem 2.1 will be based on the *multiplier method.* Our main tool is the following technical lemma which goes back essentially at least to Rellich (1940).

LEMMA 2.1 *Let u be a function having the regularity (3) and satisfying the wave equation $u'' - \Delta u = 0$ in $\Omega \times \mathbf{R}$. Fix a point $x_0 \in \mathbf{R}^n$ arbitrarily and put*

$$m(x) = x - x_0 \quad \text{and} \quad Mu := 2m \cdot \nabla u + (n-1)u \tag{6}$$

for brevity. Then for any fixed $-\infty < S < T < \infty$ the following identity holds true:

$$\begin{aligned}&\int_S^T \int_\Gamma (\partial_\nu u) Mu + (m \cdot \nu)((u')^2 - |\nabla u|^2)\ d\Gamma\ dt \\ &= \left[\int_\Omega u' Mu\ dx\right]_S^T + \int_S^T \int_\Omega (u')^2 + |\nabla u|^2\ dx\ dt.\end{aligned} \tag{7}$$

(The dot denotes the usual scalar product in $\mathbf{R}^n$.)

Proof. Integrating by parts we obtain that

$$\begin{aligned}0 &= \int_S^T \int_\Omega (u'' - \Delta u) Mu\ dx\ dt \\ &= \left[\int_\Omega u' Mu\ \mathrm{dx}\right]_S^T - \int_S^T \int_\Gamma (\partial_\nu u) Mu\ d\Gamma\ dt \\ &- \int_S^T \int_\Omega u' Mu'\ dx\ dt + \int_S^T \int_\Omega \nabla u \cdot \nabla(Mu)\ dx\ dt.\end{aligned}$$

We have

$$u' Mu' = 2u' m \cdot \nabla u' + (n-1)(u')^2 = m \cdot \nabla (u')^2 + (n-1)(u')^2$$

and

$$\begin{aligned}&\nabla u \cdot \nabla(Mu) = \partial_i u \partial_i (2m_k \partial_k u + (n-1)u) \\ &= 2(\partial_i u)(\partial_i m_k)(\partial_k u) + 2m_k(\partial_i u)(\partial_k \partial_i u) + (n-1)|\nabla u|^2 \\ &= m \cdot \nabla \left(|\nabla u|^2\right) + (n+1)|\nabla u|^2.\end{aligned}$$

In the last computation we applied the summation convention of repeated indices and we used the obvious relation $\partial_i m_k = \delta_{ik}$.

Substituting these equalities into the first identity we obtain that

$$
\begin{aligned}
0 &= \left[\int_\Omega u' Mu \,\mathrm{dx}\right]_S^T \\
&- \int_S^T \int_\Gamma (\partial_\nu u) Mu \, d\Gamma \, dt \\
&+ \int_S^T \int_\Omega -m \cdot \nabla (u')^2 - (n-1)(u')^2 + m \cdot \nabla \left(|\nabla u|^2\right) + (n+1)|\nabla u|^2.
\end{aligned}
$$

Integrating by parts again and using the relation div $m \equiv n$ the lemma follows:

$$
\begin{aligned}
0 &= \left[\int_\Omega u' Mu \,\mathrm{dx}\right]_S^T \\
&+ \int_S^T \int_\Gamma -(\partial_\nu u) Mu + (m \cdot \nu)\left(|\nabla u|^2 - (u')^2\right) \, d\Gamma \, dt \\
&+ \int_S^T \int_\Omega (u')^2 + |\nabla u|^2 \, dx \, dt.
\end{aligned}
$$

■

Note that the lemma and its proof remains valid if we replace **R** in (3) by some interval I and if $S, T \in I$.

We shall also need from Komornik (1987) the following

LEMMA 2.2 *Given $u \in H^2(\Omega)$ arbitrarily, we have the following identity:*

$$
\int_\Omega (Mu)^2 \, dx = \int_\Omega |2m \cdot \nabla u|^2 + (1-n^2)u^2 \, dx + (2n-2) \int_\Gamma (m \cdot \nu) u^2 \, d\Gamma. \tag{8}
$$

Proof. We integrate by parts and we use again the relation div m $\equiv n$ as follows:

$$
\begin{aligned}
\int_\Omega (Mu)^2 \, dx &= \int_\Omega |2m \cdot \nabla u + (n-1)u|^2 \, dx \\
&= \int_\Omega |2m \cdot \nabla u|^2 + (n-1)^2 u^2 + 4(n-1) u m \cdot \nabla u \, dx \\
&= \int_\Omega |2m \cdot \nabla u|^2 + (n-1)^2 u^2 + (2n-2) m \cdot \nabla (u^2) \, dx \\
&= \int_\Omega |2m \cdot \nabla u|^2 + (n-1)^2 u^2 - n(2n-2)u^2 \, dx + (2n-2) \int_\Gamma (m \cdot \nu) u^2 \, d\Gamma.
\end{aligned}
$$

■

Now we are ready to prove the following result obtained in Komornik (1987):

THEOREM 2.2 *Assume that Ω is of class C^2 and that it is contained in a ball $B(x_0, R)$. Then all strong solutions of (1) satisfy for all $T > 0$ the following estimates:*

$$2(T-2R)E \le \int_0^T \int_\Gamma (m\cdot\nu)(\partial_\nu u)^2 \, d\Gamma \, dt \le 2(T+2R)E. \tag{9}$$

Proof. We apply the identity (7) of Lemma 2.1 with $S = 0$. Since $u = 0$ on $\Gamma \times \mathbf{R}$, we also have $u' = 0$ and $\nabla u = (\partial_\nu u)\nu$ on $\Gamma \times \mathbf{R}$. Therefore the expression under the integral sign on the left-hand side of the identity reduces to $(m\cdot\nu)(\partial_\nu u)^2$.

Furthermore, using the definition and the conservation of the energy, the last integral on the right-hand side of this identity is equal to $2TE$, so that (7) reduces to

$$\int_0^T \int_\Gamma (m\cdot\nu)(\partial_\nu u)^2 \, d\Gamma \, dt = \left[\int_\Omega u'Mu \, dx\right]_0^T + 2TE.$$

If we prove the inequality

$$\left|\int_\Omega u'Mu \, dx\right| \le 2RE, \tag{10}$$

then the estimates (9) will follow. For the proof of (10) first we note that the identity (8) of Lemma 2.2 implies the inequality

$$\int_\Omega (Mu)^2 \, dx \le \int_\Omega |2m\cdot\nabla u|^2 \, dx \le 4R^2 \int_\Omega |\nabla u|^2 \, dx$$

because $u = 0$ on Γ, $n \ge 1$ and because $|m| \le R$ in $\overline{\Omega}$. Now (10) follows easily:

$$\int_\Omega |u'Mu| dx \le \int_\Omega R(u')^2 + (4R)^{-1}(Mu)^2 dx \le R\int_\Omega (u')^2 + |\nabla u|^2 dx = 2RE.$$

■

Proof of Theorem 2.1. We prove the theorem only under the extra hypothesis that Ω is strictly star-shaped with respect to x_0, i.e., there exists a number $r > 0$ such that $m\cdot\nu \ge r$ on Γ. In this case the theorem follows at once from the preceding one with $c_1 = 2(T-2R)/R$ and $c_2 = 2(T-2R)/r$.

In the general case we need a slight generalization of Lemma 2.1, replacing the function m by another function h whose restriction to Γ is equal to ν; see, e.g., Lions (1988a,b) or Komornik (1994). ■

2.3. A simple plate model

Now consider the problem

$$\begin{cases} u'' + \Delta^2 u = 0 & \text{in} \quad \Omega\times\mathbf{R}, \\ u = \partial_\nu u = 0 & \text{on} \quad \Gamma\times\mathbf{R}, \\ u(0) = u_0 \quad \text{and} \quad u'(0) = u_1 & \text{in} \quad \Omega. \end{cases} \tag{11}$$

For $n = 2$ this represents a very simple model describing the small transversal vibrations of a thin plate clamped along its boundary.

We recall the following facts (see, e.g., Lions and Magenes, 1968-70):

- given $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ arbitrarily, (11) has a unique *weak* solution
  $u \in C(\mathbf{R}; H_0^2(\Omega)) \cap C^1(\mathbf{R}; L^2(\Omega))$;
- the energy of the solution, defined by the formula
  $E(t) = \frac{1}{2} \int_\Omega u'(t)^2 + (\Delta u(t))^2 \, dx$,
  is in fact independent of $t \in \mathbf{R}$. We shall therefore denote it simply by E;
- if Ω is of class C^4 and if the initial data (u_0, u_1) belong to $(H^4(\Omega) \cap H_0^2(\Omega)) \times H_0^2(\Omega)$, then the corresponding *strong* solutions are smoother:
  $u \in C(\mathbf{R}; H^4(\Omega)) \cap C^1(\mathbf{R}; H^2(\Omega)) \cap C^2(\mathbf{R}; L^2(\Omega))$.

We have the

THEOREM 2.3 *Assume that Ω is of class C^4. Then for every number $T > 0$ there exist two constants c_1, $c_2 > 0$ such that*

$$c_1 E \le \int_0^T \int_\Gamma (\Delta u(t))^2 \, d\Gamma \, dt \le c_2 E$$

for all weak solutions of (11).

This theorem implies that two solutions of (11), corresponding to different initial data, can be distinguished by observing them in some neighbourhood of the boundary Γ during an arbitrarily small time interval. (There is no contradiction because in the present problem we have infinite propagation speed.)

Theorem 2.3 was proved by Lions (1988a) for a sufficiently large T and then in Komornik (1987) under the weaker condition of $T > 2R/\sqrt{\mu_1}$, where $B(x_0, R)$ is the smallest ball containing Ω and μ_1 denotes the first eigenvalue of the eigenvalue problem

$$\Delta^2 v = -\mu \Delta v, \quad v \in H_0^2(\Omega).$$

Finally, using a compactness–uniqueness argument based on Holmgren's theorem, Zuazua (1988) proved the theorem for all $T > 0$.

Another, constructive and more elementary method was developed later in Komornik (1989) in order to weaken the sufficient conditions for inverse inequalities. This approach provides a simple recipe: whenever we obtain a sufficient condition of the form $T > f(\mu_1)$ where μ_1 is the first eigenvalue of some corresponding eigenvalue problem, the inverse inequality also holds under the (usually weaker) condition $T > f(\infty)$. In the present case this leads to the condition $T > 2R/\infty = 0$. We refer to Komornik (1991) for the proof of Theorem 2.3 using this method.

3. Controllability. The Hilbert uniqueness method

3.1. The wave equation

Fix a number $T > 0$ and consider the following problem:

$$\begin{cases} y'' - \Delta y = 0 \quad \text{in} \quad \Omega \times [0,T], \\ y = v \quad \text{on} \quad \Gamma \times [0,T], \\ y(0) = y_0 \quad \text{and} \quad u'(0) = u_1 \quad \text{in} \quad \Omega. \end{cases} \tag{12}$$

We shall prove the following result of Lions (1988a):

THEOREM 3.1 *Assume that Ω is of class C^2 and that it belongs to an open ball of diameter $< T$. Then for any given $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists $v \in L^2(0,T; L^2(\Gamma))$ such that the solution of (12) satisfies*

$$y(T) = y'(T) = 0 \quad \textit{in} \quad \Omega. \tag{13}$$

In what follows we shall identify $L^2(\Omega)$ and $L^2(\Gamma)$ with their respective duals.

Let us first study the well-posedness of (12). Since we will have to use rather irregular initial and boundary data, we will define a suitable weak solution by applying the transposition method. Consider for this the problem studied in Subsections 2.1 and 2.2:

$$\begin{cases} u'' - \Delta u = 0 \quad \text{in} \quad \Omega \times \mathbf{R}, \\ u = 0 \quad \text{on} \quad \Gamma \times \mathbf{R}, \\ u(0) = u_0 \quad \text{and} \quad u'(0) = u_1 \quad \text{in} \quad \Omega. \end{cases} \tag{14}$$

If y solves (12) and u solves (14), then we can make the following formal computation for every $S \in [0,T]$:

$$\begin{aligned} 0 &= \int_0^S \int_\Omega (y'' - \Delta y) u \, dx \, dt \\ &= \left[\int_\Omega y'u - yu' \, dx \right]_0^S + \int_0^S \int_\Omega y(u'' - \Delta u) \, dx \, dt \\ &+ \int_0^S \int_\Gamma -(\partial_\nu y) u + y(\partial_\nu u) \, d\Gamma \, dt. \end{aligned}$$

Using the initial and boundary conditions in (12) and (14) we conclude that

$$\int_\Omega -y'(S)u(S) + y(S)u'(S) \, dx = \int_\Omega -y_1 u_0 + y_0 u_1 \, dx + \int_0^S \int_\Gamma v \partial_\nu u \, d\Gamma \, dt,$$

or writing in a more abstract way,

$$\begin{aligned} &\langle (-y'(S), y(S)), (u(S), u'(S)) \rangle_{H^{-1}(\Omega) \times L^2(\Omega), H_0^1(\Omega) \times L^2(\Omega)} \\ &= \langle (-y_1, y_0), (u_0, u_1) \rangle_{H^{-1}(\Omega) \times L^2(\Omega), H_0^1(\Omega) \times L^2(\Omega)} \\ &+ (v, \partial_\nu u)_{L^2(0,S; L^2(\Gamma))}. \end{aligned} \tag{15}$$

This suggests us to *define* a solution of (12) as a *continuous* function

$$(y, y') : [0, T] \to L^2(\Omega) \times H^{-1}(\Omega)$$

satisfying (15) for all $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and for all $S \in [0, T]$. This definition is justified by the

PROPOSITION 3.1 *Given*

$$(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega) \quad \textit{and} \quad v \in L^2(0, S; L^2(\Gamma))$$

arbitrarily, the problem (12) has a unique solution.

Proof. First fix $S \in [0, T]$ arbitrarily. Thanks to the second inequality in (5) the right-hand side of (15) defines a bounded linear form of

$$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega).$$

Since the problem (14) is time reversible, the application

$$(u_0, u_1) \mapsto (u(S), u'(S))$$

is an automorphism of $H_0^1(\Omega) \times L^2(\Omega)$. Therefore the right-hand side of (15) may also be considered as a bounded linear form L_S of

$$(u(S), u'(S)) \in H_0^1(\Omega) \times L^2(\Omega).$$

By the *definition* of the dual of a Hilbert space there exists a unique pair

$$(-y'(S), y(S)) \in H^{-1}(\Omega) \times L^2(\Omega)$$

satisfying (15).

Since the bounded linear form L_S depends continuously on S (which is easy to verify), the function $S \mapsto (-y'(S), y(S))$ is also continuous. ∎

Proof of Theorem 3.1. The idea is to seek a suitable control in the special form $v = \partial_\nu u$ where u solves (14) for some appropriate choice of the initial data (u_0, u_1). It is sufficient to show that if (u_0, u_1) runs over $H_0^1(\Omega) \times L^2(\Omega)$ and if y denotes the solution of the problem

$$\begin{cases} y'' - \Delta y = 0 & \text{in} \quad \Omega \times [0, T], \\ y = v & \text{on} \quad \Gamma \times [0, T], \\ y(T) = y'(T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{16}$$

then $(y(0), y'(0))$ runs over $L^2(\Omega) \times H^{-1}(\Omega)$. Indeed, then it is sufficient to choose $v = \partial_\nu u$ in (12) with $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ such that $y(0) = y_0$ and $y'(0) = y_1$.

Equivalently, it is sufficient to show that the linear map

$$\Lambda : H_0^1(\Omega) \times L^2(\Omega) \to H^{-1}(\Omega) \times L^2(\Omega)$$

defined by the formula

$$\Lambda(u_0, u_1) = (y'(0), -y(0))$$

is onto. We will show that Λ is in fact an isomorphism. Thanks to the Lax–Milgram theorem it is sufficient to show that the associated bilinear form

$$\langle \Lambda(u_0, u_1), (v_0, v_1) \rangle_{H^{-1}(\Omega)\times L^2(\Omega), H_0^1(\Omega)\times L^2(\Omega)}$$

is continuous and coercive on $H_0^1(\Omega) \times L^2(\Omega)$.

The continuity of Λ follows from the well-posedness of the problems (14) and (16). (Thanks to the time reversibility of the wave equation the well-posedness of (16) can be deduced from that of (12) by the change of variable $t \mapsto T - t$.) The coercivity of Λ will follow from Theorem 2.1 if we establish the formula

$$\langle \Lambda(u_0, u_1), (u_0, u_1) \rangle_{H^{-1}(\Omega)\times L^2(\Omega), H_0^1(\Omega)\times L^2(\Omega)} = \int_0^T \int_\Gamma |\partial_\nu u|^2 \, d\Gamma \, dt.$$

This equality follows from (15) applied with $S = T$ if we use "final" conditions in (16) and the equality $v = \partial_\nu u$ in the definition of Λ. ∎

3.2. The plate model

By applying the method of the preceding subsection, Theorem 2.3 implies the following exact controllability result for the problem

$$\begin{cases} y'' + \Delta^2 y = 0 \quad \text{in} \quad \Omega \times [0, T], \\ y = 0 \quad \text{and} \quad \partial_\nu y = v \quad \text{on} \quad \Gamma \times [0, T], \\ y(0) = y_0 \quad \text{and} \quad y'(0) = y_1 \quad \text{in} \quad \Omega : \end{cases} \tag{17}$$

THEOREM 3.2 *Assume that Ω is of class C^4 and fix $T > 0$ arbitrarily. For any given $(y^0, y^1) \in L^2(\Omega) \times H^{-2}(\Omega)$ there exists a function $v \in L^2(0, T; L^2(\Gamma))$ such that the solution of (17) satisfies*

$$y(T) = y'(T) = 0 \quad \textit{in} \quad \Omega.$$

This result is due to Lions (1988a) (for T sufficiently large) and Zuazua (1988) (for T arbitrarily small). We leave the proof to the reader.

4. Stabilization by "natural" feedbacks

In this section we consider boundary feedbacks making the system dissipative and we estimate the energy decay rate. The Lyapunov type approach used here was introduced in Komornik and Zuazua (1990) and was later developed in Komornik (1991).

4.1. The wave equation with linear feedbacks

Fix two continuous functions $a, b : \Gamma \to (0, +\infty)$ and consider the following problem:

$$\begin{cases} u'' - \Delta u = 0 \quad \text{in} \quad \Omega \times [0, +\infty), \\ \partial_\nu u + au + bu' = 0 \quad \text{on} \quad \Gamma \times [0, +\infty), \\ u(0) = u_0 \quad \text{and} \quad u'(0) = u_1 \quad \text{on} \quad \Omega. \end{cases} \tag{18}$$

This type of boundary feedback was first proposed by Russell (1978). We recall the following facts (see, e.g., Komornik, 1991; 1994):

- given $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega)$ arbitrarily, (18) has a unique *weak* solution
  $u \in C([0, +\infty); H^1(\Omega)) \cap C^1([0, +\infty); L^2(\Omega))$;
- the energy of the solution, defined by the formula
  $$E(t) = \tfrac{1}{2} \int_\Omega u'(t)^2 + |\nabla u(t)|^2 \, dx + \tfrac{1}{2} \int_\Gamma au(t)^2 \, d\Gamma, \tag{19}$$
  is nonincreasing;
- if Ω is of class C^2 and if the initial data satisfy $(u_0, u_1) \in H^2(\Omega) \times H^1(\Omega)$ and $\partial_\nu u_0 + au_0 + bu_1 = 0$ on Γ, then the corresponding *strong* solutions are smoother:
  $u \in C([0, +\infty); H^2(\Omega)) \cap C^1([0, +\infty); H^1(\Omega)) \cap C^2([0, +\infty); L^2(\Omega))$.

Let us first give a more precise result concerning the nonincreasingness of energy:

LEMMA 4.1 *The strong solutions of (18) satisfy the equalities*

$$E(S) - E(T) = \int_S^T \int_\Gamma bu'(t)^2 \, d\Gamma \, dt \tag{20}$$

for all $0 \le S < T < +\infty$.

Proof We have

$$\begin{aligned} E' &= \int_\Omega u'u'' + \nabla u \cdot \nabla u' \, dx + \int_\Gamma auu' \, d\Gamma \\ &= \int_\Omega u'\Delta u + \nabla u' \cdot \nabla u \, dx + \int_\Gamma auu' \, d\Gamma \\ &= \int_\Gamma u'(\partial_\nu u + au) \, d\Gamma = -\int_\Gamma b(u')^2 \, d\Gamma. \end{aligned}$$

∎

If Ω is a star-shaped domain of class C^2, then energy tends to zero exponentially as $t \to \infty$. For the sake of simplicity, we shall prove this here only in a very particular case and we refer to Komornik (1991, 1994), Tcheugoué Tébou (1994, 1996), Martinez (1999) for more general results. See also Aassila (1997) for strong stability theorems under weaker assumptions by a modification of the proof below.

THEOREM 4.1 *Let Ω be a unit ball $B(x_0, 1)$ in $\mathbf{R}^3$ and choose $a = b \equiv 1$. Then all solutions of (18) satisfy the estimate*

$$E(t) \leq E(0)e^{1-t/2}$$

for all $t \geq 0$.

Proof. It is sufficient to consider strong solutions; the general case then follows by an easy density argument.

By applying Lemma 2.1 (see the note following its proof) we have for all $0 \leq S < T < +\infty$ the identity

$$\int_S^T \int_\Gamma (\partial_\nu u)Mu + (m \cdot \nu)((u')^2 - |\nabla u|^2) \, d\Gamma \, dt$$
$$= \left[\int_\Omega u' Mu \, dx \right]_S^T + \int_S^T \int_\Omega (u')^2 + |\nabla u|^2 \, dx \, dt.$$

Since now $m \cdot \nu \equiv 1$, $\partial_\nu u = -u - u'$ and $Mu = 2m \cdot \nabla u + 2u$, using also the definition of the energy this identity can be rewritten in the following form:

$$\int_S^T \int_\Gamma (u')^2 - |\nabla u|^2 - 2(u + u')(m \cdot \nabla u) - 2(u + u')u + u^2 \, d\Gamma \, dt$$
$$= \left[\int_\Omega u' Mu \, dx \right]_S^T + 2 \int_S^T E \, dt. \tag{21}$$

Since $|m| = 1$ on Γ, the expression under the integral sign on the left-hand side can be majorized as follows:

$$(u')^2 - |\nabla u|^2 - 2(u + u')(m \cdot \nabla u) - 2(u + u')u + u^2$$
$$\leq (u')^2 - |\nabla u|^2 + (u + u')^2 + |m \cdot \nabla u|^2 - 2(u + u')u + u^2$$
$$\leq (u')^2 + (u + u')^2 - 2(u + u')u + u^2 = 2(u')^2.$$

Therefore, using also Lemma 4.1, we deduce from (21) the following inequality:

$$2 \int_S^T E \, dt \leq 2E(S) - 2E(T) - \left[\int_\Omega u' Mu \, dx \right]_S^T. \tag{22}$$

If we prove that

$$\left| \int_\Omega u' Mu \, dx \right| \leq 2E, \tag{23}$$

then (22) will imply that

$$2 \int_S^T E \, dt \leq 2E(S) - 2E(T) + 2E(S) + 2E(T) = 4E(S),$$

whence, letting $T \to +\infty$ we shall conclude that

$$\int_S^\infty E\ dt \le 2E(S) \quad \text{for all} \quad S \ge 0. \tag{24}$$

For the proof of (23) we use the identity (8) of Lemma 2.2:

$$\int_\Omega (Mu)^2\ dx = \int_\Omega |2m \cdot \nabla u|^2 + (1-n^2)u^2\ dx + (2n-2)\int_\Gamma (m\cdot\nu)u^2\ d\Gamma.$$

Since $|m| \le 1$ in $\overline{\Omega}$ and since $n = 3$, this implies the inequality

$$\int_\Omega (Mu)^2\ dx \le 4\int_\Omega |\nabla u|^2\ dx + 4\int_\Gamma u^2\ d\Gamma.$$

Now (23) follows easily:

$$\int_\Omega |u'Mu|\ dx \le \int_\Omega (u')^2 + (1/4)(Mu)^2\ dx$$
$$\le \int_\Omega (u')^2 + |\nabla u|^2\ dx + \int_\Gamma u^2\ d\Gamma = 2E.$$

Since the energy function is nonnegative and nonincreasing, the theorem now follows from (24) by applying the Gronwall type lemma (4.2) given below. ■

LEMMA 4.2 *Let $E : [0,+\infty) \to [0,+\infty)$ be a nonincreasing function and assume that there exists a constant $T > 0$ such that*

$$\int_t^\infty E(s)\ ds \le TE(t), \quad \forall t \ge 0. \tag{25}$$

Then

$$E(t) \le E(0)e^{1-t/T}, \quad \forall t \ge 0. \tag{26}$$

Proof. Define

$$f(x) := e^{x/T}\int_x^\infty E(s)\ ds, \quad x \ge 0;$$

then f is locally absolutely continuous and it is also nonincreasing by (25):

$$f'(x) = T^{-1}e^{x/T}\Big(\int_x^\infty E(s)\ ds - TE(x)\Big) \le 0$$

almost everywhere in $[0,+\infty)$. Hence, using (25) again,

$$f(x) \le f(0) = \int_0^\infty E(s)\ ds \le TE(0), \quad \forall x \ge 0,$$

i.e.,

$$\int_x^\infty E(s)\,ds \le TE(0)e^{-x/T}, \quad \forall x \ge 0. \tag{27}$$

Since E is nonnegative and nonincreasing, we have

$$\int_x^\infty E(s)\,ds \ge \int_x^{x+T} E(s)\,ds \ge TE(x+T).$$

Substituting into (27) we obtain that

$$E(x+T) \le E(0)e^{-x/T}, \quad \forall x \ge 0.$$

Setting $t := x + T$ hence we conclude (26) for all $t \ge T$. Finally, for $0 \le t < T$ the inequality (26) is obviously satisfied because $E(t) \le E(0)$. ■

REMARK 4.1 This lemma is taken from Haraux (1978b). See Haraux (1978a), Lagnese (1989), Komornik (1996), Martinez (1999), Laurençot (1998), for more general results.

It is natural to ask whether we can achieve arbitrarily large energy decay rates by a suitable choice of the coefficients a and b in (18). A result of Koch and Tataru (1995) shows that this is impossible. In the last section of these notes we shall construct boundary feedbacks of a different kind leading to arbitrarily high decay rates.

4.2. Nonlinear feedbacks

The method of the preceding section can be adapted to nonlinear feedbacks. We state just one result here; we refer to Komornik (1994) for proof. Various other results of this type can be found in Komornik (1994), Kouémou-Patcheu (1996), Martinez (1999) and in the references therein.

Let $g : \mathbf{R} \to \mathbf{R}$ be a nondecreasing, continuous function. Assume that there exists a real number $p > 1$ and positive constants c_i such that g satisfies the following growth conditions:

$$c_1|x|^p \le |g(x)| \le c_2|x|^{1/p} \quad \text{if} \quad |x| \le 1$$

and

$$c_3|x| \le |g(x)| \le c_4|x| \quad \text{if} \quad |x| > 1.$$

Fix a continuous function $a : \Gamma \to (0, +\infty)$ and consider the following problem:

$$\begin{cases} u'' - \Delta u = 0 & \text{in} \quad \Omega \times [0, +\infty), \\ \partial_\nu u + au + g(u') = 0 & \text{on} \quad \Gamma \times [0, +\infty), \\ u(0) = u_0 \quad \text{and} \quad u'(0) = u_1 & \text{in} \quad \Omega. \end{cases} \tag{28}$$

This problem is well-posed in $H^1(\Omega) \times L^2(\Omega)$. We define the energy of the solutions by the same formula (19) as in the preceding subsection. Then we have the

THEOREM 4.2 *Assume that Ω is a star-shaped domain of class C^2. Then the solutions of (28) satisfy the estimates*

$$E(t) \le Ct^{-2/(p-1)}$$

for all $t > 0$, where C is a constant depending on the initial energy $E(0)$.

REMARK 4.2 In the one-dimensional case the optimality of these estimates has been proved by J. Vancostenoble and P. Martinez in Vancostenoble (1999) and Martinez and Vancostenoble (1999).

5. Abstract framework

5.1. Observability implies controllability

Consider a linear evolutionary problem

$$x' = Ax + Bv, \qquad x(0) = x_0, \tag{29}$$

where A is a densely defined, closed linear operator in some Hilbert space H and B is a densely defined, closed linear operator from another Hilbert space G into $D(A^*)'$. Let us also consider the dual problem

$$\varphi' = -A^*\varphi, \qquad \varphi(0) = \varphi_0, \qquad \psi = B^*\varphi, \tag{30}$$

where A^*, B^* denote the adjoints of A and B. In control-theoretical terminology B is a *control* operator, v is a *control*, B^* is an *observation* operator, and ψ is an *observation.*

Assume that the following hypotheses are satisfied (we denote by G', H' the dual spaces of G and H):

(H1) The operator A^* generates a *group* e^{sA^*} in H';

(H2) $D(A^*) \subset D(B^*)$, and there exists a constant c such that
$$\|B^*\varphi_0\|_{G'} \le c\|A^*\varphi_0\|_{H'}$$
for all $\varphi_0 \in D(A^*)$;

(H3) There exist three numbers $T, c_1, c_2 > 0$ such that the solutions of (30) satisfy the inequalities
$$c_1\|\varphi_0\|_{H'} \le \|\psi\|_{L^2(0,T;G')} \le c_2\|\varphi_0\|_{H'}$$
for all $\varphi_0 \in D(A^*)$.

In the applications the hypothesis (H1) is usually satisfied for time-reversible problems. Hypothesis (H2) is not necessary if the operator B is bounded. It is often satisfied in boundary control problems where B is unbounded. Finally, (H3) is an abstract form of the direct and inverse inequalities.

We shall see later that the problems studied in Section 2 can be rewritten in the form (30) satisfying these hypotheses.

- It follows from (H1) that for every $\varphi_0 \in H'$ the initial value problem in (30) has a unique *weak* solution $\varphi \in C(\mathbf{R}; H')$, given by the formula $\varphi(s) = e^{-sA^*}\varphi_0$.

- Furthermore, it follows from (H1) and (H2) that for every $\varphi_0 \in D(A^*)$, (30) has a unique *strong* solution
$$\varphi \in C(\mathbf{R}; D(A^*)) \cap C^1(\mathbf{R}; H'),$$
and that $\psi \in C(\mathbf{R}; G')$; in particular, hypothesis (H3) is meaningful.
- Now the second (*direct*) inequality in (H3) allows us to define ψ as an element of $L^2(0,T;G')$ for all $\varphi_0 \in H'$, by a density argument.

Next we show that hypotheses (H1), (H2) and the second inequality in (H3) allow us to *define* by transposition the solution of (12) for every $x_0 \in H$ and $v \in L^2(0,T;G)$. Proceeding formally, if x solves (29) and φ, ψ solve (30), then for every $S \in [0,T]$ we have the identity

$$\langle x(S), \varphi(S)\rangle_{H,H'} = \langle x_0, \varphi_0\rangle_{H,H'} + \int_0^S \langle v(s), \psi(s)\rangle_{G,G'}\, ds. \tag{31}$$

Indeed, we have

$$\begin{aligned}
&\int_0^S \langle x(s), \varphi'(s) + A^*\varphi(s)\rangle_{H,H'}\, ds.\\
&= [\langle x(s), \varphi(s)\rangle_{H,H'}]_0^S + \int_0^S \langle -x'(s), \varphi(s)\rangle_{H,H'} + \langle x(s), A^*\varphi(s)\rangle_{H,H'}\, ds.\\
&= [\langle x(s), \varphi(s)\rangle_{H,H'}]_0^S + \int_0^S \langle -x'(s) + Ax(s), \varphi(s)\rangle_{H,H'}\, ds.\\
&= [\langle x(s), \varphi(s)\rangle_{H,H'}]_0^S - \int_0^S \langle Bv(s), \varphi(s)\rangle_{H,H'}\, ds.\\
&= [\langle x(s), \varphi(s)\rangle_{H,H'}]_0^S - \int_0^S \langle v(s), \psi(s)\rangle_{G,G'}\, ds.
\end{aligned}$$

Hence we *define* a solution of (29) as a *continuous* function $x : [0,T] \to H$ satisfying the identity (31) for all $\varphi_0 \in H'$ and for all $S \in [0,T]$. This definition is justified by the

PROPOSITION 5.1 *Given $x_0 \in H$ and $v \in L^2(0,T;G)$ arbitrarily, the problem (29) has a unique solution.*

Proof. Thanks to the second inequality in (H3) the right-hand side of (31) defines a bounded linear form of $\varphi_0 \in H'$. Since the map $\varphi_0 \mapsto \varphi(T)$ is an automorphism of H' by hypothesis (H1), the right-hand side of (31) is also a bounded linear form of $\varphi(S) \in H'$. Since $H'' = H$, it is uniquely represented by some $x(S) \in H$, so that (31) is satisfied. ∎

Until now we did not use the first (*inverse*) inequality in (H3), expressing the observability of the problem (30). Now we prove that the observability of (30) implies the controllability of (29):

THEOREM 5.1 *Assume (H1) to (H3). Then to every initial state $x_0 \in H$ there exists a function $v \in L^2(0,T;G)$ such that the solution of (29) satisfies the final* *condition* $x(T) = 0$. *(We say that the* control v *drives the system to rest in time* T.*)*

Proof. Thanks to hypotheses (H1) to (H3) the formula

$$(\varphi_0, \psi_0) \mapsto \int_0^T (B^* e^{-sA^*} \varphi_0, B^* e^{-sA^*} \psi_0)_{G'} \, ds$$

defines a continuous, symmetric and coercive bilinear form in H'. Applying the Riesz–Fréchet theorem, there exists a self-adjoint, positive definite isomorphism $\Lambda \in L(H', H)$ such that

$$\langle \Lambda \varphi_0, \psi_0 \rangle_{H,H'} = \int_0^T (B^* e^{-sA^*} \varphi_0, B^* e^{-sA^*} \psi_0)_{G'} \, ds$$

for all $\varphi_0, \psi_0 \in H'$.

Let us denote by $J : G' \to G$ the canonical Riesz isomorphism. Given $x_0 \in H$ arbitrarily, we claim that the control

$$v(s) := -J B^* e^{-sA^*} \Lambda^{-1} x_0$$

drives x_0 to rest in time T. Indeed, for any given $\varphi_0 \in H'$, using (30) and (31) we have

$$\begin{aligned}
\langle x(T), \varphi(T) \rangle_{H,H'} &= \langle x_0, \varphi_0 \rangle_{H,H'} + \int_0^T \langle v(s), \psi(s) \rangle_{G,G'} \, ds. \\
&= \langle x_0, \varphi_0 \rangle_{H,H'} - \int_0^T (B^* e^{-sA^*} \Lambda^{-1} x_0, B^* e^{-sA^*} \varphi_0)_{G'} \, ds. \\
&= \langle x_0, \varphi_0 \rangle_{H,H'} - \langle \Lambda \Lambda^{-1} x_0, \varphi_0 \rangle_{H,H'} = 0.
\end{aligned}$$

Since by hypothesis (H1) $\varphi(T)$ runs over the whole H' if φ_0 does, hence we conclude that $x(T) = 0$. ■

In fact, Dolecki and Russell (1977) proved that under hypotheses (H1) to (H3) the controllability of (29) is *equivalent* to the observability of (30). (See also Komornik, 1997, for a short proof.) Moreover, this duality relation remains valid if we only assume instead of (H1) that A^* generates a *semigroup* in H'; see Dolecki and Russell (1977).

Lions (1988a, b) developed a general and systematic approach for the study of exact controllability of linear distributed systems, the so-called Hilbert uniqueness method (HUM). It was based on the preceding theorem.

5.2. Application to the wave equation

We may study the problem of Subsection 3.1 in the abstract framework as follows. First, putting $\varphi = (u, u')$, $\varphi_0 = (u_0, u_1)$ and introducing the linear operators A^* and B^* by the formulas

$$D(A^*) = D(B^*) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega),$$
$$A^*(z_0, z_1) = -(z_1, \Delta z_0),$$
$$B^*(z_0, z_1) = \partial_\nu z_0,$$

we may rewrite (14) with the observation of $\partial_\nu u$ in the abstract form (30).

We claim that by choosing $H' = H_0^1(\Omega) \times L^2(\Omega)$ and $G' = L^2(\Gamma)$ we satisfy the assumptions (H1) to (H3). Indeed, (H1) is well-known and is related to the energy conservation, see, e.g., Lions and Magenes (1968-70). Property (H2) follows from the definition of A^*, B^* and from the elliptic regularity theory for $z_0 \in H^2(\Omega) \cap H_0^1(\Omega)$:

$$\|B^*(z_0, z_1)\|_{L^2(\Gamma)} = \|\partial_\nu z_0\|_{L^2(\Gamma)} \le c\|z_0\|_{H^2(\Omega)}$$
$$\le c\|\Delta z_0\|_{L^2(\Omega)} \le c\|A^*(z_0, z_1)\|_{H_0^1(\Omega)\times L^2(\Omega)}.$$

Finally, (H3) is equivalent to the inequalities proved in Theorem 2.1.

Now, by comparing the identities (15) and (31) we obtain that the dual problem (29) of (30) is just another form of the problem (12) if we introduce the notations $x = (-y', y)$, $x_0 = (-y_1, y_0)$ and if we put $G := G'' = L^2(\Gamma)$ and $H := H'' = H^{-1}(\Omega) \times L^2(\Omega)$.

By applying Theorem 5.1 and Theorem 2.1 we secure satisfaction of Theorem 3.1.

5.3. Observability implies stabilizability

Despite its elegance and relative simplicity, the method applied in Section 4 has two drawbacks. First, it can be applied only in few cases. Secondly, it does not lead to arbitrarily large decay rates. We present here another approach to the uniform stabilization, analogous to HUM. This leads to the construction of boundary feedbacks with arbitrarily large decay rates. The results are borrowed from Komornik (1997). The particular weight function e_ω below was proposed to the author by Bourquin (1998).

Let us return to the abstract framework of Subsection 5.1. Assume hypotheses (H1) to (H3) again. Fix a number $\omega > 0$, set $T_\omega = T + (2\omega)^{-1}$, define

$$e_\omega(s) = \begin{cases} e^{-2\omega s} & \text{if } 0 \le s \le T, \\ 2\omega e^{-2\omega T}(T_\omega - s) & \text{if } T \le s \le T_\omega, \end{cases}$$

and set

$$\langle \Lambda_\omega \varphi_0, \psi_0 \rangle_{H,H'} := \int_0^{T_\omega} e_\omega(s)(B^* e^{-sA^*}\varphi_0, B^* e^{-sA^*}\psi_0)_{G'}\, ds.$$

Then Λ_ω is a selfadjoint, positive definite isomorphism $\Lambda_\omega \in L(H', H)$. The following result was obtained in Komornik (1997).

THEOREM 5.2 *Assume (H1) to (H3) and fix $\omega > 0$ arbitrarily. Then the problem*

$$x' = (A - BJB^*\Lambda_\omega^{-1})x, \qquad x(0) = x_0 \tag{32}$$

is well-posed in H. Furthermore, there exists a constant M such that the solutions of (32) satisfy the estimates

$$\|x(t)\|_H \le M\|x_0\|_H e^{-\omega t} \tag{33}$$

for all $x_0 \in H$ and for all $t \ge 0$.

In other words, this theorem asserts that the *feedback law*

$$v = -JB^*\Lambda_\omega^{-1}x,$$

where $J : G' \to G$ denotes again the canonical Riesz isomorphism, uniformly stabilizes the control problem

$$x' = Ax + Bv, \qquad x(0) = x_0$$

with a decay rate at least equal to ω.

The well-posedness means here that (32) has a unique solution $x \in C(\mathbf{R}; H)$ for every $x_0 \in H$.

Sketch of the proof. We admit the well-posedness of (32) and we write Λ_ω in the following form:

$$\Lambda_\omega = \int_0^{T_\omega} e_\omega(s)e^{-sA}BJB^*e^{-sA^*}\, ds.$$

Fix $x_0 \in H$ arbitrarily and consider the solution of (32). A simple (formal) computation leads to the following identity:

$$\frac{d}{dt}\langle\Lambda_\omega^{-1}x, x\rangle_{H',H} = \langle\Lambda_\omega^{-1}x, (A\Lambda_\omega + \Lambda_\omega A^* - 2BJB^*)\Lambda_\omega^{-1}x\rangle_{H',H}. \tag{34}$$

Since

$$2\omega e_\omega(s) \le -e_\omega'(s) \quad \text{and} \quad e_\omega(T_\omega) = 0,$$

we have

$$A\Lambda_\omega + \Lambda_\omega A^* + 2\omega\Lambda_\omega \le -\int_0^{T_\omega} \frac{d}{ds}\left(e_\omega(s)e^{-sA}BJB^*e^{-sA^*}\right) ds = BJB^*.$$

Hence we obtain that

$$A\Lambda_\omega + \Lambda_\omega A^* - 2BJB^* \le -2\omega\Lambda_\omega.$$

(It means that the right-hand side minus the left-hand side is positive semidefinite.) Therefore we deduce from the identity (34) the following inequality:

$$\frac{d}{dt}\langle\Lambda_\omega^{-1}x, x\rangle_{H',H} \le -2\omega\langle\Lambda_\omega^{-1}x, x\rangle_{H',H}.$$

Hence

$$\langle \Lambda_\omega^{-1} x(t), x(t)\rangle_{H',H} \le \langle \Lambda_\omega^{-1} x_0, x_0\rangle_{H',H} e^{-2\omega t} \tag{35}$$

for all $t \ge 0$. Since $\Lambda_\omega \in L(H', H)$ is a selfadjoint, positive definite isomorphism, there exist two constants c_1 and c_2 such that

$$c_1 \|x\|_H^2 \le \langle \Lambda_\omega^{-1} x, x\rangle_{H',H} \le c_2 \|x\|_H^2$$

for all $x \in H$. Using these inequalities, (35) implies (33) with $M = \sqrt{c_2/c_1}$. ■

The above proof is correct in the finite-dimensional case, but there are some technical difficulties in the infinite-dimensional case due to the rather weak regularity of the solutions of (32). We overcome this difficulty by working with an equivalent integral equation.

Fix $\varphi_0 \in D((A^*)^2)$ and consider the solution of (30). We have

$$-\|B^*\varphi_0\|_{G'}^2 = \int_0^{T_\omega} \frac{d}{ds}(e_\omega(s)\|B^*\varphi(s)\|_{G'}^2)\, ds.$$

$$= \int_0^{T_\omega} e_\omega'(s)\|B^*\varphi(s)\|_{G'}^2\, ds - \langle \Lambda_\omega\varphi_0, A^*\varphi_0\rangle_{H,H'} - \langle A^*\varphi_0, \Lambda_\omega\varphi_0\rangle_{H',H},$$

and hence

$$-\|B^*\varphi_0\|_{G'}^2 = \int_0^{T_\omega} e_\omega'(s)\|B^*\varphi(s)\|_{G'}^2\, ds - \langle \Lambda_\omega\varphi_0, A^*\varphi_0\rangle_{H,H'} - \langle A^*\varphi_0, \Lambda_\omega\varphi_0\rangle_{H',H} \tag{36}$$

for all $\varphi_0 \in D(A^*)$, too, by a density argument. Identifying H' with H, by hypothesis (H3) we obtain existence of a nonnegative bounded selfadjoint operator $C \in L(H, H)$ (defined as a square root) such that

$$\|C\Lambda_\omega\varphi_0\|_H^2 = -\int_0^{T_\omega} e_\omega'(s)\|B^*\varphi(s)\|_{G'}^2\, ds$$

for all $\varphi_0 \in D(A^*)$. Then we conclude from (36) that Λ_ω satisfies the *algebraic Riccati equation*

$$A\Lambda_\omega + \Lambda_\omega A^* - BJB^* + \Lambda_\omega C^* C \Lambda_\omega = 0.$$

Thanks to hypotheses (H1)-(H3) we may apply a theorem of Flandoli (1987) to conclude that Λ_ω^{-1} satisfies the *dual algebraic Riccati equation*

$$\Lambda_\omega^{-1} A + A^* \Lambda_\omega^{-1} - \Lambda_\omega^{-1} BJB^* \Lambda_\omega^{-1} + C^* C = 0$$

in the following sense: the operator $A - BJB^*\Lambda_\omega^{-1}$ "generates" a strongly continuous group $U(s)$ in H, and

$$\Lambda_\omega^{-1} = U(t-s)^* \Lambda_\omega^{-1} U(t-s) + \int_s^t U(r-s)^*(C^*C + \Lambda_\omega^{-1} BJB^* \Lambda_\omega^{-1}) U(r-s)\, dr \tag{37}$$

for all $t, s \in \mathbf{R}$. (See also Komornik, 1997, for a formal justification of (37).)

Since we have

$$C^*C + \Lambda_\omega^{-1}BJB^*\Lambda_\omega^{-1} \geq C^*C \geq 2\omega\Lambda_\omega^{-1},$$

we deduce from (37) the inequality

$$\Lambda_\omega^{-1} \geq U(t-s)^*\Lambda_\omega^{-1}U(t-s) + 2\omega \int_s^t U(r-s)^*\Lambda_\omega^{-1}U(r-s)\, dr$$

for all $t \geq s$.

Now fix $x_0 \in H$ and solve (32). Putting

$$Qy := \langle \Lambda_\omega^{-1}y, y\rangle_{H',H}$$

for brevity, it follows from the preceding inequality that

$$Qx(s) \geq Qx(t) + 2\omega \int_s^t Qx(r)\, dr$$

for all $t \geq s$. If we can infer from this estimate that

$$Qx(t) \leq (Qx_0)e^{-2\omega t}$$

for all $t \geq 0$, then the proof of the theorem can be completed as above. Thus it only remains to prove the simple

LEMMA 5.1 *Let $f : [0, +\infty) \to \mathbf{R}$ be a continuous function and let $\omega > 0$ be a real number. Assume that*

$$f(s) \geq f(t) + 2\omega \int_s^t f(r)\, dr$$

for all $t > s \geq 0$. Then

$$f(t) \leq f(0)e^{-2\omega t}$$

for all $t \geq 0$.

The proof of this lemma is easy if f is of class C^1: letting $s \to t$ we obtain $f' + 2\omega f \leq 0$ in $(0, +\infty)$, whence the function $e^{2\omega t}f(t)$ is nonincreasing in $[0, +\infty)$. The general case then follows by approximating f by the sequence of functions f_k of class C^1, given by the formula

$$f_k(t) := k \int_t^{t+k^{-1}} f(s)\, ds, \quad k = 1, 2, \ldots$$

REMARK 5.1

- Bourquin (1998) gave another proof of the well posedness of (32).
- Various numerical and experimental tests were conducted by Bourquin, Briffaut and Collet (1997), Bourquin, Briffaut and Urquiza (1997), on the efficiency of these feedbacks.
- Theorem 5.2 was recently generalized by Loreti (1999) to cases where the problem (30) is only *partially* observable, i.e., the first inequality in (H3) is weakened.

5.4. Application to the wave equation

We recall from subsection 5.2. that if we write the problem

$$\begin{cases} u'' - \Delta u = 0 \quad \text{in} \quad \Omega \times \mathbf{R}, \\ u = 0 \quad \text{on} \quad \Gamma \times \mathbf{R}, \\ u(0) = u_0 \quad \text{and} \quad u'(0) = u_1 \quad \text{in} \quad \Omega, \\ \psi = \partial_\nu u \quad \text{on} \quad \Gamma \times \mathbf{R} \end{cases}$$

in the abstract form

$$\varphi' = -A^*\varphi, \qquad \varphi(0) = \varphi_0, \qquad \psi = B^*\varphi,$$

then the corresponding control problem

$$x' = Ax + Bv, \qquad x(0) = x_0$$

is equivalent to

$$\begin{cases} y'' - \Delta y = 0 \quad \text{in} \quad \Omega \times \mathbf{R}, \\ y = v \quad \text{on} \quad \Gamma \times \mathbf{R}, \\ y(0) = y_0 \quad \text{and} \quad y'(0) = y_1 \quad \text{in} \quad \Omega. \end{cases}$$

(We use here an infinite time interval instead of $[0,T]$.) Since the hypotheses (H1) to (H3) are satisfied, we can apply Theorem 5.2 It remains to identify the feedback $v = -JB^*\Lambda_\omega^{-1}x$. Writing the operator

$$\Lambda_\omega^{-1} : H^{-1}(\Omega) \times L^2(\Omega) \to H_0^1(\Omega) \times L^2(\Omega)$$

in the matrix form

$$\Lambda_\omega^{-1} = \begin{pmatrix} P & -Q \\ -R & S \end{pmatrix}$$

and using the definition of B^*, we have

$$v = -JB^*\Lambda_\omega^{-1}x = \frac{\partial}{\partial_\nu}(Py' + Qy).$$

(We identified $G = L^2(\Gamma)$ with its dual G'.) We have thus proved the

THEOREM 5.3 *Let Ω be of class C^2 and fix an arbitrarily large positive number ω. Then there exist two bounded linear maps*

$$P : H^{-1}(\Omega) \to H_0^1(\Omega), \qquad Q : L^2(\Omega) \to H_0^1(\Omega)$$

and a constant M such that the closed-loop problem

$$\begin{cases} y'' - \Delta y = 0 \quad \textit{in} \quad \Omega \times \mathbf{R}, \\ y = \partial_\nu(Py' + Qy) \quad \textit{on} \quad \Gamma \times \mathbf{R}, \\ y(0) = y_0 \quad \textit{and} \quad y'(0) = y_1 \quad \textit{in} \quad \Omega \end{cases}$$

is well-posed in $\mathcal{H} := L^2(\Omega) \times H^{-1}(\Omega)$, and its solutions satisfy the estimates

$$\|(y, y')(t)\|_{\mathcal{H}} \le M\|(y_0, y_1)\|_{\mathcal{H}} e^{-\omega t}$$

for all $t \ge 0$ and for all $(y_0, y_1) \in \mathcal{H}$.

5.5. Application to the plate model

By applying the results of Subsection 2.3, we obtain from Theorem 5.2 the

THEOREM 5.4 *Let Ω be of class C^4 and fix an arbitrarily large positive number ω. Then there exist two bounded linear maps*

$$P : H^{-2}(\Omega) \to H_0^2(\Omega), \qquad Q : L^2(\Omega) \to H_0^2(\Omega)$$

and a constant M such that the closed-loop problem

$$\begin{cases} y'' + \Delta^2 y = 0 \quad \text{in} \quad \Omega \times \mathbf{R}, \\ y = 0 \quad \text{and} \quad \partial_\nu y = \Delta(Py' + Qy) \quad \text{on} \quad \Gamma \times \mathbf{R}, \\ y(0) = y_0 \quad \text{and} \quad y'(0) = y_1 \quad \text{in} \quad \Omega \end{cases}$$

is well-posed in $\mathcal{H} := L^2(\Omega) \times H^{-2}(\Omega)$, and its solutions satisfy the estimates

$$\|(y, y')(t)\|_{\mathcal{H}} \leq M \|(y_0, y_1)\|_{\mathcal{H}} e^{-\omega t}$$

for all $t \geq 0$ and for all $(y_0, y_1) \in \mathcal{H}$.

The proof is left to the reader (or see Komornik, 1997).

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