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Boundary layer homogenization for periodic oscillating boundaries

by

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Abstract: The paper is devoted to the study of the boundary layer behaviour of solutions to partial differential equations occurring in domains with periodic oscillating boundaries, the frequency and the amplitude of the oscillations being the same. First, the transport method, a classical one from the optimal design theory, is used in order to state the problem in a fixed domain; then, an adapted two-scale boundary layer convergence is developed. Apart from this new hybrid approach, the main difference with related works is consideration of a bounded unit-cell, yielding a simple functional framework. Convergence, as well as a homogenized equation for the first order boundary layer term are given, and a first order corrector result is proved. This a priori bounding is very well suited to problems of control, and to numerical implementation considerations.

The difficulty in obtaining higher order correctors due to the bounding of the unit-cell is finally discussed.

Keywords: PDEs, boundary layer, periodic oscillating boundaries.

1. Introduction

The aim of this paper is the study of boundary layer effects that occur in problems involving periodic surface oscillations.

In many industrial areas, structures with rapidly oscillating boundaries are of very important practical use. After mastering the use of composite micro-structured materials in aerospace and solid mechanics engineering, such industries try now to master the know-how in using materials with oscillating surfaces; the "shark skin"-like coating of aeroplanes, or acoustic rooms are among many examples. To this end, "of particular importance is the analysis of the behavior of solutions near boundaries and, possibly, any associated boundary layers" as reported in the celebrated book by Bensoussan–Lions–Papanicolaou (1978).

However, the conclusion was then (in 1978) that "Relatively little seems to derived homogenized limits for very oscillating boundaries for the Laplace problem, without studying boundary layer effects. Then, in the seminal book by Sanchez-Palencia (1980), many problems related to boundary layers, oscillating boundaries and spectral perturbations due to oscillations in the boundary were studied.

Using the same techniques as Sanchez-Palencia, namely asymptotic expansion analysis, Belyaev (1988) studied the specific problem of rapidly oscillating boundaries. The study of boundary layer effects for oscillating boundaries was done by Belyaev in 1990, as reported by Friedman et al. (1997; see the reference—in Russian—cited therein). New results were obtained by Chechkin, Piatnitski and Friedman (1996). The latter author of this paper, together with Bei Hu and Yong Liu (1997) improved the results obtained by Belyaev, and directly addressed the problem of boundary layer correctors. Let us also mention the studies done by Kohn and Buttazzo (1987) and Mosco, Buttazzo and Dal Maso (1987, 1989), where the problem of oscillating thin reinforcement was adressed.

At present, a considerable amount of work is being done for the study of oscillating boundaries or interfaces, rough or very rough structures, and related boundary layer effects using asymptotic or homogenization techniques (although being not comparable to the available classical "domain" homogenization theories and results). Most of the studies are done for the case, in which the frequency of the oscillations is of a higher order than the amplitude, using asymptotic expansion techniques.

On the other hand, Allaire (1992) developed the well-known two-scale convergence method for the "domain" periodic homogenization problems. This simple and powerful technique was adapted and applied with success to many different problems. It was, in particular, adapted by Allaire and Conca (1998) to the study of the so-called two-scale boundary layers, in a special case related to fluid-structure spectral homogenization.

In the present paper, we study the problem of oscillating boundaries from a new point of view, combining an adapted two-scale boundary homogenization method, inspired by Allaire and Conca (1998), with the transport technique, well-known within the optimal design theory. The main difference with respect to related works is the choice of a *bounded* unit cell, very useful in view of numerical implementation, where boundary layer terms are expected to predominate. And our main result is to prove that this is not a bad idea (at the first order, at least).

The benefit in using the transport method is that one could then easily consider oscillations *in any arbitrary direction, with arbitrary sign.* On the other hand, consideration of a *bounded* unit-cell yields an estimation with respect to the depth of the cell, which can be related to the oscillation frequency in order to optimize the computational cost of the boundary layer terms. Another important technical benefit is the possibility of using the classical compactness design of the shape of the oscillations, and aims to prove the existence of an optimum.

In Section 2, a two-scale boundary layer approach is presented, adapted from Allaire and Conca (1998). For the sake of self-consistency, some of the original proofs are re-called, but simple and new results are also presented (thanks to a different functional framework).

Then, in Section 3, the domain transport technique is presented and the direct application to the transport of oscillations is stated, providing us with a variational framework, well suited to the application of the earlier developed two-scale approach.

To illustrate this approach, a second order elliptic equation is taken as a model problem in Section 4. We consider a particular limit case where the frequency and the amplitude of the oscillations are of the same order. The existence of boundary layer terms, and a corrector result, are proved.

Finally, a short conclusion is given in Section 5.

2. A two-scale boundary layer approach

In the following, $\Omega \subset \mathbb{R}^N$ is an open set, with a Lipschitz boundary. We focus our attention on a selected part on this boundary, denoted by Γ_0. Without any particular assumption on the whole geometry of the domain, we shall nevertheless assume that the involved boundary Γ_0 is plane, and is a union of small cells homothetic (with a ratio ϵ) to a unit-cell period denoted Y'.

Given a positive small enough real number ϵ, and a fixed positive real $L > 1$, we define the strip B_ϵ as the one obtained by normal inward increasing of the boundary Γ_0: $B_\epsilon = \Gamma_0 \times]0, L\epsilon[$, i.e. any point x of B_ϵ is of the form $x = (x', x_n)$, where x' and x_n are, respectively, the curvilinear and normal coordinates of x. The strip B_ϵ is then the union of small cells homothetic (with ratio ϵ^2) to the unit cell $G = Y' \times]0, L[$.

We emphasize *the dependance of the unit-cell G on the number L*, which is chosen *arbitrarily*. Hence, any function defined on G is implicitly depending on L. For the simplicity of the exposure, the dependence is not explicitly written, except when it is of importance to underline it.

Our aim being to study functions that have periodic oscillations along Γ_0, and are defined only in a thin neighborhood of this boundary, we naturally introduce the following functional space:

$$L^2(\Gamma_0; C_\#(\overline{G})) = \{\phi(x', y);\ x' \in \Gamma_0,\ y = (y', y_n) \in G;\ \phi(., y) \in L^2(\Gamma_0);$$
$$\phi(x', .) \in C(\overline{G})),\ \text{periodic w.r.t. } y'\}.$$

which is suitable in the present framework, due to the following:

LEMMA 2.1 *For any $\phi \in L^2(\Gamma_0; C_\#(\overline{G}))$, one has*

Proof. By density, we can take smooth enough functions. A simple scaling of the variable x_n yields:

$$\frac{1}{\epsilon}\int_{B_\epsilon}\left|\phi\left(x',\frac{x}{\epsilon}\right)\right|^2 dx = \frac{1}{\epsilon}\int_0^{L\epsilon}\int_{\Gamma_0}\left|\phi\left(x',\frac{x'}{\epsilon},\frac{x_n}{\epsilon}\right)\right|^2 dx'\,dx_n$$
$$= \int_0^L\int_{\Gamma_0}\left|\phi\left(x',\frac{x'}{\epsilon},y_n\right)\right|^2 dx'\,dy_n.$$

Then, from the classical homogenization results, see Bensoussan et al. (1978), Jikov et al. (1994), we know that

$$\lim_{\epsilon\to 0}\int_{\Gamma_0}\left|\phi\left(x',\frac{x'}{\epsilon},y_n\right)\right|^2 dx'$$
$$= \frac{1}{|Y'|}\int_{\Gamma_0\times Y'}|\phi(x',y',y_n)|^2\,dx'\,dy' \text{ for any } y_n.$$

and

$$\int_{\Gamma_0}\left|\phi\left(x',\frac{x'}{\epsilon},y_n\right)\right|^2 dx' \le \int_{\Gamma_0}\max_{z\in\overline{Y'}}|\phi(x',z,y_n)|^2\,dx'.$$

Application of the Lebesgue dominated convergence theorem yields

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{B_\epsilon}\left|\phi\left(x',\frac{x}{\epsilon}\right)\right|^2 dx = \int_0^L\frac{1}{|Y'|}\int_{\Gamma_0\times Y'}|\phi(x',y',y_n)|^2\,dx'\,dy'\,dy_n$$
$$= \frac{1}{|Y'|}\int_{\Gamma_0\times G}|\phi(x',y)|^2\,dx'\,dy.$$

We are now in a position to introduce the appropriate definition.

DEFINITION 2.1 *Let $(u_\epsilon)_{\epsilon>0}$ be a sequence in $L^2(\Omega)$. It is said to two-scale converge in the sense of boundary layers on Γ_0, if there exists a function $u_0(x',y)\in L^2(\Gamma_0\times G)$ such that*

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{B_\epsilon}u_\epsilon(x)\phi\left(x',\frac{x}{\epsilon}\right)dx = \frac{1}{|Y'|}\int_{\Gamma_0\times G}u_0(x',y)\phi(x',y)\,dx'\,dy$$

for any $\phi\in L^2(\Gamma_0;C_\#(\overline{G}))$.

Within the precise sense of the Definition 2.1 above, we shall denote such a property by: $u_\epsilon \;{}_{\mathrm{BL}}\!\rightharpoonup u_0$.

Following Allaire and Conca (1998), this definition makes sense because of the following compactness theorem:

THEOREM 2.1 *Let $(u_\epsilon)_{\epsilon>0}$ be a sequence in $L^2(\Omega)$ such that there exists a constant C independent of ϵ for which*

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Then, up to a subsequence, there exists a limit $u_0(x',y) \in L^2(\Gamma_0 \times G)$ such that u_ϵ two-scale converges in the sense of boundary layers to u_0.

From now on, the notation $L^2_\#$ stands for the space $L^2(\Gamma_0; C_\#(\overline{G}))$, when there is no possible confusion.

Proof. From the Cauchy–Schwarz inequality,

$$\left|\frac{1}{\epsilon}\int_{B_\epsilon} u_\epsilon(x)\phi\left(x', \frac{x}{\epsilon}\right) dx\right|$$
$$\leq \frac{1}{\sqrt{\epsilon}}\|u_\epsilon\|_{L^2(B_\epsilon)}.\left(\frac{1}{\epsilon}\int_{B_\epsilon}\left|\phi\left(x', \frac{x}{\epsilon}\right)\right|^2 dx\right)^{\frac{1}{2}} \leq C.\|\phi\|_{L^2_\#}.$$

Thus, the sequence

$$\mu_\epsilon : L^2_\# \longrightarrow \mathbb{R},\ \phi \longrightarrow \frac{1}{\epsilon}\int_{B_\epsilon} u_\epsilon(x)\phi\left(x', \frac{x}{\epsilon}\right) dx$$

is a bounded sequence of linear continuous forms over the separable space $L^2_\#$.

Then, one extracts a subsequence, still denoted (μ_ϵ), which converges for the weak-* topology to a limit $\mu_0 \in (L^2_\#)'$, the dual space of $L^2_\#$.

From Lemma 2.1, one has

$$|(\mu_0, \phi)| = |\lim_{\epsilon \to 0}(\mu_\epsilon, \phi)| \leq C.\|\phi\|_{L^2(\Gamma_0 \times G)}. \tag{1}$$

Since $L^2_\#$ is dense in $L^2(\Gamma_0 \times G)$, it follows that μ_0 can be extended to a linear continuous mapping over $L^2(\Gamma_0 \times G)$, it is then identified, by virtue of the Riesz theorem to a unique element $u_0(x',y) \in L^2(\Gamma_0 \times G)$ such that:

$$(\mu_0, \phi) = \frac{1}{|Y'|}\int_{\Gamma_0 \times G} u_0(x', y)\phi(x', y)\, dx'\, dy.$$

The above positive result for the $L^2(\Omega)$ case, naturally encourages us to study the extension to the case of the Sobolev space $H^1(\Omega)$.

Let us mention that within the framework of sequences in the space $H^1(\Omega)$, we are in a slightly different case than in Allaire and Conca (1998) where the reference cell G was unbounded, requiring the introduction of Deny-Lions type spaces.

The boundedness of the unit cell G in our case, allows for a simple and rather classical functional framework. However, it will be seen in Section 4.2.3 that there is a counterpart : the obtained corrector results are apparently weaker than one could expect.

Let $C^\infty_\#(G)$ be the space of infinitely differentiable functions in $\overline{G}$ which are Y'-periodic in y'. The space $C^\infty_{c\#}(G)$ is the space of infinitely differentiable functions in G, y'-periodic, which vanish over $y_n = L$. The space $H^1_\#(G)$ (respectively $H^1_{0\#}(G)$) is the Sobolev space obtained by completion of $C^\infty_\#(G)$ (respectively $C^\infty_{c\#}(G)$) with respect to the $H^1(G)$-norm.

THEOREM 2.2 *Let be a function u and a sequence $(u_\epsilon)_{\epsilon>0}$ in $H^1(\Omega)$ such that there exists a constant C independent of ϵ for which*

$$\frac{1}{\sqrt{\epsilon}}(\|u_\epsilon - u\|_{L^2(\Omega)} + \|\nabla u_\epsilon - \nabla u\|_{L^2(\Omega)}) \leq C. \tag{2}$$

Then, there exists a subsequence, still denoted (u_ϵ), and a function $u_1(x', y) \in L^2(\Gamma_0; H^1_\#(G)/\mathbb{R})$ such that:

$$\left\{ \begin{array}{l} u_\epsilon \overset{\text{BL}}{\rightharpoonup} u_\Gamma \\ \nabla(u_\epsilon - u) \overset{\text{BL}}{\rightharpoonup} \nabla_y u_1 \end{array} \right. \tag{3}$$

where u_Γ is the trace of u over the boundary Γ_0.

Moreover, if the function u belongs to $H^2(\Omega)$, then

$$\nabla u_\epsilon \overset{\text{BL}}{\rightharpoonup} \nabla u_{|\Gamma_0} + \nabla_y u_1.$$

Proof. We proceed in two steps. First, we prove that the sequence (u_ϵ) converges in the boundary layer sense to u_Γ, the trace of u over Γ_0. Then, following Allaire (1992), we prove the convergence of the gradients.

Step one. Considering the inequality (2) above and the continuity of the trace operator, we remark that the trace $u_{\epsilon|\Gamma_0}$ strongly converges to u_Γ in $L^2(\Gamma_0)$. Thus, the sequence $(u_{\epsilon|\Gamma_0})$ two-scale converges in the classical sense to u_Γ which means that

$$\int_{\Gamma_0} u_{\epsilon|\Gamma_0}(x')\phi\left(x', \frac{x'}{\epsilon}\right) dx' \to \frac{1}{|Y'|}\int_{\Gamma_0\times Y'} u_\Gamma(x')\phi(x', y')\, dx'\, dy' \tag{4}$$

for all $\phi \in L^2(\Gamma_0; C_\#(\overline{Y'}))$.

Now, for any $\psi \in L^2(\Gamma_0; C_\#(\overline{G}))$, we take $\phi(x', y') = \int_0^L \psi(x', y', y_n)\, dy_n$ and use it in the expression (4), which reads

$$\begin{aligned} &\frac{1}{\epsilon}\int_{B_\epsilon} u_{\epsilon|\Gamma_0}(x')\psi\left(x', \frac{x'}{\epsilon}, \frac{x_n}{\epsilon},\right) dx_n\, dx' \\ &\to \frac{1}{|Y'|}\int_{\Gamma_0\times G} u_\Gamma(x')\psi(x', y)\, dx'\, dy \end{aligned} \tag{5}$$

where $y = (y', y_n)$.

To conclude that u_ϵ converges in the two-scale boundary layer sense to u_Γ, namely:

$$\frac{1}{\epsilon}\int_{B_\epsilon} u_\epsilon(x)\psi\left(x', \frac{x}{\epsilon}\right) dx \to \frac{1}{|Y'|}\int_{\Gamma_0\times G} u_\Gamma(x')\psi(x', y)\, dx'\, dy, \tag{6}$$

we have to prove that the difference term (which is given a sense using the usual density argument):

$$\frac{1}{\epsilon}\int (u_\epsilon(x) - u_\epsilon(x'))\psi\left(x', \frac{x}{\epsilon}\right) dx$$

has a zero limit. We first rewrite it as follows:

$$\int_{B_\epsilon} \frac{(u_\epsilon(x) - u_{\epsilon|\Gamma_0}(x'))}{\epsilon} \psi\left(x', \frac{x}{\epsilon}\right) dx$$

and then, using the simple Hardy inequality,

$$\|u_\epsilon - u_{\epsilon|\Gamma_0}\|_{L^2(B_\epsilon)} \leq C\epsilon L \|\nabla u_\epsilon\|_{L^2(B_\epsilon)},$$

one easily obtains the upper-bound $C\|\nabla u_\epsilon\|_{L^2(B_\epsilon)}\|\psi(x', \frac{x}{\epsilon})\|_{L^2(B_\epsilon)}$.

The term $\|\nabla u_\epsilon\|_{L^2(B_\epsilon)}$ is bounded with respect to ϵ from the hypothesis (2), and from Lemma 2.1 the term $\left\|\psi(x', \frac{x}{\epsilon})\right\|_{L^2(B_\epsilon)}$ tends to zero.

REMARK 2.1 *When considering in Theorem 2.2 the case of constant sequences $v_\epsilon = v$ of $H^1(\Omega)$, the first statement affirms that any function $v \in H^1(\Omega)$ converges in the boundary layer sense to its trace v_Γ over Γ_0. Indeed, this fact is very much expected from our "boundary layer convergence" definition.*

Step two. Now, let $v_\epsilon = u_\epsilon - u$. We know from the above that v_ϵ two-scale converges in the boundary layer sense to zero.

The sequence $\frac{1}{\sqrt{\epsilon}}(\nabla v_\epsilon)$ being bounded in $L^2(B_\epsilon)^N$, there exists a function $\xi_0(x', y) \in L^2(\Gamma_0 \times G)^N$ such that:

$$\nabla v_\epsilon \ {}_{\mathrm{BL}}\!\!\rightharpoonup \xi_0 \tag{7}$$

or, in other words,

$$\frac{1}{\epsilon}\int_{B_\epsilon} (\nabla v_\epsilon(x)).\Psi\left(x', \frac{x}{\epsilon}\right) dx$$
$$\rightarrow \frac{1}{|Y'|}\int_{\Gamma_0 \times G} \xi_0(x', y).\Psi(x', y)\, dx'\, dy \text{ for any } \Psi \in (L^2_\#)^N. \tag{8}$$

Integrating by parts, we get:

$$\int_{B_\epsilon} \nabla v_\epsilon(x).\Psi\left(x', \frac{x}{\epsilon}\right) dx$$
$$= -\frac{1}{\epsilon}\int_{B_\epsilon} v_\epsilon(x) div_y \Psi\, dx - \int_{B_\epsilon} v_\epsilon(x) div_{x'} \Psi\, dx + \int_{\partial B_\epsilon} v_\epsilon(x)\Psi.n\, d\Gamma. \tag{9}$$

Then, with the particular choice of smooth functions Ψ such that

$$div_y \Psi = 0 \text{ in } G,\ \ \Psi.n = 0 \text{ over } \partial G,$$

one obtains:

$$\frac{1}{\epsilon}\int \nabla v_\epsilon(x).\Psi\left(x', \frac{x}{\epsilon}\right) dx = -\frac{1}{\epsilon}\int v_\epsilon(x) div_{x'}\Psi\, dx \tag{10}$$

Since the sequence (v_ϵ) two-scale converges to zero, the right-hand side vanishes when passing to the limit, and one gets:

$$\frac{1}{|Y'|}\int_{\Gamma_0\times G}\xi_0(x',y).\Psi(x',y)\,dx'\,dy=0. \tag{11}$$

The function $\xi_0(x',y)$ being orthogonal to the space of divergence-free functions, it is then a gradient (see, e.g., Temam, 1979, p. 15). Thus, there exists a unique function $u_1 \in L^2(\Gamma_0; H^1_\#(G)/\mathbb{R})$ such that

$$\nabla(u_\epsilon - u) \xrightarrow{\text{BL}} \xi_0(x',y) = \nabla_y u_1(x',y).$$

The last statement of the theorem is a direct application to ∇u of the first one, which states that any function of $H^1(\Omega)$ two-scale converges in the boundary layer sense to its trace over the considered boundary.

REMARK 2.2 • *The local character of the boundary layer convergence defined above implies that there is no need for any restricting condition on the whole geometry of Ω. It is clearly seen from the proof that the upper-bound condition (2) of Theorem 2.2 needs not to be satisfied in the whole domain Ω, but only in an open Lipschitzian neighborhood $\omega \subset \Omega$ independent from ϵ such that $B_\epsilon \subset \omega$ for sufficiently small ϵ.*

- *As emphasized in the introductory section, the boundary layer limit u_1 depends on the depth L.*

3. The transport method

The transport or *homotopy* method is very popular in the modern *optimal design* theory. It was developed in order to circumvent the lack of an adequate topological and algebraic structure in sets of open domains of $\mathbb{R}^N$.

The method consists in considering domains which are the images of a reference one through suitable mappings. This allows for a rigorous mathematical setting: the mappings belong to an open subset of a Banach space, where continuity and differentiability are well defined. Then, these definitions are transferred on the set of domains, see e.g. Simon and Murat (1976) for a general presentation.

By its essence, this technique does not alter the topology of the reference domain: it does not create nor removes holes, cracks... all the transported domains being topologically identical to the original reference one. The changes in topology within the optimal design framework is the concern of a recent and quite unexplored theory, the *topology optimization*, see Allaire et al. (1997) and the references therein.

The transport method allows, in particular, for domain perturbations to be defined as perturbations of the corresponding mappings (in the process leading approach to generate the oscillating domains, from an initial non-oscillating reference one.

Let Ω be an open subset of $\mathbb{R}^N$, $N = 2$ or 3, with a regular enough boundary $\partial\Omega$, e.g. piece-wise C^1. A part of this boundary, denoted Γ_0, is intended to oscillate.

We denote by Γ_1 the remaining non-oscillating part, i.e. $\partial\Omega = \Gamma_0 \cup \Gamma_1$.

Without loss of generality, we assume that Γ_0 is located in the plane $x_n = 0$, any generic point x of $\mathbb{R}^N$ being denoted $x = (x', x_n)$, $x' \in \mathbb{R}^{N-1}$.

We denote by $Y' =]0,1[^{N-1}$ the unit cell of $\mathbb{R}^{N-1}$, and by $G = Y' \times]0, L[$ the unit cell of $\mathbb{R}^N$, with the same convention as above, $y = (y', y_n) \in G$. We use the letter G instead of the standard Y because we are dealing with functions or operators with periodicity only with respect to the y' variable.

First, we detail the formal construction of the oscillating domains.

Given a Y'-periodic function, $\vec{\psi} : Y' \to \mathbb{R}^N$, such that $\vec{\psi}_{|\partial Y'} = 0$, and a positive (small) real number ϵ, we define *an oscillating perturbation* $\vec{\psi}_\epsilon$ of the boundary Γ_0 through:

$$\vec{\psi}_\epsilon : (x', 0) \in \Gamma_0 \to \vec{\psi}(x'/\epsilon) \in \mathbb{R}^N.$$

Classically, in order to define the oscillating domain Ω_ϵ, one usually considers a *harmonic extension* $\vec{V}_\epsilon$ of $\vec{\psi}_\epsilon$ on the reference domain Ω: the field $\vec{V}_\epsilon$ is the unique solution to the Laplace equation:

$$\begin{cases} -\Delta \vec{V}_\epsilon = 0 \text{ in } \Omega \\ \vec{V}_\epsilon = \vec{\psi}_\epsilon \text{ over } \Gamma_0 \\ \vec{V}_\epsilon = 0 \text{ over } \Gamma_1. \end{cases} \tag{12}$$

The oscillating domain Ω_ϵ is then defined as the image of the reference domain Ω through the mapping: $T_\epsilon = I + \epsilon^\alpha.\vec{V}_\epsilon$, where $\alpha \geq 0$ is a real non-negative parameter, which measures the relative speed of the oscillations:

$$\Omega_\epsilon = T_\epsilon(\Omega) = \{x + \epsilon^\alpha \vec{V}_\epsilon(x),\ x \in \Omega\}.$$

It is clear from the definition of Ω_ϵ that the image of Γ_0 through T_ϵ is now an oscillating boundary, which we denote by Γ_ϵ, see Fig. 1.

To complete the construction, let us note that the transport method requires that, for a fixed $\epsilon > 0$, the mapping T_ϵ should be a lipeomorphism over Ω (lipschitzian, bijective and of lipschitzian inverse). For instance, this is straightforward for any $\vec{\psi}$ which is Y'-periodic, with bounded first and second order derivatives, i.e. $\vec{\psi} \in W^{2,\infty}_{\#}(Y'; \mathbb{R}^N)$.

From now on, we only consider the case where the amplitude and the frequency of the oscillations *are of the same order*: $\alpha = 1$.

Moreover, in order to properly apply the two-scale boundary layer technique as presented in Section 2, we choose perturbation fields $\vec{V}_\epsilon$ which vanish outside

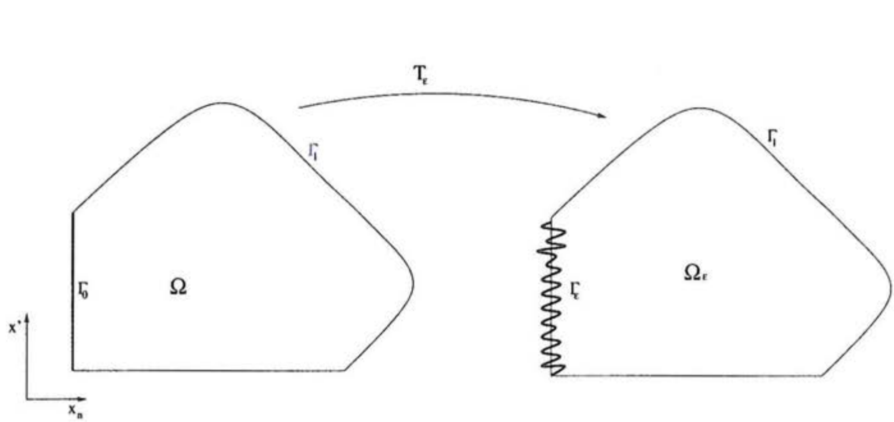

Figure 1. The reference and the oscillating domains.

large, while γ is of order of the unit, e.g. $\gamma = 1$. Such fields could be explicitly approximated by mappings of the form

$$\vec{V}_\epsilon(x', x_n) = \vec{\psi}(x'/\epsilon)F(x_n/\epsilon), \tag{13}$$

where F is a smooth mollifier, such as $F(t) = exp\left(\frac{-t}{\gamma - t}\right)$, $0 \leq t \leq \gamma$.

4. A second order elliptic equation as a model problem

In this section, we present the model with which we aim to illustrate our approach, combining the transport method and the boundary layer version of the two-scale convergence method. Following Allaire (1992) and others, we restrict ourselves to a simple partial derivative equation, which nevertheless models a large class of physical phenomena.

We consider the following second order linear elliptic problem:

$$\begin{cases} -div(A\nabla U_\epsilon) = f \text{ in } \Omega_\epsilon \\ \partial_\nu U_\epsilon = \epsilon g \text{ over } \Gamma_\epsilon \\ U_\epsilon = 0 \text{ over } \Gamma_1 \end{cases} \tag{14}$$

where A is a constant real matrix, satisfying the ellipticity (and continuity) hypothesis: there exist two constants $a_1 \geq a_0 > 0$ such that

$$a_0|\xi|^2 \leq \sum_{i,j} A_{ij}\xi_i\xi_j \leq a_1|\xi|^2 \text{ for all } \xi \in \mathbb{R}^N, \tag{15}$$

and $\partial_\nu U_\epsilon$ stands for the A-normal derivative, i.e. the trace of $(A\nabla U_\epsilon).\vec{n}$ on the boundary Γ_ϵ.

The diffusion matrix A is chosen constant in order to keep clear and simple that the present transport approach is adequate for the consideration of *stratified* media (i.e. with y'-periodicity).

The source terms f and g are assumed to be, respectively, the traces in the spaces $L^2(\Omega_\epsilon)$ and $L^2(\Gamma_\epsilon)$ of functions defined on a sufficiently large open set (containing all the oscillating domains).

Set $H(\Omega_\epsilon) = \{v \in H^1(\Omega_\epsilon),\ v = 0 \text{ over } \Gamma_1\}$. It is then well known that, for any $\epsilon > 0$, there exists a unique $U_\epsilon \in H(\Omega_\epsilon)$ solution to the *variational problem* derived from the equation (14):

$$\int_{\Omega_\epsilon} (A\nabla U_\epsilon).\nabla v\, d\Omega_\epsilon = \int_{\Omega_\epsilon} fv\, d\Omega_\epsilon + \epsilon \int_{\Gamma_\epsilon} gv\, d\Gamma_\epsilon\ \forall v \in H(\Omega_\epsilon). \tag{16}$$

Due to the *elliptic* nature of the problem, one expects that the small local oscillations, of both the boundary Γ_ϵ and the flux term ϵg (with possibly periodic $g = g(x', x'/\epsilon)$), should give rise to small oscillating effects, namely, boundary layer effects.

In our way to exhibit such a behavior, the first main step is to *transport the variational form* of the model equation.

4.1. Transport of the model problem

The purpose of this step is to define a variational equation posed over the reference domain Ω, the oscillations being transferred on the diffusion operator A, and source terms f, g.

The reference Sobolev space is defined by

$$H(\Omega) = \{v \in H^1(\Omega),\ v = 0 \text{ over } \Gamma_1\}.$$

From the definition and regularity (algebraic and topological) properties of the mapping $T_\epsilon = I + \epsilon\vec{V}_\epsilon$, the natural norms over the spaces $H(\Omega)$ and $H(\Omega_\epsilon)$ are equivalent. Hence, setting $u_\epsilon = U_\epsilon \circ T_\epsilon$, where U_ϵ is the solution to the equation (16), we get the *transported* variational form:

Find $u_\epsilon \in H(\Omega)$, such that $\forall v \in H(\Omega)$,

$$\int_\Omega (A_\epsilon \nabla u_\epsilon).\nabla v\, dx = \int_\Omega f_\epsilon v\, dx + \epsilon \int_{\Gamma_0} g_\epsilon v\, d\Gamma. \tag{17}$$

where:

$$\begin{aligned} &A_\epsilon = (DT_\epsilon)^{-1}(A \circ T_\epsilon)(DT_\epsilon)^{-T}|\det(DT_\epsilon)| \\ &f_\epsilon = (f \circ T_\epsilon)\,|\det(DT_\epsilon)| \\ &g_\epsilon = (g \circ T_\epsilon)\, S_\epsilon(\vec{\psi}),\ S_\epsilon(\vec{\psi}) = \sqrt{1 + |D_{y'}\vec{\psi}|^2_{y'=x'/\epsilon}} \end{aligned} \tag{18}$$

and

$(DT_\epsilon) = (\partial(T_\epsilon)_i/\partial x_j)_{(1 \le i,j \le N)}$ is the Jacobian matrix of T_ϵ,

$\det(DT_\epsilon) \approx 1 + \epsilon\, div(V_\epsilon)$ at first order approximation.

The reader interested in more details related to the formulas above, the

Simon and Murat (1976), Pironneau (1984), Masmoudi (1987), Habbal (1996) and the references therein.

It is an easy exercise to check that the operator A_ϵ is continuous, bounded and $H(\Omega)$-elliptic (assuming that $\vec{\psi}$ belongs to the unit ball of $W^{1,\infty}(Y')$). This latter essential property, remaining stable under transport, is no more available when the oscillation speed factor α, equal to 1 in the present case, is less than 1: the smallest eigenvalue of A_ϵ tends to zero with ϵ.

In the limit case, where α is equal to zero, the oscillations occupy a subset $\omega \subset \Omega$ of non-null measure, and instead of boundary layer effects, it is shown in Brizzi and Chalot (1978) that the open set Ω is split into two subsets, ω and Ω/ω, where two *different* equations hold.

Now, our new variable is $u_\epsilon \in H(\Omega)$, a space which is *independent* from the ϵ parameter. The first benefit from the transport method is that it is not necessary to impose a sign on the oscillating motif (in order to get all the oscillating domains as subsets of the non-oscillating one), as for instance done in Brizzi and Chalot (1978), Sanchez-Palencia (1980) and Chechkin et al. (1996).

The second benefit is that it provides us immediately with a variational form well suited to a two-scale convergence framework.

REMARK 4.1 *It is straightforward, from the transport formula of A_ϵ in (18), that the case where the relative ratio α between frequency and amplitude of the oscillations is such that $\alpha > 1$, is trivial. The sequence of operators (A_ϵ) strongly converges to the operator A. Even though the study of convergence of the sequence of fields $(\vec{V}_\epsilon)$ is interesting in itself, it is not necessary for our purpose. The explicit formula (13) is sufficient to convince the reader that, in case of $\alpha = 1$, one has only a weak convergence of $(\vec{V}_\epsilon)$, and, a fortiori, of operators (A_ϵ) (to a limit operator $\bar{A}$ different from A).*

4.2. The convergence results

Let us consider three following variational problems:

- the transported model equation: Find $u_\epsilon \in H(\Omega)$, such that $\forall v \in H(\Omega)$,

$$\int_\Omega (A_\epsilon \nabla u_\epsilon)\nabla v\,dx = \int_\Omega f_\epsilon v\,dx + \epsilon \int_{\Gamma_0} g_\epsilon v\,d\Gamma. \tag{19}$$

- the "limit"problem: Find $u \in H(\Omega)$, such that $\forall v \in H(\Omega)$,

$$\int_\Omega (A\nabla u)\nabla v\,dx = \int_\Omega f v\,dx. \tag{20}$$

- the difference between the two above:

$$\forall v \in H(\Omega)\ \int_\Omega (A_\epsilon \nabla(u_\epsilon - u))\nabla v\,dx$$
$$= \int_\Omega (f_\epsilon - f)v\,dx + \epsilon \int_{\Gamma_0} g_\epsilon v\,d\Gamma + \int_\Omega (A - A_\epsilon)\nabla u \nabla v\,dx. \tag{21}$$

In what follows, we use these three equations in order to obtain an estimation for the $H^1(\Omega)$ norm of $(u_\epsilon - u)$, which is the starting point for the application of the two-scale boundary layer tools. We then prove the existence of a boundary layer term, and give an equation satisfied by this latter. In a third step, a corrector-like result is demonstrated.

In order to successfully fulfill the program above, we need to make more precise some additional points:

- by using the explicit formula (13), we remark that the transported operator A_ϵ depends only on the fast variable $y = x/\epsilon$; we define the periodic operator $\bar{A}(y)$ as the restriction to G of the operator $A_\epsilon(x/\epsilon)$;
- the source f is assumed to belong to the space $H^1(\Omega)$, particularly to fulfill the condition of the useful lemma stated below;
- the solution u of the oscillations free problem (20) should belong to the space $H^3(\Omega) \cap H(\Omega)$; this is straightforward for our concern, since we deal with a constant diffusion operator A and a source f with the regularity assumed above.

LEMMA 4.1 *There exists a positive constant C, not depending on ϵ, such that for any $v \in H^1(\Omega)$, the following upper-bound holds:*

$$\|v\|_{L^2(S_\gamma)} \leq C\sqrt{\epsilon}\|v\|_{H^1(\Omega)}. \tag{22}$$

Proof. Let there be a generic $v \in C^\infty(\overline{\Omega})$, and let v_Γ be its trace over the boundary Γ_0.

Using the Hardy inequality

$$\|v - v_\Gamma\|_{L^2(S_\gamma)} \leq C\epsilon\|\nabla v\|_{L^2(\Omega)},$$

one has

$$\|v - v_\Gamma + v_\Gamma\|_{L^2(S_\gamma)} \leq C\epsilon\|v\|_{H^1(\Omega)} + \|v_\Gamma\|_{L^2(S_\gamma)}.$$

On the other hand, it is obvious that $\|v_\Gamma\|^2_{L^2(S_\gamma)} = \epsilon\gamma\|v_\Gamma\|^2_{L^2(\Gamma_0)}$. By density of $C^\infty(\overline{\Omega})$ in $H^1(\Omega)$, and since the trace operator is continuous from the latter onto $L^2(\Gamma_0)$, we get the desired conclusion.

4.2.1. An estimation for $(u_\epsilon - u)$

The estimation for $(u_\epsilon - u)$, stated below, is rather classical, but the proof presented here is short and simple. Similar results are proved, with slightly different approaches, in Belyaev (1988), Friedman et al. (1997) and Chechkin et al. (1996).

THEOREM 4.1 *There exists a positive constant C, independent of ϵ and L, such that*

Proof. Using the uniform ellipticity of A_ϵ in equation (21), we get, for the LHS,

$$a_0\|u_\epsilon - u\|^2_{H^1(\Omega)} \le \int_\Omega A_\epsilon \nabla(u_\epsilon - u)^2\, dx. \tag{24}$$

At the same time, the right-hand side of this equation has each of its terms upper-bounded as follows:

- The first term is re-written:
  $$\int_{S_\gamma} (f_\epsilon - f)v\, dx = \int_{S_\gamma} R(x/\epsilon) f v\, dx$$
  where the linear operator $R(x/\epsilon)$ is uniformly bounded w.r.t. ϵ. Application of Lemma 4.1 yields
  $$\left|\int_{S_\gamma} R(x/\epsilon) f v\, dx\right| \le C\|f\|_{L^2(S_\gamma)}\|v\|_{L^2(S_\gamma)}$$
  $$\le C\epsilon\|f\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}. \tag{25}$$
- The second term is immediate:
  $$\left|\epsilon\int_{\Gamma_0} g_\epsilon v\, d\Gamma\right| \le C\epsilon|S|_{L^\infty(Y')}\|g\|_{L^2(\Gamma_0)}\|v\|_{H^1(\Omega)}, \tag{26}$$
- and finally the last term turns out to be the leading one:
  $$\left|\int_{S_\gamma} (A - A_\epsilon)\nabla u \nabla v\, dx\right| \le |A - \overline{A}|_{L^\infty(G)}\|\nabla u\|_{L^2(S_\gamma)}\|v\|_{H^1(\Omega)}$$
  $$\le C\sqrt{\epsilon}\|u\|_{H^2(\Omega)}\|v\|_{H^1(\Omega)}. \tag{27}$$

Now, taking $v = u_\epsilon - u$ in the three above upper-bounds, and using the inequality (24) give the conclusion sought.

4.2.2. The boundary layer terms

From the previous section, it is clear that the condition of Theorem 2.2 is fulfilled. Then, $u_\epsilon \overset{\text{BL}}{\rightharpoonup} u_\Gamma$, the trace of u over Γ_0 and, up to a subsequence, there exists a function $u_1(x', y) \in L^2(\Gamma_0; H^1_\#(G)/\mathbb{R})$ such that $\nabla(u_\epsilon - u) \overset{\text{BL}}{\rightharpoonup} \nabla_y u_1$, or thanks to the regularity of u, $\nabla u_\epsilon \overset{\text{BL}}{\rightharpoonup} \nabla u_{|\Gamma_0} + \nabla_y u_1$.

In order to derive a variational equation for u_1, let us consider the transported equation (19) with a particular choice of the test functions: $v(x) = \phi(x'; x/\epsilon)$ where $\phi(x'; .) \in C^\infty_{c\#}(G)$. Such v, extended by zero outside the strip B_ϵ belongs to the space $H(\Omega)$.

Using the rule $\nabla_x v = 1/\epsilon \nabla_y \phi + \nabla_{x'}\phi$, the equation (19) reads:

$$\frac{1}{\epsilon}\int_{B_\epsilon} (A_\epsilon \nabla u_\epsilon)\nabla_y \phi\, dx$$
$$= -\int_{B_\epsilon} (A_\epsilon \nabla u_\epsilon)\nabla_{x'}\phi\, dx + \int_{B_\epsilon} f_\epsilon \phi\, dx + \epsilon\int_{\Gamma_0} g_\epsilon \phi\, d\Gamma. \tag{28}$$

The left-hand side can be re-written $\frac{1}{\epsilon}\int_{B_\epsilon} \nabla u_\epsilon.(A_\epsilon^T \nabla_y \phi)\,dx$, and regarding $A_\epsilon^T \nabla_y \phi(x', x/\epsilon)$ as a test function, this term converges to the limit:

$$\frac{1}{\epsilon}\int_{B_\epsilon} (A_\epsilon \nabla u_\epsilon)\nabla_y \phi\,dx \to \frac{1}{|Y'|}\int_{\Gamma_0\times G} (\nabla u_{|\Gamma_0} + \nabla_y u_1)\bar{A}^T \nabla_y \phi\,dx'\,dy. \quad (29)$$

Hence, by density, the limit equation for the two-scale boundary layer u_1 is stated as follows:

Find $u_1(x', y) \in L^2(\Gamma_0; H^1_\#(G)/\mathbb{R})$ such that for any $\phi \in L^2(\Gamma_0; H^1_{0\#}(G))$,

$$\frac{1}{|Y'|}\int_{\Gamma_0\times G} \bar{A}\nabla_y u_1 \nabla_y \phi\,dx'\,dy = -\frac{1}{|Y'|}\int_{\Gamma_0\times G} \bar{A}\nabla u_{|\Gamma_0} \nabla_y \phi\,dx'\,dy. \quad (30)$$

The equation above is ill-posed due to the lack in boundary conditions. Indeed, we shall prove in the next section that the boundary layer $u_1(x', y)$ is the sum of a uniquely defined first order corrector term $c_1(x', y)$, with a compact support in $\Gamma_0 \times [0, L[$ i.e. $c_1 \in H^1_{0\#}(G)$, and a first order "tail" term $t_1(x', y)$ *whose gradient* dies exponentially with respect to L. Roughly speaking, we shall give all the local energy induced by $\nabla u_{|\Gamma_0}$ to c_1, while reporting the unknown terms (and non-unicity) on the tail term t_1.

Let then pose $u_1(x', y) = c_1(x', y) + t_1(x', y)$ with c_1 and t_1, as outlined above, solutions to the equations:

- Find $c_1(x', y) \in L^2(\Gamma_0; H^1_{0\#}(G))$ such that for any $\phi \in L^2(\Gamma_0; H^1_{0\#}(G))$,

$$\frac{1}{|Y'|}\int_{\Gamma_0\times G} \bar{A}\nabla_y c_1 \nabla_y \phi\,dx'\,dy = -\frac{1}{|Y'|}\int_{\Gamma_0\times G} \bar{A}\nabla u_{|\Gamma_0} \nabla_y \phi\,dx'\,dy. \quad (31)$$

- Find $t_1(x', y) \in L^2(\Gamma_0; H^1_\#(G)/\mathbb{R})$ such that for any $\phi \in L^2(\Gamma_0; H^1_{0\#}(G))$,

$$\frac{1}{|Y'|}\int_{\Gamma_0\times G} \bar{A}\nabla_y t_1 \nabla_y \phi\,dx'\,dy = 0. \quad (32)$$

By equipping the space $L^2(\Gamma_0; H^1_{0\#}(G))$ with the norm $\|\phi\| = \|\nabla_y \phi\|_{L^2(\Gamma_0\times G)}$, it can be easily checked that the left-hand side of the equation (31) is a bilinear continuous and elliptic mapping, and the right-hand side is continuous (since $u \in H^2(\Omega)$), allowing for the Lax–Milgram lemma to apply. This lemma ensures the existence and uniqueness of c_1.

It will be proved in Section 4.2.3 that the functions c_1 and t_1 do effectifely behave as respectively a corrector and a tail terms, in the usual boundary layer sense. It turns out that the boundary layer equation (31) strongly resembles the one obtained for the "domain" classical homogenization case, where the technique of the so-called *local or cell* problem is well-known (see e.g. the model problem in Allaire, 1992).

A direct application of this technique to our problem yields:

$$c_1(x', y) = \sum_{i=1}^{i=N} \frac{\partial u}{\partial x_i}(x', 0) w_i(y) \quad (33)$$

where $w_i \in H^1_{0\#}(G)$ are the unique solutions of the cell problems:

$$\int_G \bar{A}(y)\nabla_y w_i(y)\nabla_y \phi \, dy = -\int_G \bar{A}(y) e_i \nabla_y \phi \, dy \ \forall \phi \in H^1_{0\#}(G). \qquad (34)$$

(here (e_i) is the canonical basis of $\mathbb{R}^N$.)

This is a very important simplification in view of the numerical computation of the boundary layer corrector term c_1.

Let us also mention that in the works done by Belyaev (1988) and Friedman et al. (1997), *harmonic functions* playing the same role as above are directly derived from an asymptotic analysis. In a case study similar to ours (but with multi-scale oscillations), the latter authors use a strong form of the weak equation (34), with an explicit form of uniformly elliptic operators involving the geometry of the oscillations (as does $\bar{A}$).

4.2.3. The corrector results

From Section 4.2.1, we can see that $\frac{1}{\sqrt{\epsilon}}\|u_\epsilon - u\|_{H^1(\Omega)}$ is merely bounded with respect to ϵ and L. The aim of the introduction of boundary layer terms (also called *correctors*) is to improve this (rate of) convergence.

Classically, one sets a mathematical framework where the cell G is a semi-infinite band ($L = +\infty$), and proves that

$$\frac{1}{\sqrt{\epsilon}}\|u_\epsilon(x) - u(x) - \epsilon c_1(x', x/\epsilon)\|_{H^1(\Omega)} \to 0. \qquad (35)$$

Our purpose now is to prove that such an improvement holds in the bounded unit-cell, more precisely:

THEOREM 4.2 *The two following statements hold:*

i) There exist two positive constants β and C_1 independent from the width L, such that:

$$\|t_1\|_{L^2(\Gamma_0; H^1_\#(G)/\mathbb{R})} \leq C_1 \exp(-\beta L). \qquad (36)$$

ii) There exists a positive constant C_2 independent of L such that:

$$\lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}}\|u_\epsilon(x) - u(x) - \epsilon c_1(x', x/\epsilon)\|_{H^1(\Omega)} \leq C_2 \exp(-\beta L/2). \qquad (37)$$

Moreover, the constant $\beta > 0$ depends only on the operator $\bar{A}$ and on the boundary Γ_0.

Before proving the theorem, let us recall that the functions c_1 and t_1 depend on the arbitrary depth parameter L. But, since it is not fixed once for all, these functions c_1 and t_1 start to play a corrector-tail role when L is chosen large enough, as results from the application of the Phragmen–Lindelöf principle. Related to the tail behavior of boundary layers, let us mention a paper of Neuss-Radu (2000) where it is shown that in general this decay property does not hold.

Proof of i). Let us consider the elliptic operator defined over the infinite cylinder $G_\infty = Y' \times [0, +\infty)$ by:

$$\mathcal{L}(y) = -div_y \bar{A}(y) \nabla_y.$$

It is always possible to construct a linear extension $\bar{t}$ of the function t_1, defined over the infinite cylinder, such that:

$$\begin{cases} \mathcal{L}(y)\bar{t} = 0 \text{ over } Y' \times]L, +\infty) \\ \bar{t}(L) = t_1(L) \\ \|\nabla_y \bar{t}\|_{L^2(\Gamma_0 \times G_\infty)} \leq 4\|\nabla_y t_1\|_{L^2(\Gamma_0 \times G)}. \end{cases} \tag{38}$$

The considered extension $\bar{t}$ is then in the nullspace of the elliptic operator $\mathcal{L}$, which is *translation invariant* with respect to the y_n variable, i.e., that if it contains $u(x', y', y_n)$, it also contains $u(x', y', y_n + \eta)$ for any positive η.

Thanks to the well-known *interior estimates* for the elliptic problems (see Browder, 1956, among many others), and to the Rellich compactness theorem, it is easily seen that the nullspace is *interior compact* in the sense of Lax (1957).

Then, in order to apply the Phragmén–Lindelöf Theorem (yielding the exponential decay) as stated by P. D. Lax in (1957), p. 379, we should establish that the norm $\|\bar{t}\|_{L^2(\Gamma_0; H^1_\#(G_\infty)/\mathbb{R})}$ is finite, or in other words, thanks to (38):

$$\|\nabla_y t_1\|_{L^2(\Gamma_0 \times G)} \leq C,$$

the constant C being independent from L. The latter condition is crucial, since the parameter L could be arbitrarily large.

Indeed, it is an easy exercise to show that the L^2-norm is weakly lower semi-continuous with respect to the two-scale boundary layer convergence. Hence, for any arbitrary L, one has:

$$\liminf_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} \|\nabla(u_\epsilon - u)\|_{L^2(\Omega)} \geq \frac{1}{\sqrt{|Y'|}} \|\nabla_y u_1\|_{L^2(\Gamma_0 \times G)}. \tag{39}$$

Thanks to Theorem 4.1, the left-hand side is bounded by a constant C independent from L.

On the other hand, and since $\nabla_y c_1$ and $\nabla_y t_1$ are orthogonal (in the sense of equation (32)), we have:

$$\int_{\Gamma_0 \times G} \bar{A}(\nabla_y u_1)^2 \, dx' \, dy$$
$$= \int_{\Gamma_0 \times G} \bar{A}(\nabla_y c_1)^2 \, dx' \, dy + \int_{\Gamma_0 \times G} \bar{A}(\nabla_y t_1)^2 \, dx' \, dy. \tag{40}$$

Then, using the ellipticity and continuity of the operator $\bar{A}$ with constants, respectively, a_0 and a_1, it is straightforward that:

$$\|\nabla_y t_1\|_{L^2(\Gamma_0 \times G)} \leq \sqrt{\frac{a_1}{a_0}} \|\nabla_y u_1\|_{L^2(\Gamma_0 \times G)} \leq C, \tag{41}$$

The Phragmén–Lindelöf Theorem then applies, it asserts that there exists a constant $\beta > 0$ which depends only on the operator $\bar{A}$ and on the boundary Γ_0, such that:

$$\|\bar{t}\|_{L^2(\Gamma_0;H^1_{\#}(Y'\times[Z,+\infty[)/\mathbb{R})} \leq C_1 \exp(-\beta Z). \tag{42}$$

(Here, $Z > 0$ is a generic number large enough.)

Then, coming back to the actual *finite width* L, setting $Z = L$ in (42), and using the continuity of the trace operator, one has:

$$\|t_1(x', y', L)\|_{L^2(\Gamma_0;H^{\frac{1}{2}}_{\#}(Y')/\mathbb{R})} = \|\bar{t}(x', y', L)\|_{L^2(\Gamma_0;H^{\frac{1}{2}}_{\#}(Y')/\mathbb{R})} \tag{43}$$

$$\leq \|\bar{t}\|_{L^2(\Gamma_0;H^1_{\#}(Y'\times[L,+\infty[)/\mathbb{R})} \tag{44}$$

$$\leq C_1 \exp(-\beta L) \tag{45}$$

Finally, we come back to the *bounded* variational problem for t_1, i.e. the problem set in the bounded cell $G = \Gamma_0 \times]0, L[$, with a homogeneous Neumann boundary condition over $y_n = 0$, a non-homogeneous Dirichlet condition over $y_n = L$, and y'-periodicity. From the classical theory of non-homogeneous elliptic boundary value problems, one finally obtains:

$$\|t_1\|_{L^2(\Gamma_0;H^1_{\#}(G)/\mathbb{R})} \leq C\|t_1(x', y', L)\|_{L^2(\Gamma_0;H^{\frac{1}{2}}_{\#}(Y')/\mathbb{R})}$$
$$\leq Const. \exp(-\beta L), \tag{46}$$

all the involved constants being independent from L.

Proof of ii). Let us remark that $\epsilon c_1(x', x/\epsilon)$, seen as an element of the space $H^1(B_\epsilon)$, extended by zero outside the strip B_ϵ, belongs to the space $H(\Omega)$. Indeed, from the expression of $c_1(x', y)$ in (33), it can be noticed that the regularity of $c_1(x', y)$ with respect to the boundary variable x' is leaded by that of the non-oscillating solution u. As we assumed earlier that $u \in H^3(\Omega)$, it follows that the trace of the gradient over Γ_0 is continuous. The homogeneous Dirichlet condition set over the complementary of Γ_0 implies that the trace of the gradient is null for any $x' \notin \Gamma_0$ (a *local* surrounding of Γ_0 by a Dirichlet boundary condition is sufficient). Whence the possible extension of c_1 by zero outside the strip.

Using the uniform ellipticity of A_ϵ, we have:

$$\frac{1}{\epsilon}\int_\Omega A_\epsilon \nabla(u_\epsilon - u - \epsilon c_1)^2 \, dx \geq \frac{C}{\epsilon}\|u_\epsilon - u - \epsilon c_1\|^2_{H^1(\Omega)}. \tag{47}$$

And, simultaneously, the left-hand side of the inequality (47) above is equal to:

$$\frac{1}{\epsilon}\int_\Omega A_\epsilon \nabla(u_\epsilon - u)^2 \, dx - \frac{2}{\epsilon}\int_{B_\epsilon} A_\epsilon \nabla(u_\epsilon - u)\nabla_y c_1 \, dx$$
$$+ \frac{1}{\epsilon}\int_{B_\epsilon} A_\epsilon (\nabla_y c_1)^2 \, dx. \tag{48}$$

Each of the three terms in the expression (48) is processed as follows:

- From the classical convergence of periodic functions, the third term yields:

$$\frac{1}{\epsilon}\int_{B_\epsilon} A_\epsilon(\nabla_y c_1)^2\,dx \to \frac{1}{|Y'|}\int_{\Gamma_0\times G} \bar{A}(\nabla_y c_1)^2\,dx'\,dy. \tag{49}$$

- From the boundary layer convergence of $\nabla(u_\epsilon - u)$ to $\nabla_y u_1$, the second term yields:

$$-\frac{2}{\epsilon}\int_{B_\epsilon} A_\epsilon \nabla(u_\epsilon - u)\nabla_y c_1\,dx$$
$$\to -\frac{2}{|Y'|}\int_{\Gamma_0\times G} \bar{A}(\nabla_y u_1)(\nabla_y c_1)\,dx'\,dy. \tag{50}$$

The right-hand side reduces to

$$-\frac{2}{|Y'|}\int_{\Gamma_0\times G} \bar{A}(\nabla_y c_1)^2\,dx'\,dy$$

thanks to the orthogonality of $\nabla_y t_1$ and $\nabla_y c_1$ (put $\phi = c_1$ in equation (32)).

- The first term is less straightforward than the two above. We proceed in three steps.

Step one. From the variational equation (21), we get

$$\frac{1}{\epsilon}\int_\Omega A_\epsilon \nabla(u_\epsilon - u)^2\,dx = \frac{1}{\epsilon}\int_{S_\gamma} R(x/\epsilon)f(u_\epsilon - u)\,dx$$
$$+\int_{\Gamma_0} g_\epsilon(u_\epsilon - u)\,d\Gamma + \frac{1}{\epsilon}\int_{S_\gamma}(A - A_\epsilon)\nabla u\nabla(u_\epsilon - u)\,dx. \tag{51}$$

Step two. From the inequalities (25) and (26), the sum of the first and second terms of equation (51) is bounded by

$$(C_1\|f\|_{H^1(\Omega)} + C_2\|g\|_{L^2(\Gamma_0)})\|u_\epsilon - u\|_{H^1(\Omega)},$$

which tends to zero with the rate $\sqrt{\epsilon}$. Note that the positive constants C_1 and C_2 do not depend on L.

In the third term, replacing (∇u) by $(\nabla u - (\nabla u)_{|\Gamma_0} + (\nabla u)_{|\Gamma_0})$ (using the usual density argument in order to give a sense to the use of $(\nabla u)_{|\Gamma_0}$ as a function over B_ϵ), and by means of the Hardy inequality, yields:

$$\frac{1}{\epsilon}\int_{S_\gamma}(A - A_\epsilon)(\nabla u - (\nabla u)_{|\Gamma_0})\nabla(u_\epsilon - u)\,dx$$
$$\leq C\|u\|_{H^2(\Omega)}\|u_\epsilon - u\|_{H^1(\Omega)}, \tag{52}$$

which also tends to zero with the rate $\sqrt{\epsilon}$.

Finally, the function $(A - A_\epsilon)(\nabla u)_{|\Gamma_0}$ can be taken as a test-function for the boundary layer convergence, giving then:

$$\frac{1}{\epsilon}\int_{S_\gamma}(A - A_\epsilon)(\nabla u)_{|\Gamma_0}\nabla(u_\epsilon - u)\,dx$$

Step three. Passing to the limit in the expression (48), and using the results (49)–(53) above, we obtain, when summing up the limits:

$$-\frac{1}{|Y'|}\int_{\Gamma_0\times G}\bar{A}(\nabla_y c_1+\nabla u_{|\Gamma_0})\nabla_y c_1\,dx'\,dy$$
$$+\frac{1}{|Y'|}\int_{\Gamma_0\times G}A(\nabla u)_{|\Gamma_0}\nabla_y c_1\,dx'\,dy$$
$$+\frac{1}{|Y'|}\int_{\Gamma_0\times G}(A-\overline{A})(\nabla u)_{|\Gamma_0}\nabla_y t_1\,dx'\,dy. \tag{54}$$

The first term above is equal to zero since c_1 solves the boundary layer variational equation (31).

The second term also tends to zero (with the rate ϵ): if we re-write the equation (20) (the limit model) with test functions of the form $v(x) = \phi(x', x/\epsilon)$ (as done in Section 4.2.2), it reads:

$$\frac{1}{\epsilon}\int_{B_\epsilon}(A\nabla u)\nabla_y\phi\,dx=\int_{B_\epsilon}f\phi\,dx. \tag{55}$$

Since A is constant, and $(A\nabla u)$ belongs to $H^1(\Omega)^N$, it converges in the boundary layer sense to its trace over Γ_0:

$$A\nabla u \overset{\text{BL}}{\rightharpoonup} A(\nabla u)_{|\Gamma_0},$$

and the right-hand side of equation (55) tends to zero thanks to Lemma 4.1. Finally, the last term is upper-bounded as follows:

$$\left|\int_{\Gamma_0\times G}(A-\overline{A})(\nabla u)_{|\Gamma_0}\nabla_y t_1\,dx'\,dy\right|$$
$$\le \|A-\overline{A}\|_{L^\infty(G)}\|(\nabla u)_{|\Gamma_0}\|_{L^2(\Gamma_0)}\|\nabla_y t_1\|_{L^2(\Gamma_0\times G)}. \tag{56}$$

The conclusion of the proof of (ii) follows then by application of the statement (i).

From the L^2-convergence of oscillating periodic functions to their mean-value at the rate $\sqrt{\epsilon}$, and carefully observing the rate of convergence of the sequences involved in the proof above, one obtains that the convergence of the correction terms takes place at the rate $\epsilon^{3/4}$, or in other words:

$$\|u_\epsilon(x)-u(x)-\epsilon c_1(x', x/\epsilon)\|_{H^1(\Omega)}\le\ C_1\epsilon^{3/4}+C_2\sqrt{\epsilon}\exp(-\beta L/2). \tag{57}$$

This rate is less good than the one (rate ϵ) obtained by Belyaev (1988) or Friedman et al. (1997). The loss in the rate of convergence is a consequence of bounding of the unit cell G in our case; conditions to impose in order to get rid of lower powers in ϵ, emphasized in Friedman et al. (1997), are not compatible with the boundedness of G.

Nevertheless, this drawback is largely compensated for by the adequacy of the functional framework to a standard numerical analysis and, from a computational viewpoint, the straightforward possibility of using standard finite element methods to solve the cell equations, avoiding the use of the expensive boundary

Let us also remark that, from the upper-bound in (57), the cut-off error induced by the bounding of the unit cell, balances the limit rate $\epsilon^{3/4}$ as soon as the width L is of order $|\log(\epsilon)|$, which is much better (from the point of view of the numerical computation cost) than the usual $\sqrt{\epsilon^{-1}}$ boundary layer width, and indeed better than any ϵ^{-a}, $a > 0$.

REMARK 4.2 *From the boundary layer equation (30) and the upper-bound (57), it is seen that when the trace* $\nabla u_{|\Gamma_0}$ *is equal to zero, then the first order boundary layer term vanishes, and one has a better estimate for* $\|u_\epsilon - u\|_{H^1(\Omega)}$ *which is at the rate* $\epsilon^{3/4}$ *instead of* $\sqrt{\epsilon}$ *as given by Theorem 4.1.*

4.2.4. A remark on higher order terms

While we think that the present framework is well adapted to problems where at most first order corrector results are sought, it turns out that obtaining higher order approximations do require working within the classical unbounded unit cell $Y' \times [0, +\infty)$.

This fact could be explained from a physical point of view: one could consider hierarchical boundary layer effects, starting from the macroscopic trace effect u_Γ, then the first order two-scale boundary layer effect u_1, which predominate quite naturally in the strip B_ϵ (and this fact is proved throughout this paper). Then higher order effects follow, having no reason to concentrate on the strip (think of the dust of a comet tail), and cannot then be captured using the bounded cell G.

Thus, working in a bounded unit-cell (which is essentially different from deriving estimates in bounded subsets of the domain, but with boundary layer terms defined in unbounded unit-cell, see Friedman et al. (1997), p. 90) seems not to be of particular interest for more than first order corrector problems.

5. Conclusion

Within the present paper, we studied the boundary layer behavior of solutions to partial differential equations occurring in a domain with a periodic oscillating boundary, the frequency and the amplitude of the oscillations being of the same order.

The oscillating domains were considered as lipschitzian perturbations of a reference non-oscillating one. Using then the classical domain transport method, a variational transported formulation was established, and with the choice of a bounded unit-cell, the two-scale homogenization tools were applied.

We obtained the existence of a two-scale boundary layer limit, which is shown to be the sum of a corrector term, and an exponentially dying in the gradient tail term. Uniqueness as well as a homogenized equation for the first order corrector term were derived. Then, a corrector result was proved. We also to exploit the classical *cell equations* method in order to simplify the numerical computation of the boundary layer.

At this step, we think that combining the transport method and the two-scale boundary layer homogenization can be easily applied to more general elliptic models, as well as to parabolic problems. In particular, our framework is well suited for the elasticity problem (and one knows the importance of concentrating the mechanical stress along the boundary of structures). Transported equations, and domain sensitivities (derivatives) are well-known for these problems, so that choosing concentrated perturbation fields would easily lead to variational formulations ready for the two-scale analysis.

The choice of a bounded unit-cell has the advantage of a very simple and classical functional framework, as well as the ability to use standard finite element solvers in order to compute the first order corrector term; while its main drawback is the difficulty (impossibility?) of capturing the whole boundary layer behavior, or in other words, higher than the first order correctors are out of reach.

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