

Approximate controllability for semilinear heat equations with globally Lipschitz nonlinearities<sup>1</sup>

by

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**Abstract:** We consider the semilinear heat equation involving gradient terms in a bounded domain of  $\mathbf{R}^n$ . It is assumed that the non-linearity is globally Lipschitz. We prove that the system is approximately controllable when the control acts on a bounded subset of the domain. The proof uses a variant of a classical fixed point method and is a simpler alternative to the earlier proof existing in the literature by means of the penalization of an optimal control problem. We also prove that the control may be built so that, in addition to the approximate controllability requirement, it ensures that the state reaches exactly a finite number of constraints.

**Keywords:** controllability, systems governed by PDEs, nonlinear PDEs of parabolic type

## 1. Introduction and main results

Let  $\Omega$  be a bounded domain of  $\mathbf{R}^n$ ,  $n \geq 1$  of class  $C^2$  and consider the semilinear heat equation

$$\begin{cases} u_t - \Delta u + f(u, \nabla u) = v1_\omega & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) = u^0(x) & \text{in } \Omega. \end{cases} \quad (1)$$

In (1)  $1_\omega$  denotes the characteristic function of an open non-empty subset  $\omega$  of  $\Omega$ .

The function  $f : \mathbf{R} \times \mathbf{R}^n \rightarrow \mathbf{R}$  is assumed to be *globally Lipschitz* all along the paper, i.e.

$$\exists L > 0 : |f(y, \xi) - f(z, \eta)| \leq L [|y - z| + |\xi - \eta|],$$

$$\forall y, z \in \mathbf{R}; \xi, \eta \in \mathbf{R}^n. \quad (2)$$

In (1)  $u = u(x, t)$  is the state and  $v = v(x, t)$  is the control function which acts on the system through the subset  $\omega$ .

The problem of the approximate controllability of (1) can be formulated as follows: Given  $T > 0$ ,  $u^0, u^1 \in L^2(\Omega)$  and  $\varepsilon > 0$ , to find a control  $v \in L^2(\omega \times (0, T))$  such that the solution  $u$  of (1) satisfies

$$\|u(T) - u^1\|_{L^2(\Omega)} \leq \varepsilon. \quad (3)$$

In other words, the problem of the approximate controllability of (1) consists in studying whether the range of solutions of (1) at time  $T$ ,

$$R(u^0, T) = \{u(\cdot, T) : u \text{ solution of (1) with } v \in L^2(\omega \times (0, T))\}, \quad (4)$$

is dense in  $L^2(\Omega)$  or not.

In this paper we shall also study a stronger version of this control problem that we shall refer to as the *finite-approximate control* problem. Given  $E$ , a finite-dimensional subspace of  $L^2(\Omega)$ , the rest of the parameters of the problem being unchanged, we look for a control  $v \in L^2(\omega \times (0, T))$  such that the solution  $u$  of (1) satisfies

$$\begin{cases} \|u(T) - u^1\|_{L^2(\Omega)} \leq \varepsilon, \\ \pi_E[u(T)] = \pi_E[u^1], \end{cases} \quad (5)$$

$\pi_E$  being the orthogonal projection from  $L^2(\Omega)$  over  $E$ .

Note that in (5), in addition to the approximate controllability requirement (3), the control is requested to be such that the projection over  $E$  of the state  $u(T)$  and the target  $u^1$  coincide.

As proved in Appendix B of Lions and Zuazua (1997), in the context of linear control systems, finite-approximate controllability is a consequence of approximate controllability. However in the nonlinear context one property may not be deduced as a consequence of the other one.

These problems have been the object of intensive research in the past few years. Fabre, Puel and Zuazua (1993, 1995) adapted the fixed point method of Zuazua (1991) to prove the approximate controllability of (1) in the particular case in which  $f = f(y)$ ,  $f$  being globally Lipschitz. Note that the nonlinearity was not allowed to depend on the gradient of the state in this result. Later on in Zuazua (1997) the notion of finite-approximate controllability above was introduced and it was shown that it holds when  $f = f(y)$ ,  $f$  being globally Lipschitz. The complete case where  $f = f(y, \nabla y)$  was addressed by Fernández and Zuazua (1997) by means of the optimal control approach introduced by Lions (1991). In Fernández and Zuazua (1997) it was shown that under the globally Lipschitz assumption (2) the system is both approximately and finite-

and Zuazua (1997) was the recent unique-continuation result by Fabre (1996) on the linear heat equation

$$\varphi_t - \Delta\varphi + a\varphi + \operatorname{div}(b\varphi) = 0 \quad (6)$$

with bounded potentials  $a \in L^\infty(\Omega \times (0, T))$ ,  $b \in (L^\infty(\Omega \times (0, T)))^n$ , and without further regularity assumptions on  $b$ .

The goal of this paper is to show how the fixed point approach may be adapted to address these two controllability problems for the complete system (1) in which the nonlinearity is allowed to depend both on the state and its gradient. Approximate and finite-approximate controllability of the system will be proved. None of these results is new, since, as we said above, they were proved previously in Fernández and Zuazua (1997). However, the new proof we present here is simpler and may be easily adapted to other situations (see Section 5 below). In particular, the boundary control problem may be addressed in a similar way, as an alternative to Zuazua (1997b) in which the optimal control approach was applied; the  $L^p$ -versions can be easily handled; quasi bang-bang controls may be built, etc.

There is a clear limitation in the application of this fixed point method: the globally Lipschitz assumption (2) on the non-linearity. But this condition is, roughly speaking, necessary. Indeed, a well-known example of A. Bamberger (see for instance Henry, 1978) shows that system (1) is not approximately controllable when  $f(y) = |y|^{p-1}y$  for any  $p > 1$ . This counterexample does not show that the globally Lipschitz assumption (2) is sharp but it does it in the context of nonlinearities that grow at infinity as a power of  $y$ . Note, however, that, as proved by Fernández-Cara (1997), when  $f = f(y)$ , null controllability holds under the weaker growth condition

$$|f(s)| \leq C|s| \log |s| \quad \text{as } |s| \rightarrow \infty.$$

This condition has been more recently relaxed in Fernández-Cara and Zuazua (1999) to

$$|f(s)| \leq C(s) \log^{\frac{3}{2}} |s| \quad \text{as } |s| \rightarrow \infty, \quad (7)$$

with  $C > 0$  small enough. In this work approximate controllability is also proved under the condition (7). Therefore, one may expect the results of this paper to hold in a slightly more general class of nonlinearities than (2).

We have mentioned above some works that are closely related to the present one. But many others have also been published. We refer, for instance, to the work by Naito and Seidman (1991) on the invariance of reachable sets under nonlinear perturbations, Limaco and Medeiros (1998) on the approximate controllability in non-cylindrical domains, Bezerra (1999), Teresa (1998), and Teresa and Zuazua (1999) on the case of unbounded domains, etc. We refer to

We do not address here the problem of null-controllability. Let us recall briefly its formulation. Assume for simplicity that  $f(0,0) = 0$ . System (1) is said to be null controllable if for any  $u^0 \in L^2(\Omega)$  there exists a control  $v \in L^2(\omega \times (0, T))$  such that the solution  $u$  of (1) satisfies

$$u(x, T) = 0 \text{ in } \Omega. \quad (8)$$

In the context of linear heat equations with time independent coefficients Russell (1973) proved that the null controllability of the heat equation is a consequence of the exact controllability of the wave equation. More recently, Lebeau and Robbiano (1995) proved the null controllability without any assumption on the control subdomain  $\omega$  using Fourier series developments and sharp estimates on the eigenfunctions of the Laplacian. Similar results but in a more general context including time-dependent coefficients were proved by Fursikov and Imanuvilov (1996) using global Carleman inequalities for the heat equation. In Fursikov and Imanuvilov (1996) local null controllability results were also proved for semilinear heat equations. More recently, the connections between null and approximate controllability were investigated in Fernández-Cara and Zuazua (1998, 1999). We refer to the bibliography for a more complete list of references.

After this work had been completed the author was informed by O. Yu. Imanuvilov about a work in collaboration with M. Yamamoto in which observability inequalities are obtained for equations of the form (6) with  $a \in L^\infty(\Omega \times (0, T))$ ,  $b \in (L^\infty(\Omega \times (0, T)))^n$  (see Imanuvilov and Yamamoto, 1998). These observability estimates, combined with the fixed point technique described in this paper, allow to prove null-controllability results for equations of the form (1). On the other hand, one may expect that, combining these estimates with the techniques in Fernández-Cara and Zuazua (1998), explicit estimates on the size of the controls that are needed to achieve (5) will also be obtained. Using these estimates, the null controllability of system (1) for some nonlinearities growing at infinity in a superlinear way has been demonstrated by Anita and Barbu (2000).

## 2. Description of the fixed point method

As we said in the introduction, the method developed in this article is a variant of the fixed point method introduced in Zuazua (1991) in the context of the wave equation and adapted in Fabre, Puel and Zuazua (1993, 1995) to deal with the semilinear heat equation.

We observe that for any  $y \in L^2(0, T; H_0^1(\Omega))$  the following identity holds:

$$f(y, \nabla y) - f(0, 0) = \int_0^1 \frac{d}{d\sigma} (f(\sigma y, \sigma \nabla y)) d\sigma \quad (9)$$

In (9)  $\partial f/\partial y$  and  $\partial f/\partial \eta$  denote, respectively, the partial derivatives of  $f$  with respect to the variables  $y$  and  $\nabla y$ .

We introduce the notation

$$\begin{cases} F(y) = \int_0^1 \frac{\partial f}{\partial y}(\sigma y, \sigma \nabla y) d\sigma \\ G(y) = \int_0^1 \frac{\partial f}{\partial \eta}(\sigma y, \sigma \nabla y) d\sigma. \end{cases} \tag{10}$$

In view of the globally Lipschitz assumption (2) on  $f$ ,  $F$  and  $G$  map  $L^2(0, T; H_0^1(\Omega))$  into a bounded set of  $L^\infty(\Omega \times (0, T))$ . Moreover,

$$\| F(y) \|_{L^\infty(\Omega \times (0, T))} \leq L, \forall y \in L^2(0, T; H_0^1(\Omega)), \tag{11}$$

$$\| G(y) \|_{(L^\infty(\Omega \times (0, T)))^n} \leq L, \forall y \in L^2(0, T; H_0^1(\Omega)), \tag{12}$$

$L$  being the Lipschitz constant of  $f$ .

Using these notations the system (1) can be rewritten as follows

$$\begin{cases} u_t - \Delta u + F(y)u + G(y) \cdot \nabla u + f(0, 0) = v1_\omega & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) = u^0(x) & \text{in } \Omega. \end{cases} \tag{13}$$

Given  $y \in L^2(0, T; H_0^1(\Omega))$  we now consider the ‘‘linearized’’ system

$$\begin{cases} u_t - \Delta u + F(y)u + G(y) \cdot \nabla u + f(0, 0) = v1_\omega & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) = u^0(x) & \text{in } \Omega. \end{cases} \tag{14}$$

Observe that (14) is a linear system on the state  $u$  with potentials  $a = F(y) \in L^\infty(\Omega \times (0, T))$  and  $b = G(y) \in (L^\infty(\Omega \times (0, T)))^n$  satisfying the following uniform bound

$$\| a \|_{L^\infty(\Omega \times (0, T))} \leq L, \| b \|_{(L^\infty(\Omega \times (0, T)))^n} \leq L. \tag{15}$$

With this notation the system (14) may be rewritten in the form

$$\begin{cases} u_t - \Delta u + au + b \cdot \nabla u + f(0, 0) = v1_\omega & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) = u^0(x) & \text{in } \Omega. \end{cases} \tag{16}$$

We now fix the initial datum  $u^0 \in L^2(\Omega)$ , the target  $u^1 \in L^2(\Omega)$ ,  $\varepsilon > 0$  and the finite-dimensional subspace  $E$  of  $L^2(\Omega)$ .

Using the variational approach to approximate controllability introduced by Lions (1992), further developed in Fabre, Puel and Zuazua (1993) and adapted to the problem of finite-approximate controllability in Zuazua (1997a), we build a control  $v$  for the linear system (16) such that

$$\| u(T) - u^1 \|_{L^2(\Omega)} \leq \varepsilon.$$

The control  $v \in L^2(\Omega \times (0, T))$  satisfying (17) is not unique but the variational approach mentioned above provides the unique one of minimal  $L^2(\omega \times (0, T))$ -norm.

Thus, for any  $y \in L^2(0, T; H_0^1(\Omega))$  this allows us to define a control  $v = v(x, t; y) \in L^2(\omega \times (0, T))$  such that the solution  $u = u(x, t; y) \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$  of (14) satisfies (17). This allows to build a non-linear mapping

$$\mathcal{N} : L^2(0, T; H_0^1(\Omega)) \rightarrow L^2(0, T; H_0^1(\Omega)), \mathcal{N}(y) = u. \tag{18}$$

We claim that the problem is then reduced to finding a fixed point of  $\mathcal{N}$ . Indeed, if  $y \in L^2(0, T; H_0^1(\Omega))$  is such that  $\mathcal{N}(y) = u = y$ ,  $u$  solution of (14) is actually solution of (13). Then, the control  $v = v(y)$  is the one we were looking for since, by construction,  $u = u(y)$  satisfies (17).

As we shall see, the nonlinear map  $\mathcal{N} : L^2(0, T; H_0^1(\Omega)) \rightarrow L^2(0, T; H_0^1(\Omega))$  satisfies the following two properties:

$$\mathcal{N} \text{ is continuous and compact ;} \tag{19}$$

$$\left\{ \begin{array}{l} \text{the range of } \mathcal{N} \text{ is bounded, i.e. } \exists R > 0 : \\ \| \mathcal{N}(y) \|_{L^2(0, T; H_0^1(\Omega))} \leq R, \forall y \in L^2(0, T; H_0^1(\Omega)). \end{array} \right. \tag{20}$$

In view of these two properties and as a consequence of Schauder’s fixed point theorem, the existence of a fixed point of  $\mathcal{N}$  follows immediately.

The uniform bound (20) on the range of  $\mathcal{N}$  is a consequence of the uniform bound (15) on the potentials  $a$  and  $b$  which, in turn, is a consequence of the globally Lipschitz assumption (2).

Roughly speaking, the control problem for the semilinear equation (1) or (13) through this fixed point method is reduced to obtaining a uniform controllability result for the family of linear control problems (14) under the constraints (11)-(12). At this level the unique continuation result of Fabre (1996) for equations of the form

$$\varphi_t - \Delta\varphi + a\varphi + \text{div}(b\varphi) = 0$$

with  $L^\infty$ -coefficients  $a$  and  $b$  plays a crucial role.

### 3. Controllability of the linearized systems

Given  $L^\infty$ -potentials  $a \in L^\infty(\Omega \times (0, T))$ ,  $b \in (L^\infty(\Omega \times (0, T)))^n$  and a real constant  $\lambda \in \mathbf{R}$  we consider the control problem

$$\left\{ \begin{array}{ll} u_t - \Delta u + au + b \cdot \nabla u + \lambda = v1_\omega & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \end{array} \right. \tag{21}$$

Let  $E$  be a finite-dimensional subspace of  $L^2(\Omega)$ . Given  $u^0 \in L^2(\Omega)$ ,  $u^1 \in L^2(\Omega)$  and  $\varepsilon > 0$  we look for a control  $v \in L^2(\omega \times (0, T))$  such that the solution  $u$  of (21) satisfies

$$\begin{cases} \| u(T) - u^1 \|_{L^2(\Omega)} \leq \varepsilon \\ \pi_E(u(T)) = \pi_E(u^1). \end{cases} \tag{22}$$

The following holds:

**THEOREM 3.1** *Let  $T > 0$ . Then, there exists a control  $v \in L^2(\omega \times (0, T))$  such that the solution  $u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$  of (21) satisfies (22).*

*Moreover, for any  $R > 0$  there exists a constant  $C(R) > 0$  such that*

$$\| v \|_{L^2(\omega \times (0, T))} \leq C(R) \tag{23}$$

for any  $a \in L^\infty(\Omega \times (0, T))$ ,  $b \in (L^\infty(\Omega \times (0, T)))^n$  satisfying

$$\| a \|_{L^\infty(\Omega \times (0, T))} \leq R, \| b \|_{(L^\infty(\Omega \times (0, T)))^n} \leq R. \tag{24}$$

**REMARK 3.1** *Theorem 3.1 does not provide any estimate on how the norm of the control  $v$  depends on  $E$ ,  $u^0$ ,  $u^1$  and  $\varepsilon > 0$ . However, (23) guarantees that  $v$  remains uniformly bounded when the potentials  $a$ ,  $b$  remain bounded in  $L^\infty$ .*

*The control  $v$  is not unique. The construction we develop below provides the control of minimal  $L^2$ -norm. It is this control of minimal norm which satisfies the uniform boundedness condition (23).*

**Proof of Theorem 3.1.**

Without loss of generality we may assume that  $\lambda = 0$  and  $u^0 \equiv 0$ . Indeed, otherwise we consider the solution  $z$  of

$$\begin{cases} z_t - \Delta z + az + b \cdot \nabla z + \lambda = 0 & \text{in } \Omega \times (0, T) \\ z = 0 & \text{on } \partial\Omega \times (0, T) \\ z(0) = u^0 & \text{in } \Omega. \end{cases} \tag{25}$$

Then, the solution  $u$  of (21) may be decomposed as

$$u = w + z \tag{26}$$

where  $w$  solves

$$\begin{cases} w_t - \Delta w + aw + b \cdot \nabla w = v1_\omega & \text{in } \Omega \times (0, T) \\ w = 0 & \text{on } \partial\Omega \times (0, T) \\ w(0) = 0 & \text{in } \Omega. \end{cases} \tag{27}$$

Then, (22) is equivalent to

$$\| w(T) - (u^1 - z(T)) \|_{L^2(\Omega)} \leq \varepsilon$$

Therefore, in the sequel we shall assume that  $\lambda = 0$  and  $u^0 \equiv 0$ .  
The regularizing effect of the heat equation allows to show that

$$\begin{cases} z(T) \text{ remains in a relatively compact set of } L^2(\Omega) \\ \text{when the potentials } a \text{ and } b \text{ vary in the class (24).} \end{cases} \quad (29)$$

This will be important when deriving the uniform bound (23).

Consider the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi + a\varphi - \operatorname{div}(b\varphi) = 0 & \text{in } \Omega \times (0, T) \\ \varphi = 0 & \text{on } \partial\Omega \times (0, T) \\ \varphi(T) = \varphi^0 & \text{in } \Omega. \end{cases} \quad (30)$$

Taking into account that the potentials  $a$  and  $b$  are bounded it is easy to see that for any  $\varphi^0 \in L^2(\Omega)$  system (30) has a unique solution in the class  $\varphi \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ .

We now consider the functional  $J : L^2(\Omega) \rightarrow \mathbf{R}$  defined as follows:

$$J(\varphi^0) = \frac{1}{2} \int_0^T \int_{\omega} \varphi^2 dx dt + \varepsilon \| (I - \pi_E)\varphi^0 \|_{L^2(\Omega)} - \int_{\Omega} u^1 \varphi^0 dx. \quad (31)$$

It is easy to see that

$$J : L^2(\Omega) \rightarrow \mathbf{R} \text{ is continuous ;} \quad (32)$$

$$J : L^2(\Omega) \rightarrow \mathbf{R} \text{ is convex .} \quad (33)$$

Moreover

$$J : L^2(\Omega) \rightarrow \mathbf{R} \text{ is strictly convex .} \quad (34)$$

This property is a consequence of the following unique continuation result due to Fabre (1996):

**PROPOSITION 3.1** (Fabre, 1996). *Assume that  $a \in L^\infty(\Omega \times (0, T))$  and  $b \in (L^\infty(\Omega \times (0, T)))^n$ . Let  $\varphi^0 \in L^2(\Omega)$  be such that the solution  $\varphi$  of (30) satisfies*

$$\varphi = 0 \text{ in } \omega \times (0, T). \quad (35)$$

*Then, necessarily,  $\varphi^0 \equiv 0$ .*

The functional  $J : L^2(\Omega) \rightarrow \mathbf{R}$  is also coercive. More precisely, the following holds:

**PROPOSITION 3.2** *Under the assumptions above*

$$\liminf \frac{J(\varphi^0)}{\|\varphi^0\|} > \varepsilon. \quad (36)$$

**Proof of Proposition 3.2.** The proof of this proposition follows the argument in Fabre, Puel and Zuazua (1993) and Zuazua (1997a) combined with the unique-continuation result of Proposition 3.1. Let us recall it for the sake of completeness.

Let  $\{\varphi_j^0\}$  be a sequence in  $L^2(\Omega)$  such that

$$\|\varphi_j^0\|_{L^2(\Omega)} \rightarrow \infty \text{ as } j \rightarrow \infty. \tag{37}$$

We denote by  $\{\varphi_j\}$  the corresponding sequence of solutions of (30).

We also set

$$\widehat{\varphi}_j^0 = \varphi_j^0 / \|\varphi_j^0\|_{L^2(\Omega)}, \widehat{\varphi}_j = \varphi_j / \|\varphi_j^0\|_{L^2(\Omega)}. \tag{38}$$

Obviously  $\widehat{\varphi}_j$  is the solution of (30) with the normalized initial data  $\widehat{\varphi}_j^0$ .

We have

$$\begin{aligned} \frac{J(\varphi_j^0)}{\|\varphi_j^0\|_{L^2(\Omega)}} &= \frac{\|\varphi_j^0\|_{L^2(\Omega)}}{2} \int_0^T \int_{\omega} |\widehat{\varphi}_j|^2 dxdt \\ &+ \varepsilon \|(I - \pi_E)\widehat{\varphi}_j^0\|_{L^2(\Omega)} - \int_{\Omega} u^1 \widehat{\varphi}_j^0 dx. \end{aligned} \tag{39}$$

We distinguish the following two cases:

**Case 1.**  $\liminf_{j \rightarrow \infty} \int_0^T \int_{\omega} |\widehat{\varphi}_j|^2 dxdt > 0;$

**Case 2.**  $\liminf_{j \rightarrow \infty} \int_0^T \int_{\omega} |\widehat{\varphi}_j|^2 dxdt = 0.$

In the first case, due to (37), the first term in (39) tends to  $+\infty$  while the other two remain bounded. We deduce that

$$\liminf_{j \rightarrow \infty} \frac{J(\varphi_j^0)}{\|\varphi_j^0\|_{L^2(\Omega)}} = +\infty$$

in this case.

Let us now analyze the second case. Let us consider a subsequence (still denoted by the index  $j$  to simplify the notation) such that

$$\int_0^T \int_{\omega} |\widehat{\varphi}_j|^2 dxdt \rightarrow 0, \text{ as } j \rightarrow \infty. \tag{40}$$

By extracting subsequences we may deduce that

$$\widehat{\varphi}_j^0 \rightharpoonup \widehat{\varphi}^0 \text{ weakly in } L^2(\Omega). \tag{41}$$

Consequently

where  $\widehat{\varphi}$  is the solution of (30) with datum  $\widehat{\varphi}^0$ . According to (41) we deduce that

$$\widehat{\varphi} \equiv 0 \text{ in } \omega \times (0, T)$$

and, as a consequence of Proposition 3.1, that  $\widehat{\varphi}^0 \equiv 0$ . Therefore, if (40) holds, necessarily

$$\widehat{\varphi}_j^0 \rightharpoonup 0 \text{ weakly in } L^2(\Omega). \tag{43}$$

But then,  $E$  being finite-dimensional,

$$\pi_E \widehat{\varphi}_j^0 \rightarrow 0 \text{ in } L^2(\Omega)$$

and therefore

$$\|(I - \pi_E)\widehat{\varphi}_j^0\|_{L^2(\Omega)} \rightarrow 1 \tag{44}$$

since  $\|\widehat{\varphi}_j^0\|_{L^2(\Omega)} = 1$  for all  $j$ .

As a consequence of (43) and (44) we deduce that

$$\liminf_{j \rightarrow \infty} \frac{J(\varphi_j^0)}{\|\varphi_j^0\|_{L^2(\Omega)}} \geq \liminf_{j \rightarrow \infty} \left[ \varepsilon \|(I - \pi_E)\widehat{\varphi}_j^0\|_{L^2(\Omega)} - \int_{\Omega} u^1 \widehat{\varphi}_j^0 dx \right] = \varepsilon.$$

This concludes the proof of Proposition 3.2. ■

In view of properties (32), (34) and (36) of the functional  $J$  we deduce that  $J$  achieves its minimum at a unique  $\widehat{\varphi}^0 \in L^2(\Omega)$ , i.e.

$$\begin{cases} J(\widehat{\varphi}^0) = \min_{\varphi^0 \in L^2(\Omega)} J(\varphi^0) \\ J(\widehat{\varphi}^0) < J(\varphi^0), \forall \varphi^0 \in L^2(\Omega), \varphi^0 \neq \widehat{\varphi}^0. \end{cases} \tag{45}$$

It is easy to see that the control

$$v = \widehat{\varphi} \text{ in } \omega \times (0, T), \tag{46}$$

$\widehat{\varphi}$  being the solution of (30) with the minimizer  $\widehat{\varphi}^0$  as datum is such that the solution  $u$  of

$$\begin{cases} u_t - \Delta u + au + b \cdot \nabla u = v1_{\omega} & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u(0) = 0 & \text{in } \Omega \end{cases} \tag{47}$$

satisfies (22) (see Zuazua, 1997a, for the details of the proof).

This concludes the proof of the finite-approximate controllability.

In order to prove the uniform bound (23), we first observe, as indicated in (29), that the problem may be reduced to the case  $u^0 \equiv 0$  and  $\lambda = 0$ , provided  $u^1$  is allowed to vary in a relatively compact set of  $L^2(\Omega)$ .

**PROPOSITION 3.3** *Let  $R > 0$  and  $K$  be a relatively compact set of  $L^2(\Omega)$ . Then, the coercivity property (36) holds uniformly on  $u^1 \in K$  and potentials  $a$  and  $b$  satisfying (24).*

**REMARK 3.2** *Note that the functional  $J$  depends on the potentials  $a$  and  $b$  and the target  $u^1$ . Proposition 3.3 guarantees the uniform coercivity of these functionals when  $u^1 \in K$ ,  $K$  being a compact set of  $L^2(\Omega)$  and the potentials  $a$  and  $b$  are uniformly bounded.*

As a consequence of Proposition 3.3 we deduce that the minimizers  $\widehat{\varphi}^0$  of the functionals  $J$  are uniformly bounded when  $u^1 \in K$  and the potentials  $a$  and  $b$  are uniformly bounded. Consequently, the controls  $v = \widehat{\varphi}$  are uniformly bounded as well.

Therefore, in order to complete the proof of Theorem 3.1 it is sufficient to prove Proposition 3.3.

**Proof of Proposition 3.3.** The proof is similar to that of Proposition 3.2. We argue by contradiction. If the coercivity property (36) does not hold uniformly, we deduce the existence of sequences

$$u_j^1 \in K \tag{48}$$

$$\varphi_j^0 \in L^2(\Omega) : \|\varphi_j^0\|_{L^2(\Omega)} \rightarrow \infty \tag{49}$$

and

$$\begin{cases} a_j \in L^\infty(\Omega \times (0, T)); b_j \in (L^\infty(\Omega \times (0, T)))^n \\ \|a_j\|_{L^\infty(\Omega \times (0, T))} \leq R; \|b_j\|_{(L^\infty(\Omega \times (0, T)))^n} \leq R \end{cases} \tag{50}$$

such that

$$\frac{J_j(\varphi_j^0)}{\|\varphi_j^0\|_{L^2(\Omega)}} \leq \varepsilon - \delta \tag{51}$$

for some  $0 < \delta < \varepsilon$ .

Here and in the sequel  $J_j$  denotes the functional corresponding to the target  $u_j^1$  and to the potentials  $a_j, b_j$ .

As in the proof of Proposition 3.2 we set

$$\widehat{\varphi}_j^0 = \varphi_j^0 / \|\varphi_j^0\|_{L^2(\Omega)}; \widehat{\varphi}_j = \varphi_j / \|\varphi_j^0\|_{L^2(\Omega)}. \tag{52}$$

We have

$$\frac{J(\varphi_j^0)}{\|\varphi_j^0\|_{L^2(\Omega)}} = \frac{\|\varphi_j^0\|_{L^2(\Omega)}}{2} \int_0^T \int_\omega |\widehat{\varphi}_j|^2 dxdt \tag{53}$$

In view of (49) and (51) we immediately deduce that

$$\int_0^T \int_{\omega} |\widehat{\varphi}_j| dx dt \rightarrow 0, \text{ as } j \rightarrow \infty. \quad (54)$$

Extracting subsequences we also have

$$\widehat{\varphi}_j^0 \rightharpoonup \widehat{\varphi}^0 \text{ in } L^2(\Omega) \text{ weakly} \quad (55)$$

$$\widehat{\varphi}_j \rightharpoonup \widehat{\varphi} \text{ in } L^2(0, T; H_0^1(\Omega)) \text{ weakly} \quad (56)$$

$$u_j^1 \rightarrow u^1 \text{ strongly in } L^2(\Omega) \quad (57)$$

$$a_j \rightharpoonup a \text{ weakly } - * \text{ in } L^\infty(\Omega \times (0, T)) \quad (58)$$

$$b_j \rightharpoonup b \text{ weakly } - * \text{ in } (L^\infty(\Omega \times (0, T)))^n. \quad (59)$$

In view of (54) we have

$$\widehat{\varphi} = 0 \text{ in } \omega \times (0, T). \quad (60)$$

On the other hand,  $\widehat{\varphi}$  solves

$$\begin{cases} -\widehat{\varphi}_t - \Delta \widehat{\varphi} + a\widehat{\varphi} - \operatorname{div}(b\widehat{\varphi}) = 0 & \text{in } \Omega \times (0, T) \\ \widehat{\varphi} = 0 & \text{on } \partial\Omega \times (0, T) \\ \widehat{\varphi}(T) = \widehat{\varphi}^0 & \text{in } \Omega, \end{cases} \quad (61)$$

the potentials  $a$  and  $b$  being the limits in (58)-(59) and the datum  $\widehat{\varphi}^0$  the limit in (55).

In order to obtain (61) we have to show that

$$a_j \widehat{\varphi}_j \rightharpoonup a\widehat{\varphi} \text{ weakly in } L^2(\Omega \times (0, T)) \quad (62)$$

$$b_j \widehat{\varphi}_j \rightharpoonup b\widehat{\varphi} \text{ weakly in } (L^2(\Omega \times (0, T)))^n. \quad (63)$$

This can be done easily since

$$\partial_t \widehat{\varphi}_j \text{ is bounded in } L^2(0, T; H^{-1}(\Omega)). \quad (64)$$

Combining (56) with (64) and Aubin-Lions compactness lemma we deduce that

$$\widehat{\varphi}_j \text{ is relatively compact in } L^2(\Omega \times (0, T)). \quad (65)$$

Consequently

$$\widehat{\varphi}_j \rightarrow \widehat{\varphi} \text{ in } L^2(\Omega \times (0, T)). \quad (66)$$

Convergences (62)-(63) follow immediately from (58), (59) and (66). This jus-

According to Proposition 3.1, (60)-(61) yield that  $\widehat{\varphi} \equiv 0$ . Going back to (55) this shows that

$$\widehat{\varphi}_j^0 \rightharpoonup 0 \text{ weakly in } L^2(\Omega).$$

This, together with (57), implies that

$$\int_{\Omega} u_j^1 \widehat{\varphi}_j^0 dx \rightarrow 0.$$

On the other hand, as indicated in the proof of Proposition 3.2, we also have

$$\|(I - \pi_E)\widehat{\varphi}_j^0\|_{L^2(\Omega)} \rightarrow 1.$$

Therefore

$$\frac{J_j(\varphi_j^0)}{\|\varphi_j^0\|_{L^2(\Omega)}} \geq \varepsilon \|(I - \pi_E)\widehat{\varphi}_j^0\|_{L^2(\Omega)} - \int_{\Omega} u_j^1 \widehat{\varphi}_j^0 dx \rightarrow \varepsilon.$$

This is in contradiction with (51).

This completes the proof of Proposition 3.3 and consequently that of Theorem 3.1. ■

*REMARK 3.3 Note that the proof of Theorem 3.1 provides not only for the existence of the control  $v$  but also a constructive way of finding it, and choosing it in a unique way.*

### 4. The semilinear control problem

This section is devoted to proving the following result:

**THEOREM 4.1** *Assume that  $f$  satisfies (2). Then, for all  $T > 0$ , system (1) is finite-approximately controllable.*

*More precisely, for any finite-dimensional subspace  $E$  of  $L^2(\Omega)$ ,  $u^0, u^1 \in L^2(\Omega)$  and  $\varepsilon > 0$  there exists a control  $v \in L^2(\omega \times (0, T))$  such that the solution  $u$  of (1) satisfies (5).*

*REMARK 4.1 As indicated in the introduction, this result is not new. It was proved in Fernández and Zuazua (1997) by means of a suitable penalization of an optimal control problem. However, we believe that the proof presented here is simpler, easier to adapt to other situations and that it brings new light to the approximate controllability problem.*

#### Proof of Theorem 4.1.

We proceed by means of the fixed point method described in Section 2.

Given a finite-dimensional subspace  $E$  of  $L^2(\Omega)$ ,  $u^0, u^1 \in L^2(\Omega)$  and  $\varepsilon > 0$ , the nonlinear map

is well defined in view of Theorem 3.1.

As indicated in Section 2, in order to conclude the existence of a fixed point of  $\mathcal{N}$  by means of Schauder’s fixed point method (and therefore to conclude the proof of Theorem 4.1) it is sufficient to check the following three facts:

$$\mathcal{N} : L^2(0, T; H_0^1(\Omega)) \rightarrow L^2(0, T; H_0^1(\Omega)) \text{ is continuous ;} \tag{67}$$

$$\mathcal{N} : L^2(0, T; H_0^1(\Omega)) \rightarrow L^2(0, T; H_0^1(\Omega)) \text{ is compact ;} \tag{68}$$

$$\exists R > 0 : \|\mathcal{N}(y)\|_{L^2(0, T; H_0^1(\Omega))} \leq R, \forall y \in L^2(0, T; H_0^1(\Omega)). \tag{69}$$

Let us prove these three properties.

**Continuity of  $\mathcal{N}$ .** Assume that  $y_j \rightarrow y$  in  $L^2(0, T; H_0^1(\Omega))$ . Then, the potentials  $F(y_j), G(y_j)$  are such that

$$F(y_j) \rightarrow F(y) \text{ in } L^p(\Omega \times (0, T)) \tag{70}$$

$$G(y_j) \rightarrow G(y) \text{ in } (L^p(\Omega \times (0, T)))^n \tag{71}$$

for all  $1 \leq p < \infty$  and

$$\|F(y_j)\|_{L^\infty(\Omega \times (0, T))} \leq L; \|G(y_j)\|_{(L^\infty(\Omega \times (0, T)))^n} \leq L. \tag{72}$$

According to Theorem 3.1 the corresponding controls are uniformly bounded:

$$\|v_j\|_{L^2(\omega \times (0, T))} \leq C, \forall j \geq 1 \tag{73}$$

and, more precisely,

$$v_j = \widehat{\varphi}_j \text{ in } \omega \times (0, T) \tag{74}$$

where  $\widehat{\varphi}_j$  solves

$$\begin{cases} -\varphi_t - \Delta\varphi + F(y_j)\varphi - \operatorname{div}(G(y_j)\varphi) = 0 & \text{in } \Omega \times (0, T) \\ \varphi = 0 & \text{on } \partial\Omega \times (0, T) \\ \varphi(T) = \widehat{\varphi}_j^0 & \text{in } \Omega \end{cases} \tag{75}$$

with the datum  $\widehat{\varphi}_j^0$  minimizing the corresponding functional  $J_j$ . We also have

$$\|\widehat{\varphi}_j^0\|_{L^2(\Omega)} \leq C. \tag{76}$$

By extracting subsequences we have

$$\widehat{\varphi}_j^0 \rightharpoonup \widehat{\varphi}^0 \text{ weakly in } L^2(\Omega) \tag{77}$$

and in view of (70)-(71), arguing as in the proof of Proposition 3.3, we deduce that

where  $\widehat{\varphi}$  solves

$$\begin{cases} -\varphi_t - \Delta\varphi + F(y)\varphi - \operatorname{div}(G(y)\varphi) = 0 & \text{in } \Omega \times (0, T) \\ \varphi = 0 & \text{on } \partial\Omega \times (0, T) \\ \varphi(T) = \widehat{\varphi}^0 & \text{in } \Omega. \end{cases} \tag{79}$$

We also have that

$$\partial_t \widehat{\varphi}_j \text{ is bounded in } L^2(0, T; H^{-1}(\Omega)), \tag{80}$$

and, once again, by Aubin-Lions compactness lemma, it follows that

$$\widehat{\varphi}_j \rightarrow \widehat{\varphi} \text{ strongly in } L^2(\Omega \times (0, T)). \tag{81}$$

Consequently

$$v_j \rightarrow v \text{ in } L^2(\omega \times (0, T)) \tag{82}$$

where

$$v = \widehat{\varphi} \text{ in } \omega \times (0, T). \tag{83}$$

It is then easy to see that

$$u_j \rightarrow u \text{ in } L^2(0, T; H_0^1(\Omega)) \tag{84}$$

where

$$\begin{cases} u_t - \Delta u + F(y)u + G(y) \cdot \nabla u + f(0, 0) = v1_\omega & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u(0) = u^0 & \text{in } \Omega \end{cases} \tag{85}$$

and

$$\begin{cases} \|u(T) - u^1\|_{L^2(\Omega)} \leq \varepsilon, \\ \pi_E(u(T)) = \pi_E(u^1). \end{cases} \tag{86}$$

To conclude the continuity of  $\mathcal{N}$  it is sufficient to check that the limit  $\widehat{\varphi}^0$  in (77) is the minimizer of the functional  $J$  associated with the limit control problem (85)-(86).

To do this, given  $\psi^0 \in L^2(\Omega)$  we have to show that

$$J(\widehat{\varphi}^0) \leq J(\psi^0).$$

But this is immediate since, by lower semicontinuity, we have

$$J(\widehat{\varphi}^0) \leq \liminf_{j \rightarrow \infty} J_j(\widehat{\varphi}_j^0),$$

on the one hand,

$$J(\psi^0) = \liminf J(\psi^0)$$

on the other one, and finally

$$J_j(\widehat{\varphi}_j^0) \leq J_j(\psi^0)$$

since  $\widehat{\varphi}_j^0$  is the minimizer of  $J_j$ .

**Compactness of  $\mathcal{N}$ .** The arguments above show that when  $y$  lies in a bounded set  $B$  of  $L^2(0, T; H_0^1(\Omega))$ ,  $u = \mathcal{N}(y)$  also lies in a bounded set of  $L^2(0, T; H_0^1(\Omega))$ . We have to show that  $\mathcal{N}(B)$  is relatively compact in  $L^2(0, T; H_0^1(\Omega))$ . But this can be obtained easily by means of the regularizing effect of the heat equation.

Indeed, we have

$$\begin{cases} u_t - \Delta u = h & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u(0) = u^0 & \text{in } \Omega, \end{cases}$$

with

$$h = v1_\omega - F(y)u - G(y) \cdot \nabla u - f(0, 0)$$

which is uniformly bounded in  $L^2(\Omega \times (0, T))$ .

Then,  $u$  can be decomposed as

$$u = p + q$$

where

$$\begin{cases} p_t - \Delta p = 0 & \text{in } \Omega \times (0, T) \\ p = 0 & \text{on } \partial\Omega \times (0, T) \\ p(0) = u^0 & \text{in } \Omega \end{cases}$$

and

$$\begin{cases} q_t - \Delta q = h & \text{in } \Omega \times (0, T) \\ q = 0 & \text{on } \partial\Omega \times (0, T) \\ q(0) = 0 & \text{in } \Omega. \end{cases}$$

Obviously,  $p$  is a fixed element of  $L^2(0, T; H_0^1(\Omega))$ . On the other hand, by classical regularity results on the heat equation we deduce that  $q$  lies in a bounded set of  $L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$ , which, as a consequence of Aubin-Lions compactness lemma, is a relatively compact set of  $L^2(0, T; H_0^1(\Omega))$ .

This completes the proof of the compactness of  $\mathcal{N}$ .

**Boundedness of the range of  $\mathcal{N}$ .** Theorem 3.1 shows that there exists  $C > 0$  such that the control  $v = v(y)$  satisfies

$$\|v(y)\|_{L^2(\omega \times (0, T))} \leq C.$$

The classical energy estimates for system (85) show that

$$\|u(y)\|_{L^2(0, T; H_0^1(\Omega))} \leq C$$

as well, since the potentials involved in it are uniformly bounded.

## 5. Further comments and results

The method presented may be easily adapted to deal with various variants of the problem addressed here. We present briefly below some of them. Note that we systematically assume the nonlinearity  $f$  to be globally Lipschitz.

### 5.1. Boundary control

Consider the semilinear heat equation with boundary control

$$\begin{cases} u_t - \Delta u + f(u, \nabla u) = 0 & \text{in } \Omega \times (0, T) \\ u = v & \text{on } \partial\Omega \times (0, T) \\ u(0) = u^0 & \text{in } \Omega. \end{cases} \quad (87)$$

This problem was addressed in Zuazua (1997a) by means of the penalization technique of an optimal control problem. The finite-approximate controllability of (87) may be easily proved by the fixed point method we have presented here. This completes the results in Fabre, Puel and Zuazua (1993) on the case where  $f = f(u)$ .

### 5.2. The $L^p$ -setting

All along this paper we have worked in the  $L^2$ -setting. Similar results may be proved in  $L^p$  for  $1 \leq p < \infty$  or in  $C_0(\Omega)$ . We refer to Fabre, Puel and Zuazua (1993) for a careful analysis of the case  $f = f(u)$ . One can easily combine the developments in Fabre, Puel and Zuazua (1993) and in the present paper to deal with the more general case  $f = f(u, \nabla u)$ .

### 5.3. Quasi bang-bang controls

The problem of finding quasi bang-bang controls was addressed in Fabre, Puel and Zuazua (1993, 1995) in the case where  $f = f(u)$ . In a similar way, using the fixed point argument of the present paper, the case where  $f = f(u, \nabla u)$  may be addressed as well. We recall that quasi bang-bang controls are of the form  $v \in \lambda \operatorname{sgn}(\varphi)$ ,  $\lambda$  being a real number and  $\varphi$  a solution of an adjoint heat equation.

### 5.4. Control in the initial data

Let us consider the semilinear heat equation

$$\begin{cases} u_t - \Delta u + f(u, \nabla u) = 0 & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \end{cases} \quad (88)$$

When  $f = f(u)$ ,  $f$  being globally Lipschitz, it was proved in Fabre, Puel and Zuazua (1995a) that the range of the semigroup

$$R(T) = \{u(T) : u^0 \in L^2(\Omega)\}$$

is dense in  $L^2(\Omega)$ . This may be interpreted as an approximate controllability result, the control being the initial datum  $u^0$ .

By combining the developments of the present paper and Fabre, Puel and Zuazua (1995b) this result may be easily extended to the case where  $f = f(u, \nabla u)$ ,  $f$  being globally Lipschitz.

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