

An algebraic analysis approach to 2-D discrete problems

by

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Abstract: A model of algebraic analysis for the 2-index sequences (of the type 2-D) is considered. For difference operators of the form

$$D\{x_{m,n}\} := \{x_{m+1,n+1} - a_{m,n}x_{m,n}\}$$

the right inverses and the corresponding initial operators are constructed. Having already known the initial operators, one can determine solutions of the corresponding initial value problems.

Keywords: algebraic analysis, 2-D systems, difference operator, right invertible operator, right inverse, initial operator, forward shift, backward shift.

1. Foundations of algebraic analysis

Let X be a linear space over a field \mathbb{F} of scalars (of the characteristic zero). Let $L(X)$ be the set of all linear operators A whose domains $\text{dom } A$ and sets of values $\text{range } A = A \text{ dom } A$ are linear subsets of the space X . Write

$$L_0(X) := \{A \in L(X) : \text{dom } A = X\}.$$

An operator $D \in L(X)$ is said to be *right invertible* if there is an operator $R \in L_0(X)$ such that $RX \subset \text{dom } D$ and $DR = I$, where $I \in L_0(X)$ is the identity operator. The operator R is a *right inverse* of D . Denote by \mathcal{R}_D the set of all right inverses of D . Clearly, $\mathcal{R}_D \subset L_0(X)$.

An operator $F \in L_0(X)$ is said to be an *initial operator* for D corresponding to an $R \in \mathcal{R}_D$ if

$$F^2 = F, \quad FX = \ker D \quad \text{and} \quad FR = 0,$$

It can be shown (cf. Theorem 2.2.1 in Przeworska-Rolewicz, 1988) that F is an initial operator for D corresponding to R if and only if

$$Fx = x - RDx \text{ for } x \in \text{dom } D.$$

We therefore conclude that the family $\mathcal{R}_D = \{R_\gamma\}_{\gamma \in \Gamma}$ of right inverses of D induces a family $\mathcal{F}_D = \{F_\gamma\}_{\gamma \in \Gamma}$ of initial operators for D , where

$$F_\gamma = I - R_\gamma D \text{ on } \text{dom } D, \gamma \in \Gamma.$$

EXAMPLE 1 (see Example 5.3 in Przeworska-Rolewicz, 1998) Let $X := (s)_{\mathbb{F}}$ be the space of all sequences $x = \{x_n\}$, where $x_n \in \mathbb{F}$, $n \in \mathbb{N}$, with the usual coordinatewise addition of sequences and multiplication of sequences by scalars belonging to \mathbb{F} .

Consider the forward shift

$$D\{x_n\} = \{x_{n+1}\}, \{x_n\} \in \text{dom } D = (s)_{\mathbb{F}},$$

which is right invertible. Indeed, $\ker D = \{c\delta : c \in \mathbb{F}\} \neq \{0\}$, where $\delta := \{\delta_1^n\}$ and δ_1^n is the Kronecker symbol, i.e.

$$\delta_1^n = \begin{cases} 1 & \text{for } n = 1, \\ 0 & \text{for } n \neq 1. \end{cases}$$

A right inverse of D is the backward shift:

$$R_1\{x_n\} = \{x_{n-1}\}, \text{ where } x_0 := 0,$$

which determines the initial operator

$$F_1\{x_n\} = x_1\delta.$$

Let $i \in \mathbb{N}$ and let

$$F_i\{x_n\} := (x_1 + \cdots + x_i)\delta, \quad i > 1.$$

Observe that every operator defined by the following formula:

$$R_i x := R_1 x - F_{i-1} x, \quad x \in X, \quad i > 1,$$

is a right inverse of D , since $F_{i-1} X = \ker D$, $i > 1$. Moreover, F_i , $i > 1$, is an initial operator for D corresponding to R_i , $i > 1$. Indeed,

$$\begin{aligned} x - R_i D x &= x - (R_1 D x - F_{i-1} D x) = (x - R_1 D x) + F_{i-1} D x \\ &= F_1 x + F_{i-1} D x = x_1 \delta + (x_2 + \cdots + x_i) \delta \\ &= (x_1 + \cdots + x_i) \delta = F_i x, \quad i > 1, \end{aligned}$$

for every $x \in X$. Hence, to the family $\{R_i\}_{i \in \mathbb{N}} \subset \mathcal{R}_D$ of right inverses of D

2. Formulation of the problem

The set $(s)_{\mathbb{F}}$ of all sequences $x = \{x_{m,n}\}$, where $x_{m,n} \in \mathbb{F}$, $m, n \in \mathbb{N}$, with the coordinatewise addition and multiplication by scalars

$$x + y := \{x_{m,n} + y_{m,n}\}, \lambda x := \{\lambda x_{m,n}\},$$

where $x = \{x_{m,n}\} \in (s_2)_{\mathbb{F}}$, $y = \{y_{m,n}\} \in (s_2)_{\mathbb{F}}$, $\lambda \in \mathbb{F}$, is a linear space over the field \mathbb{F} . Moreover, if for $x, y \in (s_2)_{\mathbb{F}}$ we define the coordinatewise multiplication

$$xy := \{x_{m,n}y_{m,n}\},$$

then $(s_2)_{\mathbb{F}}$ is a commutative algebra over \mathbb{F} with the unit $e = \{e_{m,n}\}$, where $e_{m,n} = 1$ for all $m, n \in \mathbb{N}$.

Suppose therefore that $X := (s)_{\mathbb{F}}$ is an algebra with the structure operations defined as above. Suppose, moreover, that there are given sequences $\alpha, \beta \in (s)_{\mathbb{F}}$ and $a \in (s)_{\mathbb{F}}$ with the property

$$a_{m,n} \neq 0 \text{ for every } m, n \in \mathbb{N}.$$

The second author of the present paper posed in the paper Wysocki (2002) a problem, which can be presented in the algebraic analysis approach as follows:

Determine for the operator

$$D\{x_{m,n}\} := \{x_{m+1,n+1} - a_{m,n}x_{m,n}\} \tag{1}$$

a right inverse $R \in \mathcal{R}_D$ and the corresponding initial operator F induced by the following conditions:

$$x_{m,n_0} = \alpha_m, x_{m_0,n} = \beta_n, m, n \in \mathbb{N}, \tag{2}$$

where $a_{m_0} = \beta_{n_0}$ and m_0, n_0 are fixed positive integers such that $m_0 \neq n_0$.¹

3. Solution of the problem

We shall solve the problem posed in Section 2 in the case when instead of Conditions (2) the following conditions are imposed:

$$x_{m,1} = \alpha_m^1, x_{m,2} = \alpha_m^2, \dots, x_{m,n_0} = \alpha_m^{n_0}, m \in \mathbb{N} \tag{3}$$

and

$$x_{1,n} = \beta_n^1, x_{2,n} = \beta_n^2, \dots, x_{m_0,n} = \beta_n^{m_0}, n \in \mathbb{N}, \tag{4}$$

where given sequences

$$\alpha^j = \{\alpha_m^j\} \in (s)_{\mathbb{F}}, j \in \overline{1, n_0}^2,$$

$$\beta^i = \{\beta_n^i\} \in (s)_{\mathbb{F}}, i \in \overline{1, m_0},$$

¹The case $m_0 = n_0$ has been considered in Wysocki (2002).

This implies

$$y_{m,n} = \begin{cases} z_{m,n} \sum_{k=1}^m \frac{x_{k,n-m+k}}{z_{k,n-m+k}} & \text{for } m \leq n, \\ z_{m,n} \sum_{l=1}^n \frac{x_{m-n+l,l}}{z_{m-n+l,l}} & \text{for } m > n, \end{cases} \quad m, n \in \mathbb{N}. \quad (11)$$

Finally, we get

$$(I - R_1 R_2 A)^{-1} \{x_{m,n}\} = \{y_{m,n}\},$$

where the sequence $\{y_{m,n}\}$ is given by Formulae (11).

5. Algebraic description of 2-D systems in control theory

The generalized 2-D model considered in control theory is the state-space model with deviating arguments, Kaczorek (1993),

$$\begin{aligned} E_{m+1,n+1} x_{m+1,n+1} &= A_{m,n}^0 x_{m,n} + A_{m+1,n}^1 x_{m+1,n} + A_{m,n+1}^2 x_{m,n+1} \\ &+ B_{m,n}^0 x_{m-k,n-l} + B_{m+1,n}^1 x_{m-k+1,n-l} + B_{m,n+1}^2 x_{m-k,n-l+1} \\ &+ C_{m,n}^0 u_{m,n} + C_{m+1,n}^1 u_{m+1,n} + C_{m,n+1}^2 u_{m,n+1}, \end{aligned} \quad (12)$$

$$y_{m,n} = G_{m,n} x_{m,n} + H_{m,n} u_{m,n}, \quad (13)$$

where $x_{m,n} \in \mathbb{R}^p$ is the local state vector at the point $(m, n) \in \mathbb{N} \times \mathbb{N}$, $u_{m,n} \in \mathbb{R}^q$ is the input vector, $y_{m,n} \in \mathbb{R}^r$ is the output vector and the variable coefficients $E_{m,n}, A_{m,n}^0, A_{m,n}^1, A_{m,n}^2, B_{m,n}^0, B_{m,n}^1, B_{m,n}^2, C_{m,n}^0, C_{m,n}^1, C_{m,n}^2, G_{m,n}, H_{m,n}$ for each $(m, n) \in \mathbb{N} \times \mathbb{N}$ are the real matrices of appropriate dimensions.

Upon admitting

$$\begin{aligned} x &:= [x_\mu], \quad x_\mu \in (s_2)_{\mathbb{R}}, \quad \mu \in \overline{1, p}, \\ u &:= [u_\nu], \quad u_\nu \in (s_2)_{\mathbb{R}}, \quad \nu \in \overline{1, q} \end{aligned}$$

we get $x = \{x_{m,n}\}$, $u = \{u_{m,n}\}$.

Using the operators considered in this paper, the state equation (12) is transformed into the vector-matrix 'partial integro-differential equation'

$$\begin{aligned} D_1 D_2 (E x) &= A_0 x + D_1 (A_1 x) + D_2 (A_2 x) \\ &+ B_0 R_1^k R_2^l x + D_1 (B_1 R_1^k R_2^l x) \\ &+ D_2 (B_2 R_1^k R_2^l x) + C_0 u + D_1 (C_1 u) + D_2 (C_2 u), \end{aligned} \quad (14)$$

where operators $D_i, R_i, E, A_j, B_j, C_j$ are defined in the following way

$$D_i x := [D_i x_\mu], \quad R_i x := [R_i x_\mu], \quad i = 1, 2, \quad E x := E_{m,n} x_{m,n},$$

In the case, when the coefficients of the equation (12) are real-valued constant matrices, from (14) we obtain the first and the second Fornasini–Marchesini model, respectively (see Fornasini and Marchesini, 1978; Antoniou and Emmons, 2000)

$$\begin{aligned}
 D_1 D_2 x &= A_1 D_1 x + A_2 D_2 x + C_0 u, \\
 D_1 D_2 x &= A_1 D_1 x + A_2 D_2 x + C_1 D_1 u + C_2 D_2 u
 \end{aligned}$$

and the generalized linear model (see Kaczorek, 1985; Dzieliński, 1993)

$$D_1 D_2 x = A_0 x + A_1 D_1 x + A_2 D_2 x + C_0 u + C_1 D_1 u + C_2 D_2 u. \tag{15}$$

EXAMPLE 3 Consider the scalar system described by the following state equation

$$x_{m+1,n+1} = x_{m,n} + (m + 1)(n + 1), \quad m, n \in \mathbb{N}. \tag{16}$$

It corresponds a particular case of the model (15) and it can be presented in the form

$$D_1 D_2 x = x + u$$

or

$$\tilde{D}x = u, \tag{17}$$

where $x = \{x_{m,n}\}$, $u = \{(m + 1)(n + 1)\}$.

We determine the solution of the equation (17) with the condition

$$F^{(m_0,n_0)}x = 0, \tag{18}$$

for a fixed $m_0, n_0 \in \mathbb{N} \setminus \{1\}$.

The condition (18) will be satisfied if

$$\begin{aligned}
 x_{1,n} + x_{2,n} + \dots + x_{m_0-1,n} &= 0, \quad x_{m_0,n} = 0, \quad n \in \mathbb{N}, \\
 x_{m,1} + x_{m,2} + \dots + x_{m,n_0-1} &= 0, \quad x_{m,n_0} = 0, \quad m \in \mathbb{N}.
 \end{aligned}$$

The solution of the initial value problem (17), (18) is given by the formula

$$x = R^{(m_0,n_0)}u,$$

i.e.

$$\begin{aligned}
 x &= (I - R_1 R_2)^{-1} R_1^{(m_0)} R_2^{(n_0)} u \\
 &= (I - R_1 R_2)^{-1} (R_1 - F_1^{(m_0-1)}) (R_2 - F_2^{(n_0-1)}) u \\
 &= (I - R_1 R_2)^{-1} (R_1 - F_1^{(m_0-1)}) (R_2 - F_2^{(n_0-1)}) u
 \end{aligned}$$

Applying the form of the operators $R_1, R_2, F_1^{(m_0)}, F_2^{(n_0)}, (I - R_1 R_2)^{-1}$, we finally obtain

$$x_{m,n} = \begin{cases} \frac{m(m+1)(3n-m+1)}{6} - \frac{(n-m+1)(m_0-1)(m_0+2)}{2} & \text{for } m < n, \\ \frac{m(m+1)(2m+1)}{6} - \frac{(m_0-1)(m_0+2)}{2} - \frac{(n_0-1)(n_0+2)}{2} + \frac{(m_0-1)(m_0+2)(n_0-1)(n_0+2)}{4} & \text{for } m = n, \\ \frac{n(n+1)(3m-n+1)}{6} - \frac{(m-n+1)(n_0-1)(n_0+2)}{2} & \text{for } m > n, \end{cases}$$

where $m, n \in \mathbb{N}$.

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