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MARZENA IWANISZYN[1], MIECZYSŁAW JAROSZYŃSKI[1], JOANNA OCHOŃSKA[2], JOANNA ŁOJEWSKA[2], ANDRZEJ KOŁODZIEJ[1,3]

HEAT AND MASS TRANSFER ANALOGY FOR THE LAMINAR FLOW: DISCUSSION OF THE PROBLEM

1 - Institut of Chemical Engineering PAN, ul. Bałtycka 5, 44-100 Gliwice
2 - Jagiellonian University, Faculty of Chemistry, ul. Ingardena 3, 33-060 Kraków
3 - Opole University of Technology, Faculty of Civil Engineering, ul. Katowicka 48, 45-061 Opole

The study deals with the heat and mass transfer analogy developed for both laminar and turbulent flows. The Chilton-Colburn and Lévêque analogies are discussed. The Fourier-Kirchhoff equations covering the heat and mass transport as well as their theoretical solutions are also described. The ratio of Schmidt-to-Prandtl number for gases and liquids, obtained using the film and the penetration models, are discussed.

Praca poświęcona jest tematyce analogii transportu ciepła i masy dla przepływu laminarnego i turbulentnego. Omówiono analogie Chiltona-Colburna oraz Lévêque'a. Opisano także równania Fouriera-Kirchhoffa dla transportu ciepła i masy oraz ich teoretyczne rozwiązania. Przedstawiono stosunek liczby Schmidta do Prandtla dla gazów i cieczy uzyskany przy zastosowaniu modeli filmu i penetracji.

1. CHILTON-COLBURN AND LÉVÊQUE ANALOGIES

Since the basic mechanisms of heat, mass and momentum transport are essentially the same, it is sometimes possible to directly relate the heat and mass transfer coefficients, as well as friction factors, by means of analogies. The heat-mass transfer analogy is used to estimate mass transport from the experimentally derived heat transfer results (or vice versa). The Chilton-Colburn analogy is the most popular one commonly used in the literature. Strictly, the Chilton-Colburn analogy is valid only for the fully developed turbulent flow, however, it is also applied for packed columns, laminar flow, flow perpendicular to the pipe, flow through the dumped bed, flow around various elements, etc. In fact, the use of the Chilton-Colburn analogy for the cases is not theoretically supported and might not be very accurate.

The general forms of the experimentally derived equations governing the heat and mass transfer coefficient for forced turbulent flow are:

$$Nu = C\,Re^{A}\,Pr^{B} \tag{1}$$

$$Sh = C\,Re^{A}\,Sc^{B} \tag{2}$$

and the Stanton numbers for heat and mass transport are:

$$St^{H} = \frac{Nu}{Re\,Pr} = C\,Re^{A-1}\,Pr^{B-1} \tag{3}$$

$$St^{M} = \frac{Sh}{Re\,Sc} = C\,Re^{A-1}\,Sc^{B-1} \tag{4}$$

Comparison of the two above equations and assumption of B=1/3 leads to the Chilton-Colburn analogy [1–3]:

$$j = St^{H}\,\mathrm{Pr}^{\frac{2}{3}} = St^{M}\,Sc^{\frac{2}{3}} = C\,\mathrm{Re}^{A-1} \tag{5}$$

or in a more general form:

$$St^{H}\,\mathrm{Pr}^{1-B} = St^{M}\,Sc^{1-B} = C\,\mathrm{Re}^{A-1} \tag{6}$$

The analogy is based on the experimental data and correlations rather than on the assumptions about transport mechanisms. Thus, the analogy is more empirical than theoretical in nature. Rearrangement leads to a more comfortable form:

$$\frac{Sh}{Nu} = \left(\frac{Sc}{\mathrm{Pr}}\right)^{B} \tag{7}$$

The complete Chilton-Colburn analogy combines transport dimensionless numbers into one expression [1,2]:

$$\begin{aligned} j^{H} &= St^{H}\,Pr^{\frac{2}{3}} = \frac{Nu}{Re\,Pr}Pr^{\frac{2}{3}} = \frac{Nu}{Re\,Pr^{\frac{1}{3}}} = \frac{f}{2} = \\ &= j^{M} = St^{M}\,Sc^{\frac{2}{3}} = \frac{Sh}{Re\,Sc}Sc^{\frac{2}{3}} = \frac{Sh}{Re\,Sc^{\frac{1}{3}}} \end{aligned} \tag{8}$$

The Fanning friction factor is defined as:

$$f = \frac{\Delta P}{L}\frac{\varepsilon^2 D_h}{2 w_0^2 \rho} \tag{9}$$

Another analogy combining heat, mass and momentum transfer is based on the Lévêque's heat transfer equation [4] for a developing thermal boundary layer in a hydraulically developed laminar flow. This analogy, often referred to as the Lévêque analogy, is essentially valid for the laminar flow, however, it seems accepted in the literature, that the analogy may also be applied to the turbulent flow, as long as the thermal boundary layer is thinner than the viscous sublayer [5,6]. The Lévêque analogy in the generalized form may be written as [6]:

$$\frac{Nu}{\Pr^{\frac{1}{3}}} = \frac{Sh}{Sc^{\frac{1}{3}}} = 0{,}404\left(\frac{4 f \,\mathrm{Re}^2 D_h}{L}\right) \tag{10}$$

When the Hagen-Poiseuille formula for Fanning friction factor in the laminar flow is introduced (f=16/Re), the equation yields the classic Lévêque equation:

$$Nu = 1{,}615\left(\frac{\mathrm{Re}\Pr d}{L}\right)^{\frac{1}{3}} \tag{11}$$

According to the literature [6], the analogy may be used to e.g. flow through packed beds, flow perpendicular to the tubes, tube bundles or wire meshes. The important remark is, however, that the Fanning friction factor *f* that appears in eq. (10) must reflect the viscous friction forces only. Thus, the drag forces, that appear e.g. for a flow around immersed objects especially for higher Reynolds numbers, have to be substracted.

Martin [6] introduced new dimensionless group called Lévêque number Lq:

$$Lq = \left(\frac{4 f \,\mathrm{Re}^2 \Pr D_h}{L}\right) \tag{12}$$

2. COMPARISON OF THE SC/PR RATIO FOR THE FILM AND PENETRATION MODELS

There are two the most popular models applied in the mass transfer literature, namely the film model and the penetration model.

Traditionally, the film model is usually ascribed to the gas-phase side mass transfer in packed columns while the penetration model to the liquid-phase. The exponent B at the Schmidt number is usually close to 1/3 (the film model and the classic Chilton-Colburn analogy) or 1/2 (the penetration model). The exponents do not differ substantially, however, for some cases the classic Chilton-Colburn analogy may fail.

Table 1 compares the ratios of the Schmidt-to-Prandtl number for both the film and penetration models. Schmidt numbers for gases and vapors diffusing through air at normal conditions (t=25ºC, p=1 atm) and for liquids dissolved in the water at t=20ºC, p=1 atm are tabularized in [2]. The Prandtl number for air and water are 0.72 and 7.06, respectively.

Table 1. Comparison of ratio of Schmidt-to-Prandtl number for both film and penetration models
Tabela 1. Porównanie stosunku liczby Schmidta do Prandtla dla modeli filmu i penetracji

System	Pr	Sc	Sc/Pr	Film model $(Sc/Pr)^{1/3}$	Average value	Penetration model $(Sc/Pr)^{1/2}$	Average value
Gas phase	0.72	0.22–2.6	0.31–3.61	0.68–1.53	1.11±38%	0.56–1.9	1.23±54%
Liquid phase	7.06	196–2720	27.76–385.27	3.03–7.28	5.16±41%	5.27–19.63	12.45±58%

Figure 1 compares the ratio of Sherwood-to-Nusselt number for both film and penetration models for gases and liquids.

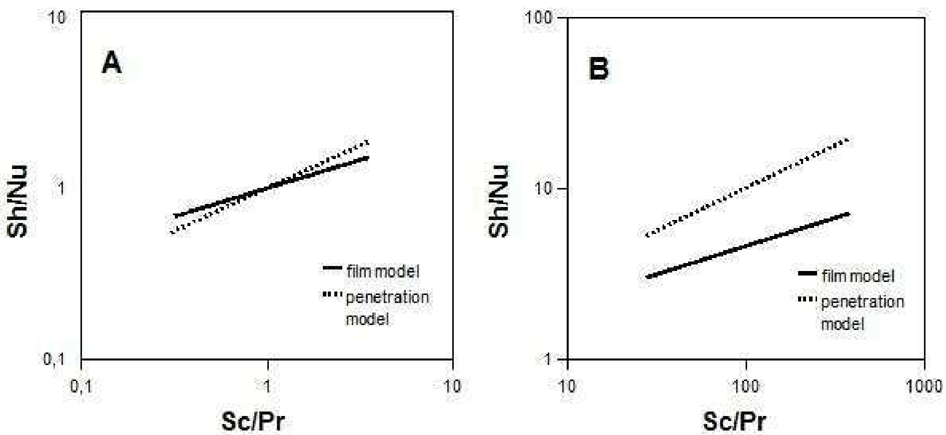

Fig. 1. Comparison of the ratio of Sherwood-to-Nusselt number for both film and penetration models for turbulent flow. A – gas phase; B – liquid phase
Rys. 1. Porównanie stosunku liczby Sherwooda do Nusselta dla modeli filmu i penetracji dla przepływu burzliwego. A – faza gazowa; B – faza ciekła

The ratios of the Schmidt-to-Prandtl number, thus of the Sherwood-to-Nusselt number, for both film and penetration models differ rather slightly for the gas phase on the contrary to the liquid phase where significant differences may appear.

When comparing both the film and penetration models for the gas phase, the differences are not substantial. Therefore the use of the classic Chilton-Colburn analogy (B=1/3) may lead to minor errors comparing with the experimental data. In fact, the Chilton-Colburn analogy is mainly used to calculate the heat transfer coefficients in the distillation columns, based on the experimental results of the height equivalent to the theoretical plate (HETP) results. Then, the whole mass transfer resistance is attributed to the gas phase, that may sometimes not be true. However, the approach usually leads to reasonable results. The Chilton-Colburn analogy applying to the liquid phase, either in the distillation columns or for any other case, has to be done very carefully. For a high value of the Sc/Pr ratio, the use of an inadequate exponent B may lead to substantial errors. The exact estimation of the exponent B is rather difficult (it requires to use several different media during the experiment), yet it is necessary for the proper use of the heat and mass transfer analogy for the liquid phase.

The comparison of the Sh/Nu ratio for the gas phase transfer (Pr=0.72 and typical Schmidt numbers) is presented in Figure 2. The scatter between the film and penetration models (B=1/3 and 1/2, respectively) is not large thus enabling the use of the Chilton-Colburn analogy.

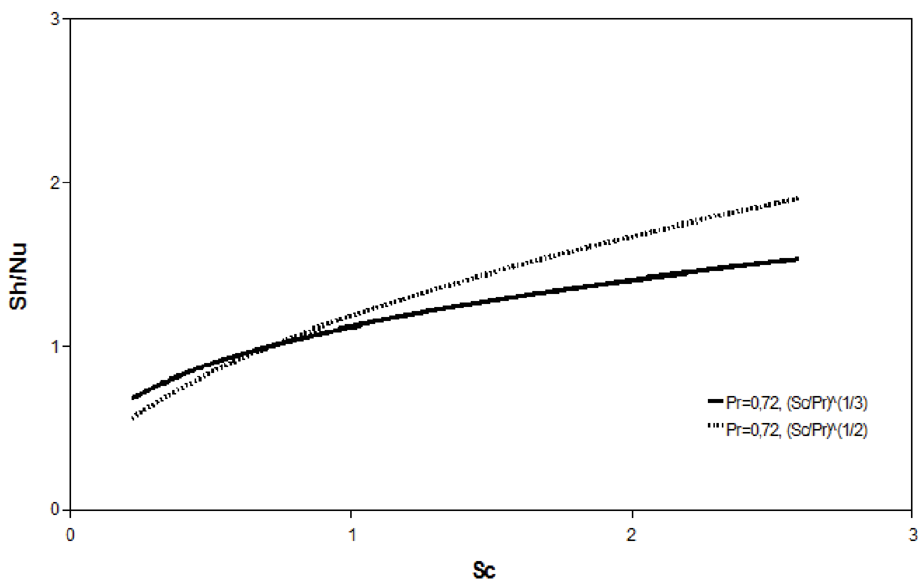

Fig. 2. Comparison of the ratio of Sherwood-to-Nusselt number for different Schmidt number and Pr=0.72 for both film and penetration models for gases and turbulent flow

Rys. 2. Porównanie stosunku liczby Sherwooda do Nusselta dla różnych liczb Schmidta i liczby Prandtla Pr=0,72 dla gazów i przepływu burzliwego dla modeli filmu i penetracji

3. ANALOGY FOR THE LAMINAR FLOW BASED ON THE FOURIER-KIRCHHOFF EQUATIONS

The heat and mass transfer analogy for a more general case may be based on the same form of the equations describing the transport processes in a flowing fluid. In fact, the Fourier-Kirchhoff equations in the forms describing heat or mass transfer are valid for any flow mechanism, thus the analogy derived here is valid independently of the actual flow mechanism. These equations have solutions which can be represented by adequate mathematical functions. It would be a mathematical equation, graph or table. Also the way to solve the function may be either analytical or numerical. For example, comparison of equations governing heat and mass transport for laminar flow is presented in Table 2.

Table 2. Equations governing heat and mass transfer in laminar flow
Tabela 2. Równania opisujące wymianę ciepła i masy w przepływie laminarnym

Heat transfer	Mass transfer
Fourier-Kirchhoff equation: $\frac{\partial T}{\partial \tau} + \vec{w} \cdot \nabla T = a_T \nabla^2 T$	Mass analogy of Fourier-Kirchhoff equation: $\frac{\partial C_A}{\partial \tau} + \vec{w} \cdot \nabla C_A = D_A \nabla^2 C_A$
Similarity modules: Nu, Re, Pr $L^* = \frac{L}{D_h \mathrm{Re}\,\mathrm{Pr}} = \frac{\pi}{4}\frac{1}{Gz}$	Similarity modules: Sh, Re, Sc $L^{*M} = \frac{L}{D_h \mathrm{Re}\, Sc} = \frac{\pi}{4}\frac{1}{Gz^M}$
Theoretical solution: $Nu_t = f(Gz)$ or $Nu_t = f(L^*)$	Theoretical solution: $Sh_t = f(Gz^M)$ or $Sh_t = f(L^{*M})$

Theoretically derived solutions for the Fourier-Kirchhoff equations are identical, assuming the same geometries of the channel, the boundary conditions and the flow velocity field. Functions f(Gz) and f(Gz^M), also f(L*) and f(L^{*M}), have the same figure. The only difference is the replacement of Gz by Gz^M or L* by L^{*M}, as well as Nu by Sh and Pr by Sc. It follows that the transition from theoretical results, obtained for the laminar flow heat transport, to solutions for the mass transport is based on the replacement of the "heat" parameters by the "mass" parameters.

Following the line of thought presented during derivation of the equation (7), the heat-mass transfer analogy based on theoretical solutions given in Table 2 is:

$$\frac{Sh_t}{Nu_t} = \frac{f(Gz^M)}{f(Gz)} \tag{13}$$

or

$$\frac{Sh_t}{Nu_t} = \frac{f(L^{*M})}{f(L^*)} \tag{14}$$

In fact, the right-hand sides of equations (13) or (14) are the functions of Schmidt and Prandtl numbers, although they might be rather complicated sometimes.

The similar discussion on the analogy problem is presented in Kołodziej study [7], where experimentally derived heat transfer coefficients obtained for wire gauzes were transform to the mass transfer. The experimental heat and mass transport representations are shown in Figure 3 [7]. Validity of the transition from the heat to the mass transfer representation was confirmed based on the reactive experiments presented in [7].

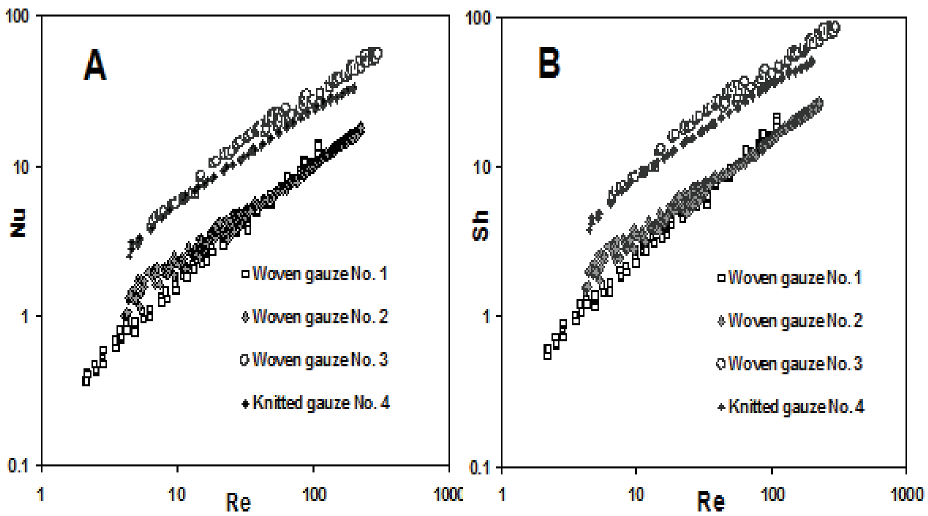

Fig. 3. Nusselt and Sherwood numbers vs. Reynolds number for laminar flow for the case of wire gauzes. A – experimental heat transfer data; B – mass transfer data derived from the analogy for the laminar flow, eq. (14)

Rys. 3. Liczby Nusselta i Sherwooda w funkcji liczb Reynoldsa dla przepływu laminarnego w przypadku siatek. A – wyniki eksperymentalne dla transportu ciepła; B – wyniki dla transportu masy uzyskane przez zastosowanie analogii dla przepływu laminarnego, równ. (14)

4. CONCLUSIONS

The Chilton-Colburn analogy is a very useful and adequate tool, but it is strictly valid for the fully developed turbulent flow. The best way to find accurate mathematical functions describing the laminar flow behavior is to use the solutions of the Fourier-Kirchhoff equations. The use of a heat and mass transfer analogy is a very convenient and simple method.

A comparison of the ratio of Schmidt to Prandtl number for both the film and penetration models shows that there are slight differences for gases and vapors diffusing through air while for liquids phase the differences are more significant. Therefore, the use of the Chilton-Colburn analogy for the gas phase is rather secure. However, for the liquid phase the use of the Chilton-Colburn analogy may be rather risky and an exact value of the exponent B of Schmidt (Prandtl) number should be carefully derived based on the experiments or the solution of the Fourrier-Kirchhoff equation.

SYMBOLS – OZNACZENIA

a_T – thermal diffusivity, $m^2\ s^{-1}$
współczynnik przewodzenia temperatury

B, C – constants
stałe

C_A – eagent concentration, mol m^{-3}
stężenie reagenta

c_p – specific heat, J kg^{-1} K^{-1}
ciepło właściwe

D_A – diffusivity, $m^2\ s^{-1}$
współczynnik dyfuzji

D_h – hydraulic diameter, m
średnica hydrauliczna

f – Fanning friction factor
współczynnik oporu hydrodynamicznego Fanninga, bezwymiarowy

G – mass stream, kg s^{-1}
strumień masowy

Gz, Gz^M – Graetz number, $Gz=G\ c_p\ \lambda^{-1}\ L^{-1}$, $Gz^M=G\ \rho^{-1}\ D_A^{-1}\ L^{-1}$
liczba Graetza cieplna lub masowa

j – Colburn factor, $j^H=Nu\ Re^{-1}\ Pr^{-1/3}$, $j^M=Sh\ Re^{-1}\ Sc^{-1/3}$
czynnik wnikania ciepła lub masy Colburna

k_C – mass transfer coefficient, m s^{-1}
współczynnik wnikania masy

L – bed length, m
długość złoża

L^*, L^{*M} – dimensionless length for the entrance region, $L^*=L\ D_h^{-1}\ Re^{-1}\ Pr^{-1}$, $L^{*M}= L\ D_h^{-1}\ Re^{-1}\ Sc^{-1}$
bezwymiarowa długość dla przepływu rozwijającego się cieplna lub masowa

Lq – Lévêque number, $Lq=4f\ Re^2\ Pr\ D_h\ L^{-1}$
liczba Lévêque'a

Nu – Nusselt number, $Nu=\alpha D_h \lambda^{-1}$
liczba Nusselta
Pr – Prandtl number, $Pr=c_p \eta \lambda^{-1}$
liczba Prandtla
Re – Reynolds number, $Re=w D_h \rho \eta^{-1}$
liczba Reynoldsa
Sc – Schmidt number, $Sc=\eta \rho^{-1} D_A^{-1}$
liczba Schmidta
Sh – Sherwood number, $Sh=k_C D_h D_A^{-1}$
liczba Sherwooda
St^H, St^M – Stanton number, $St^H=\alpha w_0^{-1} \rho^{-1} c_p^{-1}=Nu Re^{-1} Pr^{-1}$, $St^M=k_C w_0^{-1}=Sh Re^{-1} Sc^{-}$
liczba Stantona cieplna lub masowa
T – temperature, K
temperatura
w – interstitial fluid velocity, $m s^{-1}$
prędkość rzeczywista płynu
w_0 – superficial fluid velocity, $m s^{-1}$
prędkość średnia płynu (liczona na przekrój pustego aparatu)
α – heat transfer coefficient, $W m^{-2} K^{-1}$
współczynnik wnikania ciepła
ε – void volume
wolna objętość
η – dynamic viscosity, Pa s
dynamiczny współczynnik lepkości
λ – thermal conductivity, $W m^{-1} K^{-1}$
współczynnik przewodzenia ciepła
ρ – density, $kg m^{-3}$
gęstość
τ – time, s
czas

SUBSCRIPTS AND SUPERSCRIPTS – NDEKSY DOLNE I GÓRNE

t – refers to the theoretical solution of Fourier-Kirchhoff equation
rozwiązanie teoretyczne równania Fouriera-Kirchhoffa

REFERENCES – PIŚMIENNICTWO CYTOWANE

[1] HOBLER T., *Heat Transfer and Heat Exchangers*. WNT Warszawa, 1968.
[2] HOBLER T., Mass *Transfer and Absorbers*. WNT Warszawa, 1976.
[3] CHILTON T.H., COLBURN A.P., *Mass transfer (absorption) coefficients: prediction from data on heat transfer and fluid friction*. Ind. Eng. Chem., 1934, 21, 1183.
[4] LÉVÊQUE A., *Les lois de la transmission de chaleur par convection.* Annales des Mines 1928, 13, 201, 305, 381. Cited after [6].
[5] SCHLÜNDER A., *Die wissenschaftliche Theorie der Wärmeübertragung-geschichtliche Entwicklung und heutiger Stand.* Chem. Ing. Tech., 1970, 42 (14), 905. Cited after [6].
[6] MARTIN H., *The generalized Lévêque equation and its practical use for the prediction of heat and mass transfer rates from pressure drop.* Chem. Eng. Sci., 2002, 57, 3217.

[7] KOŁODZIEJ A., *Strukturalne wypełnienia krótkokanałowe w reaktorach katalitycznych o intensywnym transporcie masy.* Prace Naukowe IICh PAN 2009, Rozprawy Naukowe, Z. 3, dostępne w literaturze: www.iich.gliwice.pl

[8] KOŁODZIEJ A., ŁOJEWSKA J., *Short-Channel Structured Reactor for Catalytic Combustion: Design and Evaluation.* Chem. Eng. Process., 2007, 46, 637.

[9] KOŁODZIEJ A., ŁOJEWSKA J., *Mass transfer for woven and knitted wire gauze substrates: Experiments and modelling.* Catal. Today, 2009, 147S (suppl.), S120.

MARZENA IWANISZYN, MIECZYSŁAW JAROSZYŃSKI, JOANNA OCHOŃSKA, JOANNA ŁOJEWSKA, ANDRZEJ KOŁODZIEJ

ANALOGIA TRANSPORTU CIEPŁA I MASY: DYSKUSJA PROBLEMU

Analogia Chiltona-Colburna (równ. (8)) opiera się na podobieństwie zjawisk przenoszenia pędu, ciepła i masy (równ. 8). Obowiązuje ona ściśle dla przepływu burzliwego, ale może także być używana np. do opisu kolumn z wypełnieniem, przepływu laminarnego, przepływu prostopadłego do rur, przepływu przez złoże ziaren, opływu różnych elementów.

Analogia Chiltona-Colburna umożliwia ocenę wartości współczynników transportu masy w oparciu o wyznaczone wartości współczynników wnikania ciepła (możliwa jest też relacja odwrotna). Analogia Lévêque'a jest kolejną analogią wiążącą transport pędu, ciepła i masy. Analogia ta opiera się na równaniu Lévêque'a [4] dla rozwijającej się termicznie warstwy laminarnej w hydrodynamicznie rozwiniętym przepływie laminarnym. Obowiązuje ona ściśle dla przepływu laminarnego, ale może też być używana do opisu przepływu burzliwego do momentu, w którym termiczna warstwa graniczna będzie cieńsza od podwarstwy laminarnej [5,6]. Zastosowanie prawa Hagen-Poiseuille dla przepływu laminarnego do ogólnego równania Lévêque'a (równ. 9) prowadzi do uzyskania klasycznego równania Lévêque'a (równ. 10).

Porównano stosunek liczby Schmidta do Prandtla, a stąd stosunek liczby Sherwooda do Nusselta dla modeli filmu i penetracji dla fazy ciekłej i gazowej (tabela 1, rys. 1). Porównanie wykazało, że w przypadku fazy gazowej różnice w wartościach są niewielkie i analogia Chiltona-Colburna może być bezpiecznie stosowana. Natomiast w fazie gazowej różnice w wartościach są dość znaczne, dlatego analogię Chiltona-Colburna należy stosować ostrożnie.

Analogia dla przepływu laminarnego opiera się na podobieństwie równań Fouriera-Kirchhoffa opisujących transport ciepła i masy (tabela 2). Rozwiązania tych równań będą miały identyczną postać, a różnicą będzie zamiana parametrów cieplnych (Nu, Pr) na masowe (Sh, Sc).