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SUCCESS PROBABILITY MODEL OF PHASED MISSION SYSTEMS WITH LIMITED SPARES

MODEL PRAWDOPODOBIEŃSTWA SUKCESU SYSTEMÓW O ZADANIACH OKRESOWYCH Z OGRANICZONĄ LICZBĄ CZĘŚCI ZAMIENNYCH

This paper builds a model to analyze the success probability of phased mission systems (PMS) with given limited spares. The configuration and success criteria of phased mission may vary from phase to phase. Most reliability analysis techniques and tools of phased mission systems assume that there is no spare replacement during the phased mission or the component repair times are neglected. However, for some phased missions, failed components can be replaced by spares during the mission or in the interval of the phases and the spare replacement times are generally not negligible. By considering minimal spare replacement policy (MSRP) which is often used in military exercise, this paper presents a mathematical model for success probability analysis of phased mission which is based on minimal path set and system state analysis methods. Then, the model was demonstrated and validated by an example of military exercise.

Keywords: *success probability; phased mission systems; spare replacement; minimal path set; state transition probability.*

W niniejszej pracy skonstruowano model do analizy prawdopodobieństwa sukcesu systemów o zadaniach (misjach) okresowych (ang. phased mission systems, PMS) z daną, ograniczoną liczbą części zamiennych. Konfiguracja systemu oraz kryteria sukcesu zadania okresowego mogą być różne dla różnych faz zadania. Większość technik i narzędzi służących do analizy systemów o zadaniach okresowych nie zakłada wymiany części podczas zadania okresowego lub nie bierze pod uwagę czasu wykonania napraw elementów składowych. Tymczasem, w niektórych zadaniach okresowych istnieje możliwość wymiany elementów składowych na zapasowe bądź to w trakcie trwania zadania bądź też w przerwach pomiędzy fazami, a czas takiej wymiany zazwyczaj nie jest bez znaczenia. Biorąc pod uwagę politykę minimalnej wymiany części (ang. minimal spare replacement policy, MSRP), często stosowaną podczas ćwiczeń wojskowych, w niniejszym artykule przedstawiono matematyczny model do analizy prawdopodobieństwa sukcesu zadania okresowego, oparty na dwóch metodach: minimalnych ścieżek zdatności oraz analizy stanu systemu. Możliwość wykorzystania modelu zilustrowano i zweryfikowano na podstawie przykładowych ćwiczeń wojskowych.

Słowa kluczowe: *prawdopodobieństwo sukcesu, systemy o zadaniach okresowych, wymiana części, minimalna ścieżka zdatności, prawdopodobieństwo przejść między stanami.*

1. Introduction

Military exercise is a very important way for increasing the operational skills of operators and commanders of weapon systems. It is also used to evaluate the battle or support effectiveness of a troop. Most of exercise missions consist of several phases that must be accomplished in sequence as phased mission systems (PMS). The system configuration, success criteria, and component behavior may vary from phase to phase. How to evaluate the success probability of an exercise mission with maintenance resources plan given is very important.

In many military exercises, the weapon systems will be transferred to a position which is far away from base camp and they will stay there for a period of time. Because of the limited capability of maintenance and requirement of rapid recovering in exercise, normally spare replacement is the main maintenance type in exercises. Fortunately most units of weapon systems are designed to be the line replacement units (LRU) which can be replaced easily on line. And in order to recover the failed system as quickly as possible, the minimal spare replacement policy (MSRP) will be used in practice mostly. Under MSRP, the last component whose failure causes the system failure directly will be replaced firstly. The modern automatic fault diagnosis system makes it possible and the failed components will not be repaired in the mission normally.

Here is a real example. There are two power subsystems in the surface-to-air missile system. Both of them have components of the same type. One called electric generator produces electricity for other subsystems and it has one component of this type. The other one is able to not only produce electricity but also provide power for moving and it has two components of this type in parallel. The typical exercise of this kind of systems can be divided into three phases: move only, still tracking&shooting and move with tracking&shooting. In the move only phase, the power subsystem for moving should be working and the other subsystem doesn't need to be working. In the still tracking&shooting phase, at least one power subsystem should be working. In the move with tracking&shooting phase, both two power subsystems should be working. The phase fault trees for this exercise mission are shown in Fig.1, which is the relationship between those three components of two subsystems. Under MSRP, the first failed component supposing C_1 will not be replaced in phase 1. But after the second one C_2 fails, it will be replaced if there are enough spares left. In phase 2, only the last failed one should be replaced. However, before the phase 3 begins, the component C_3 should be replaced firstly if it has failed in phase 2. It also should be replaced after its failure in phase 3. Most exercises have schedules of the exercise include the beginning time and deadline of every phase. Normally, phase mission can be finished before the deadline. So there is free time for these activities of spares replacement. The mission will fail if the total maintenance time goes beyond the maximal free time in any phase.

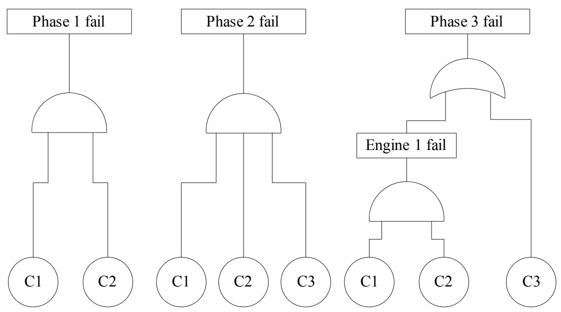

Fig.1. Phase fault trees for a mission of the surface-to-air missile system.

Such a weapon system in exercise can be considered as a PMS and some methods advanced in PMS can be used to evaluate the success probability. In recent years, many models and methods have been put forward to deal with the reliability analysis of PMS, such as Markov-chain[1], combinatorial models [2], fault tree methods [3-5] and Petri-nets [6,7] etc. BDD is more efficient for Boolean expression manipulation and the reliability analysis based on the BDD representation of the system structure function is fast, and straightforward. A BDD-based algorithm that greatly improves the computation efficiency of the PMS reliability solution was proposed in [8, 9, 10]. Considering the repair activities in the mission, a hierarchical modeling approach for the reliability analysis of phased-mission systems with repairable components was advanced in [11]. It didn't take into account spare replacements. S.P. Chew [7] describes the use of a Petri net (PN) to model the reliability of phased missions with maintenance-free operating periods (MFOP). There is no any maintenance in MFOP. Following each MFOP is a period, known as a maintenance recovery period (MRP), where the system is repaired to such a level that it is capable of completing the next MFOP. It is appropriate for some systems whose maintenance can't be done during the missions, such as the aircraft systems. But there are many exceptions, such as the move only phase of the surface-to-air missile system mentioned above. Some models of reliability analysis of PMS considering spare replacement are advanced in [12,13]. However, unfortunately the spare replacement times are neglected and these approaches have not taken into account MSRP.

In this paper, a mathematical model for success probability analysis of PMS is advanced. In this model, MSRP is considered and the spare replacement time is not neglected. The spares used in MSRP is cold standby. The rest of the paper is organized as follows. Section 2 describes the nomenclatures, problem and assumptions. In Section 3, the success probability model for phased mission systems with spares replacement is described. Section 4 gives the calculation of system state transition probability. In Section 5, a practical example of a military exercise and the experimental results are given. In Section 6, a conclusion is given.

2. Problem description

2.1 Nomenclatures

n number of phases of PMS.

T_i^{Work} work time of phase i, $i=1,2...,n$ . .

T_i^{Max} maximal duration time of phase i, $i=1,2...,n$ $T_i^{Max} \geq T_i^{Work}$

T_i^{Free} free time of phase i, $i=1,2,...,n$. $T_i^{Free}=T_i^{Max}-T_i^{Work}$

When $T_i^{Max}=T_i^{Work}$, any spare replacement activities are not permitted in the i^{th} phase. When $T_i^{Max}>T_i^{Work}$, spare replacement activities are permitted in the ith phase but the total replacement time should be less than the free time T_i^{Free}.

N number of components in PMS.

M number of component types in PMS.

Y_i i^{th} component type, .

N_i number of components of the i^{th} type, .

λ_i failure rate of components of the i^{th} type, .

T^R vector of the spare replacement times $T^R=(t_1^R, t_2^R,...,t_M^R)$, t_i^R, is the spare replacement time of the i^{th} component.

C_i state of the ith component, i. $i=1,2,...,N$, $C_i = \begin{cases} 1, & \text{the ith component is up} \\ 0, & \text{the ith component is down} \end{cases}$

F system success logic function, $F=f(C_1, C_2, ..., C_N)$.

S system state vector , $S = (s_1, s_2, \ldots, s_N)$, s_i is the *i*th component state.

S_B system state vector at the beginning of phase.

S_E system state vector at the end of phase.

$S_B^{(i)}$ system state vector at the beginning of operation phase i .

$S_E^{(i)}$ system state vector at the end of phase i , $S_E^{(i)} = S_B^{(i+1)}$.

θ system state vector when all components are down, so $\theta = (0, 0, \ldots, 0)$.

I system state vector when all components are up, so $I = (1, 1, \ldots, 1)$.

$C(S)$ set of components whose states are up in system state S , $C(S) = \{C_i \mid s_i = 1\}$. For example, when $S = (1, 0, 1, 1)$, $C(S) = \{C_1, C_3, C_4\}$.

$|S|$ count of components whose states are up in system state S . For example, when $S = (1, 0, 1, 1)$, $|S| = 3$.

$|S|_{Y_i}$ count of components of the *i*th component type whose states are up in system state S . $\|I\|_{Y_i} = N_i$.

The system state vector operations are defined as follows:

$$s_i + s_j = \begin{cases} 1, & s_i = 1,\ s_j = 1\ ;\ s_i = 0,\ s_j = 1\ ;\ s_i = 1,\ s_j = 0 \\ 0, & s_i = 0,\ s_j = 0 \end{cases}$$ is equivalent to the logical OR.

$$s_i - s_j = \begin{cases} 0, & s_i = 1,\ s_j = 1\ ;\ s_i = 0,\ s_j = 1\ ;\ s_i = 0,\ s_j = 0 \\ 1, & s_i = 1,\ s_j = 0 \end{cases}$$

$S^{(i)} + S^{(j)} = (s_1^{(i)} + s_1^{(j)}, s_2^{(i)} + s_2^{(j)}, \ldots, s_N^{(i)} + s_N^{(j)})$.

$S^{(i)} - S^{(j)} = (s_1^{(i)} - s_1^{(j)}, s_2^{(i)} - s_2^{(j)}, \ldots, s_N^{(i)} - s_N^{(j)})$.

$S^{(i)} \le S^{(j)} : S^{(i)} - S^{(j)} = \theta$.

$S^{(i)} < S^{(j)} : S^{(i)} - S^{(j)} = \theta$ and $\left|S^{(j)} - S^{(i)}\right| > 0$.

$S^{(i)} \ge S^{(j)} : S^{(j)} - S^{(i)} = \theta$.

$S^{(i)} > S^{(j)} : S^{(j)} - S^{(i)} = \theta$ and $\left|S^{(i)} - S^{(j)}\right| > 0$.

$L_f(F)$ set of the system state vectors whose system state is up and system success logic function is F . For example, when $F = C_1C_2 + C_1C_3$, $L_f(F) = \{(1, 1, 1), (1, 0, 1), (1, 1, 0)\}$. It can be obtained by the path sets.

$L_m(F)$ set of the system state vectors whose system state is up, but it will be down if any up component fails, and system success logic function is F . $L_m(F)$ is a subset of $L_f(F)$. For example, when $F = C_1C_2 + C_1C_3$, $L_m(F) = \{(1, 0, 1), (1, 1, 0)\}$. It can be obtained by the minimal path sets.

$\bar{X} = (x_1, x_2, \ldots, x_M)$ vector of the amount of spares. x_i stands for the amount of spares for the *i*th component type. The vector operations are defined as follows:

$\bar{X}^{(m)} \ge \bar{X}^{(n)}$ $\left(x_1^{(m)}, x_2^{(m)}, \ldots, x_M^{(m)}\right) \ge \left(x_1^{(n)}, x_2^{(n)}, \ldots, x_M^{(n)}\right) : x_i^{(m)} \ge x_i^{(n)}$ for $i = 1, 2, \ldots, M$.

$\bar{X}^{(m)} > \bar{X}^{(n)}$ $\left(x_1^{(m)}, x_2^{(m)}, \ldots, x_M^{(m)}\right) > \left(x_1^{(n)}, x_2^{(n)}, \ldots, x_M^{(n)}\right) : \bar{X}^{(m)} \ge \bar{X}^{(n)}$ and $x_i^{(m)} > x_i^{(n)}$ for at least one i .

$\bar{X}^*$ vector of the amount of initial spares, $\bar{X}^* = \left(x^*_1, x^*_2, \ldots, x^*_M\right)$.

$\bar{X}_B^{(i)}$ vector of the amount of available spares in the beginning of phase i , $\bar{X}_B^{(1)} = \bar{X}^*$.

$\bar{X}_E^{(i)}$ vector of the amount of available spares in the end of phase i , $\bar{X}_E^{(i)} = \bar{X}_B^{(i+1)}$.

$P\left(S_B, S_E, \bar{X}, F, T^{Work}, T^{Max}\right)$ system state transition probability from S_B to S_E and the vector of the amount of consumed spares equal to $\bar{X}$ when the system success logic function is F , the total work time is T^{Work} and the maximal duration time is T^{Max} .

$\bar{X}_D(S, F)$ vector of the amount of spares which will be used to make the system functional when the current system state is S and the system success logic function is F . We know easily $\bar{X}_D(S, F) = \bar{0}$ if $S \in L_f(F)$ under MSRP. $\bar{X}_D(S, F) > \bar{0}$, where $S \notin L_f(F)$. For example, $F = C_1C_2 + C_1C_3$, C_1 is one type and C_2, C_3 are another type. $S = (1, 0, 0)$. We can get $\bar{X}_D(S, F) = (0, 1)$ under MSRP and $\bar{X}_D(S, F) = (0, 2)$ under perfect maintenance policy.

$S_D(S, F)$ vector of system state after the spare replacement when the current system state is S and the system success logic function is F . $S_D(S, F) \in L_f(F)$, $S_D(S, F) = S$, where $S \in L_f(F)$. As above example, we can get $S_D(S, F) = (1, 0, 1)$ under MSRP and $S_D(S, F) = (1, 1, 1)$ under perfect maintenance policy.

F_i system success logic function in phase i, i =1,2,...,n.

For example, in Fig. 1, F , $F_1 = F_3 = C_1C_2 + C_1C_3 + C_2C_3$, $F_2 = C_1 + C_2 + C_3$.

2.2 Problem description

The aim is to evaluate the mission success probability when the system initial state is $S_0 \in L_f(F_1)$, the amount of initial spares is $\bar{X}^*$ and the system success logic function and the work time of each phase are given.

The phased mission system discussed in this paper is required to satisfy the following assumptions:

1) The failures of different components are statistically independent.
2) All failed components are not repairable in the mission.
3) All components are characterized by a negative exponential distribution.
4) The spare replacement policy in phased mission is MSRP.

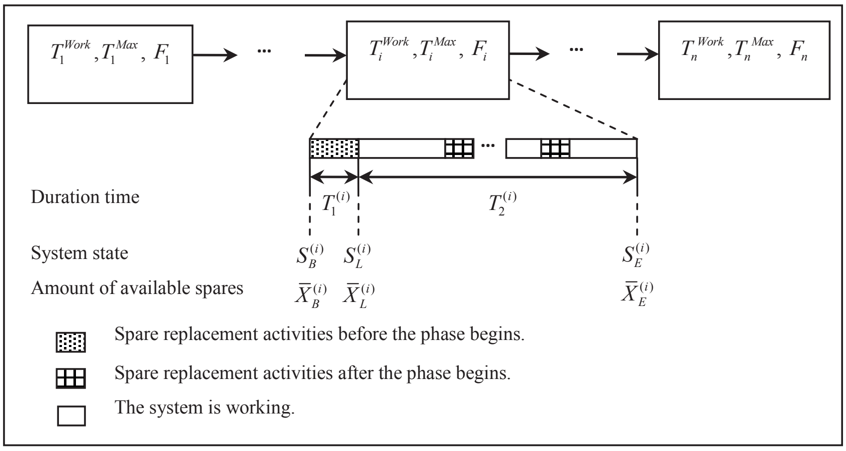

Figure 2 – Process analysis of phased mission system with spare replacement

3. Success probability model of phased mission systems with spares replacement

Fig. 2 shows the processes of the phased mission system and some nomenclatures which will be used in our model.

3.1 Spare replacement process analysis

Before the ith phase mission begins, the spare replacement may need to be done because the system success logic functions of different phases are different. This process is called as the check and spare replacement process. If there are enough spares for replacement, the system state transits from $S_B^{(i)}$ to $S_l^{(i)}$ after this process. Hence,

$$S_L^{(i)} = S_D\left(S_B^{(i)},\ F_i\right) \text{ and } S_L^{(i)} \in L_f(F_i) \tag{1}$$

It is known that $\bar{X}_D(S_B^{(i)},\ F_i)$ is the vector of the amount of spares which will be used when the current system state is $S_B^{(i)}$ and the system success logic function is F_i. So

$$\bar{X}_L^{(i)} = \bar{X}_B^{(i)} - \bar{X}_D(S_B^{(i)},\ F_i) \tag{2}$$

The duration time of spare replacement activities before the next phase begins is as below.

$$T_1^{(i)} = \bar{X}_D(S_B^{(i)},\ F_i)T^R \tag{3}$$

After the ith phase mission begins, the system state transits from $S_L^{(i)}$ to $S_E^{(i)}$. $S_E^{(i)} = S_B^{(i+1)}$. Let $\bar{X}^{(i)}$ be the vector of the amount of consumed spares in the ith phase. The duration time will be given as below.

$$T_2^{(i)} = T_i^{Work} + \bar{X}^{(i)}T^R \tag{4}$$

3.2 Calculation of success probability

If the whole mission can be completed successfully, the following conditions should be satisfied.

1) All phases have to be finished successfully, so the end system state of any phase should be in the set of system up states. Hence, $\forall\ S_E^{(i)}, i = 1,2,...,n$, $S_E^{(i)} \in L_f(F_i)$.
2) Because of the fact that there exists space replacements only when the system fails under MSRP, the end system state will not be better than the state when the system has been just made to be functional. Hence, $S_E^{(i)} \le S_D\left(S_E^{(i-1)}, F_i\right)$.
3) In any phase, the total replacement time is not more than the free time. $T_1^{(i)} + T_2^{(i)} - T_i^{Work} \le T_i^{Free}$.
4) The total amount of spare usage $\bar{X}$ is not more than $\bar{X}^*$

$$\bar{X} = \sum_{i=1}^{n}\left(\bar{X}_D\left(S_E^{(i-1)},\ F_i\right) + \bar{X}^{(i)}\right) \le \bar{X}^*$$

,   .

Let $\left(S_E^{(1)},\ S_E^{(2)},\ ...,\ S_E^{(n)}, \bar{X}^{(1)}, \bar{X}^{(2)}, ...,, \bar{X}^{(n)}\right)$ be a system state sequences. Let SB be the set of all feasible system state combinations and spare usage scenarios. Because all phases have to be finished successfully, then

$$SB = \left\{\left(S_E^{(1)},\ S_E^{(2)},\ ...,\ S_E^{(n)}, \bar{X}^{(1)}, \bar{X}^{(2)}, ...,, \bar{X}^{(n)}\right) \middle| \begin{array}{l} \forall S_E^{(i)}, i = 1,2,...,n, S_E^{(i)} \in L_f(F_i) \text{ and} \\ S_E^{(i)} \le S_D\left(S_E^{(i-1)}, F_i\right) \text{and} \\ \bar{X}_D(S_B^{(i)},\ F_i)T^R + \bar{X}^{(i)}T^R \le T_i^{Free} \text{and} \\ \sum_{i=1}^{n}\left(\bar{X}_D\left(S_E^{(i-1)},\ F_i\right) + \bar{X}^{(i)}\right) \le \bar{X}^* \end{array} \right\} \tag{5}$$

Hence, to obtain the success probability, the probabilities are summed over all feasible system state combinations and spare usage scenarios. So

$$P_{mission} = \sum_{\left(S_0, S_E^{(1)},\ S_E^{(2)},\ ...,\ S_E^{(n)}\right) \in SB} P\left(S_E^{(1)},\ S_E^{(2)},\ ...,\ S_E^{(n)}, \bar{X}^{(1)}, \bar{X}^{(2)}, ..., \bar{X}^{(n)}\right) \tag{6}$$

Where $P\left(S_E^{(1)},\ S_E^{(2)},\ ...,\ S_E^{(n)}, \bar{X}^{(1)}, \bar{X}^{(2)}, ..., \bar{X}^{(n)}\right)$ is the probability of that the total amount sequences of consumed spares are $\bar{X}^{(1)}, \bar{X}^{(2)}, ...,, \bar{X}^{(n)}$ and the system state sequences are $S_0, S_E^{(1)},\ S_E^{(2)},\ ...,\ S_E^{(n)}$. It is calculated as shown below,

$$P\left(S_E^{(1)},\ S_E^{(2)},\ ...,\ S_E^{(n)},\bar{X}^{(1)},\bar{X}^{(2)},...,\bar{X}^{(n)}\right)=$$
$$=\prod_{i=1}^{n}P\left(S_D(S_E^{(i-1)},F_i),\ S_E^{(i)},\ \bar{X}^{(i)},\ F_i,\ T_i^{Work},\ T_i^{Max}\right) \quad (7)$$

The calculation of $P\left(S_B,\ S_E,\ \bar{X},\ F,\ T^{Work},\ T^{Max}\right)$ will be described in section 4.

4. Calculation of system state transition probability

Some lemmas are given before discussing how to calculate the system state transition probability. All Lemmas as below satisfy the following conditions.

1) The system success logic function is F.
2) The system total work time is T^{Work}.
3) The maximal duration time is T^{Max}.
4) The system state is initially and it transits to S_E finally.
5) $S_B\in L_f\left(F\right)$, $S_E\in L_f\left(F\right)$.

Let $S_1=S_B-S_E-\sum_{L_k\in L_m(F)}L_k$, $S_2=S_E-\sum_{L_k\in L_m(F)}L_k$,

$$S_3=\left(s_1^{(3)},s_2^{(3)},...,s_N^{(3)}\right)\mid s_i^{(3)}=\begin{cases}1 & s_i^{(E)}=1\ and\ S_E\left(s_i^{(E)}\to 0\right)\notin L_f\left(F\right)\\ 0 & other\end{cases},i=1,2,...,N$$

$S_4=S_E-S_2-S_3$, $S_5=S_B-S_1-S_2-S_3-S_4$,

$$H_j=\left\{i\left|\ \left|S_j\right|_{Y_i}>0,\ i=1,\ 2,\ ...,\ M\right.\right\},j=1,2,...,5$$

Where $S_E\left(s_i^{(E)}\to 0\right)$ stands for the system state vector after setting the ith component state $s_i^{(E)}$ of S_E to zero.

$S_E\left(s_i^{(E)}\to 0\right)=\left(s_1^{(E)},s_2^{(E)},...,s_{i-1}^{(E)},0,s_{i+1}^{(E)},...,s_{N-1}^{(E)},s_N^{(E)}\right)$.

Lemma 1. $C\left(S_1\right)$ is the set of all components which satisfy the following condition.

1) It does not belong to any minimal path sets.
2) It fails in the mission.
3) It has not been replaced by any spares in the mission.

Proof. Let $S_L=\sum_{L_k\in L_m(F)}L_k$, $\forall C_i\in C\left(S_1\right)$, so $s_i^{(1)}=1$.

It means and $s_i^{(B)}=1$ and $s_i^{(E)}=0$. So the ith component state changes from 1 to 0 and it failed in the mission. $s_i^{(L)}=0$ is equivalent to that the ith component does not belong to any minimal path sets. So it does not have to be replaced if it fails under MSRP.

Lemma 2. $C\left(S_2\right)$ is the set of all components which satisfy the following condition.

1) It does not belong to any minimal path sets.
2) It is always up during the mission.
3) It has not been replaced by any spares in the mission.

Proof. Let $S_L=\sum_{L_k\in L_m(F)}L_k$, $\forall C_i\in C\left(S_2\right)$, so $s_i^{(2)}=1$.

It means $s_i^{(E)}=1$ and $s_i^{(L)}=0$. $s_i^{(L)}=0$ is equivalent to that the ith component does not belong to any minimal path sets. So it does not have to be replaced if it fails under MSRP. $s_i^{(E)}=1$ is equivalent to that the ith component is up when the mission is over. Hence, it is always up during the mission.

Lemma 3. $C\left(S_3\right)$ is the set of all components which satisfy the following condition.

1) It is up when the mission is over.
2) It has to be replaced if it fails after the system state is changed to be , otherwise the system will be down.

Proof. $\forall C_i\in C\left(S_3\right)$, so $s_i^{(3)}=1$. It means $s_i^{(E)}=1$ and $S_E\left(s_i^{(E)}=0\right)\notin L_f\left(F\right)$. $s_i^{(E)}=1$ is equivalent to that the ith component is up when the mission is over. $S_E\left(s_i^{(E)}=0\right)\notin L_f\left(F\right)$ is equivalent to that the system will be down if it fails after the system state is changed to be S_E. So the ith component should be replaced if it fails after the system state is changed to be S_E, otherwise the system will be down.

Lemma 4. $C\left(S_4\right)$ is the set of all components which satisfy the following condition.

1) It is up when the mission is over.
2) It does not need to be replaced if it fails in the phase.

Proof. $\forall C_i\in C\left(S_4\right)$, so $s_i^{(4)}=1$. It means $s_i^{(E)}=1$ and $s_i^{(2)}=0$ and $s_i^{(3)}=0$. $s_i^{(E)}=1$ is equivalent to that the *i*th component is up when the mission is over. $S_4=S_E-S_2-S_3$ means $C\left(S_4\right)\cap C\left(S_2\right)\neq\varnothing$ and $C\left(S_4\right)\cap C\left(S_3\right)\neq\varnothing$. Because $C\left(S_3\right)$ is the set of all components which should be replaced after its failure by Lemma 3. So $C\left(S_4\right)$ is the set of all components which do not need to be replaced if they fail.

Lemma 5. $C\left(S_5\right)$ is the set of all components which satisfies the following condition.

1) It does not belong to all minimal path sets but it belongs to at least one minimal path set.
2) It fails in the mission without spare replacement.

Proof. $\forall C_i\in C\left(S_5\right)$, so $s_i^{(5)}=1$. It means $s_i^{(B)}=1$ and $s_i^{(1)}=0$ and $s_i^{(2)}=0$ and $s_i^{(3)}=0$ and $s_i^{(4)}=0$. So the ith component is down when the mission is over. $s_i^{(E)}=0$. Let $S_L=\sum_{L_k\in L_m(F)}L_k$, $s_i^{(1)}=0$ and $s_i^{(E)}=0$ means $s_i^{(L)}=1$. So ith the component belongs to at least one minimal path set but it does not belong to all minimal path sets because of $s_i^{(E)}=0$ and $S_E\in L_f\left(F\right)$.

Lemma 6. $C(S_B)=C(S_1+S_2+S_3+S_4+S_5)$ and $C(S_1),\ C(S_2),\ C(S_3),\ C(S_4),\ C(S_5)$ are disjoint.

Proof

$S_1+S_2+S_3+S_4+S_5=S_1+S_2+S_3+S_4+S_B-S_1-S_2-S_3-S_4=S_B$, it means that $C(S_B)=C(S_1+S_2+S_3+S_4+S_5)$.

$S_5=S_B-S_1-S_2-S_3-S_4$, so $C(S_5)\cap C(S_1)=\varnothing$, $C(S_5)\cap C(S_2)=\varnothing$, $C(S_5)\cap C(S_3)=\varnothing$, $C(S_5)\cap C(S_4)=\varnothing$. $S_4=S_E-S_2-S_3$, so , . By Lemma 1, the components in $C(S_1)$ is down when mission is over. By Lemma 2,3,4, the components in the components in $C(S_2)$ or $C(S_3)$ or $C(S_4)$ is up when mission is over. So $C(S_1)\cap C(S_2)=\varnothing$, $C(S_1)\cap C(S_3)=\varnothing$, $C(S_1)\cap C(S_4)=\varnothing$. By Lemma 2 and Lemma 3, $C(S_2)\cap C(S_3)=\varnothing$. Hence, $C(S_1),\ C(S_2),\ C(S_3),\ C(S_4),\ C(S_5)$ are disjoint.

The components in the set of $C(S_1)$ failed and have not been replaced by any spares in the phase by Lemma 1. The state transition probability of these components P_1 is

$$P_1=\prod_{i\in H_1}\left(1-e^{-\lambda_i T^{Work}}\right)^{|S_1|_{Y_i}} \tag{8}$$

The components in the set of $C(S_2)$ did not fail in the phase by Lemma 2. The state transition probability of these components P_2 is

$$P_2=\prod_{i\in H_2}e^{-|S_2|_{Y_i}\lambda_i T^{Work}} \tag{9}$$

When $|S_3|>0$ and $|S_5|>0$, the components in $C(S_3)$ should be replaced after its failure after all components in $C(S_5)$ fail by lemma 3. Let $\varphi(t)$ is the density function of when all components in $C(S_5)$ fail. It is calculated by

$$\varphi(t)=\left(\prod_{i\in H_5}|S_5|_{Y_i}\lambda_i e^{-\lambda_i t}\left(1-e^{-\lambda_i t}\right)^{|S_5|_{Y_i}-1}\right)\cdot\left(\prod_{j\in H_3}e^{-|S_3|_{Y_j}\lambda_j t}\right) \tag{10}$$

In the remain time of $T^{Work}-t$, the components in $C(S_3)$ should be replaced after failure and the number of spares is consumed to be X. Hence, the state transition probability $P_{3,5}(\bar{X})$ of the components in $C(S_3)$ and $C(S_5)$ is calculated by

$$P_{3,5}(\bar{X})=\prod_{i\in H_3}\int_0^{T^{Work}}\varphi(t)\cdot\frac{1}{x_i\,!}\left(|S_3|_{Y_i}\lambda_i\left(T^{Work}-t\right)\right)^{x_i}e^{-|S_3|_{Y_i}\lambda_i(T^{Work}-t)}dt \tag{11}$$

When $|S_5|=0$ and $|S_3|>0$, the components in $C(S_3)$ should be replaced after its failure directly by lemma 3. So, $P_{3,5}(\bar{X})$ is calculated by

$$P_{3,5}(\bar{X})=\prod_{i\in H_3}\frac{1}{x_i\,!}\left(\left(|S_3|_{Y_i}\right)\lambda_i T^{Work}\right)^{x_i}e^{-|S_3|_{Y_i}\lambda_i T^{Work}} \tag{12}$$

When $|S_3|=0$ and $|S_5|>0$, the components in $C(S_5)$ fail in the phase without spare replacement. Hence, the state transition probability $P_3(\bar{X})$ of the components in $C(S_3)$ and $C(S_5)$ is calculated as shown below

$$P_{3,5}(\bar{X})=\begin{cases}\prod_{i\in H_3}\int_0^{T^{Work}}\varphi(t)\cdot\frac{1}{x_i\,!}\left(|S_3|_{Y_i}\lambda_i\left(T^{Work}-t\right)\right)^{x_i}e^{-|S_3|_{Y_i}\lambda_i(T^{Work}-t)}dt, & |S_3|>0\ and\ |S_5|>0\\ \prod_{i\in H_3}\frac{1}{x_i\,!}\left(\left(|S_3|_{Y_i}\right)\lambda_i T^{Work}\right)^{x_i}e^{-|S_3|_{Y_i}\lambda_i T^{Work}}, & |S_3|>0\ and\ |S_5|=0\\ \prod_{i\in H_5}\left(1-e^{-|S_5|_{Y_i}\lambda_i T^{Work}}\right), & |S_3|=0\ and\ |S_5|>0\\ 1, & |S_3|=0\ and\ |S_5|=0\end{cases} \tag{13}$$

When $|S_4|>0$, the components in $C(S_4)$ do not need to be replaced in the phase and they are up when the mission is over by Lemma 4. So we know these components did not fail in the phase. Hence, the state transition probability P_4 of the components in $C(S_4)$ is calculated by

$$P_4=\prod_{i\in H_4}e^{-|S_4|_{Y_i}\lambda_i T^{Work}} \tag{14}$$

By Lemma 6, $C(S_1),\ C(S_2),\ C(S_3)+C(S_5),C(S_4)$ are disjoint. $P\left(S_B,\ S_E,\ \bar{X},\ F,T^{Work},\ T^{Max}\right)$ is calculated by the product of the state transition probabilities of these three parts as shown below

$$P\left(S_B,\ S_E,\ \bar{X},\ F,T^{Work},\ T^{Max}\right)=\begin{cases}P_1P_2P_{3,5}(\bar{X})P_4, & \bar{X}T^R\le T^{Max}-T^{Work}\\ 0, & \bar{X}T^R>T^{Max}-T^{Work}\end{cases} \tag{15}$$

5. An example of military exercise

Based on the example of the surface-to-air missile system which is mentioned in Section 1, this example will take into account the electronic convertor. The electronic convertor has two components of this type in parallel. It should be working when the weapon system is in any tracking&shooting phase. Three components ($C_1,\ C_2,\ C_3$) of types X_1 and two components ($C_4,\ C_5$) of type X_2 are taken into account in this example. The failure rates are $\lambda_1=0.0007$ and $\lambda_2=0.0005$. The vector of the replacement time is $T^R=(2,\ 3)$. Now a surface-to-air missile system will execute an exercise mission with four phases whose phase fault trees are shown in Fig.3. The fourth phase is the same with the first phase.

We can get the minimal path sets and the system success logic functions of four phases. $F_1 = F_4 = C_1 + C_2$, $F_2 = C_1C_4 + C_2C_4 + C_3C_4 + C_1C_5 + C_2C_5 + C_3C_5$ and $F_3 = C_1C_3C_4 + C_2C_3C_4 + C_1C_3C_5 + C_2C_3C_5$. In the beginning of the whole mission, all components are in up states. So the vector of the initial system state is $S_0 = S_B^{(1)} = (1,\ 1,\ 1,\ 1,\ 1)$ .. The work times are $T_1^{Work} = T_4^{Work} = 45$, $T_2^{Work} = 160$ and $T_3^{Work} = 400$. The maximal duration times are $T_1^{Max} = T_4^{Max} = 60$, $T_2^{Max} = 165$ and $T_3^{Max} = 420$.

We can get $L_m(F)$ of each phase easily form the system success logic functions of three phases. The sets are as shown in table 1.

Based MSRP and perfect maintenance policy, we can get the system state after spare replacement $S_D(S,\ F_i)$ as shown in table 2. We calculated the mission success probability using the algorithm as below.

Experimental results of this example under different initial spares scenario are reported in Table 3 and Fig. 4.

From the exprimental results of this example, we can find easily the minimal initial spares scenario [3,1] if the mission success probability should be more than 0.95 under MSRP.

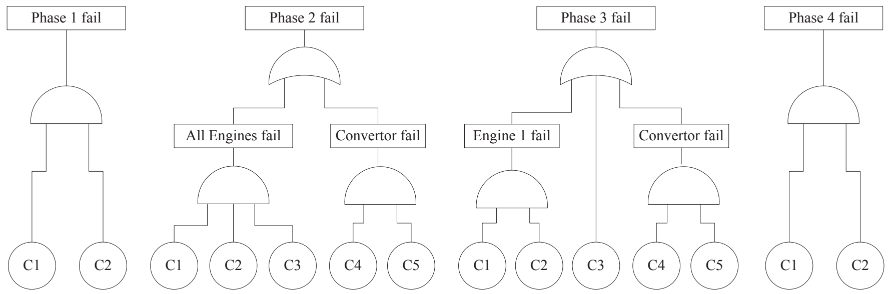

Fig.3. Phase fault trees of the example

Table 1. Set of the vectors in minimal path sets of each phase.

Number of phase	$L_m(F)$
1, 4	(0, 1), (1, 0)
2	(0, 0, 1, 0, 1), (0, 1, 0, 0, 1), (1, 0, 0, 0, 1) (0, 0, 1, 1, 0), (0, 1, 0, 1, 0), (1, 0, 0, 1, 0)
3	(0, 1, 1, 0, 1), (1, 0, 1, 0, 1), (0, 1, 1, 1, 0), (1, 0, 1, 1, 0)

Table 2. System state after spare replacement of each phase

Number of phase	S	$S_D(S,\ F_i)$ using MSRP	$S_D(S,\ F_i)$ using perfect policy
1, 4	(0, 0, 0/1, 0/1, 0/1)	(1, 0, 0/1, 0/1, 0/1)	(1,1,1,1,1)
2	(0, 0, 0, 0, 0)	(0, 0, 1, 1, 0)	
	(0, 0, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0)	(0, 0, 1, 1, 0)	
	(0, 0, 0, 0, 1)	(0, 0, 1, 0, 1)	
	(0, 1, 0, 0, 0)	(0, 1, 0, 1, 0)	
	(0, 0, 0, 1, 1)	(0, 0, 1, 1, 1)	
	(1, 0, 0, 0, 0)	(1, 0, 0, 1, 0)	
	(1, 1, 0, 0, 0)	(1, 1, 0, 1, 0)	
	(0, 1, 1, 0, 0)	(0, 1, 1, 1, 0)	
	(1, 0, 1, 0, 0)	(1, 0, 1, 1, 0)	
	(1, 1, 1, 0, 0)	(1, 1, 1, 1, 0)	
3	(1, 0, 0, 1, 0), (0, 0, 0, 1, 0), (0, 0, 1, 1, 0), (1, 0, 1, 0, 0), (0, 0, 1, 0, 0), (1, 0, 0, 0, 0), (0, 0, 0, 0, 0)	(1, 0, 1, 1, 0)	
	(1, 0, 0, 0, 1), (0, 0, 0, 0, 1), (0, 0, 1, 0, 1)	(1, 0, 1, 0, 1)	
	(0, 0, 1, 1, 1), (1, 0, 0, 1, 1), (0, 0, 0, 1, 1)	(1, 0, 1, 1, 1)	
	(0, 1, 1, 0, 0), (0, 1, 0, 1, 0), (0, 1, 0, 0, 0)	(0, 1, 1, 1, 0)	
	(0, 1, 0, 0, 1)	(0, 1, 1, 0, 1)	
	(0, 1, 0, 1, 1)	(0, 1, 1, 1, 1)	
	(1, 1, 1, 0, 0), (1, 1, 0, 1, 0), (1, 1, 0, 0, 0)	(1, 1, 1, 1, 0)	
	(1, 1, 0, 0, 1)	(1, 1, 1, 0, 1)	
	(1, 1, 0, 1, 1)	(1, 1, 1, 1, 1)	

Calculation_of_mission_success_probability (X)
for phase_num from 1 to n
scenarios_of_system_state (phase_num) = Get_states_of_path_set(phase_num);
end
scenarios_of_spares_usage =Get_ feasible_scenarios_of_spares_usage(X);
feasible_scenarios = Cartesian(scenarios_of_system_state, scenarios_of_spares_usage);
for each feasible_scenario(i) in feasible_scenarios
if (it satisfies the condition in equation (5)) then
for p from 1 to n
$P_{phase}\left(p\right)$ = *Calculation_of_state_transition_probability(feasible_scenario(i));*
// equation (15)
end
$P_{mission}\left(i\right) = \prod_{p=1}^{n} P_{phase}\left(p\right)$;
end
end
$P_{mission} = \sum_{i} P_{mission}\left(i\right)$; *// equation (6)*
end

Table 3. Mission success probabilities under different amount of initial spares using MSRP

number of X_2 spares / number of X_1 spares	0	1	2	3
0	0.528606	0.563374	0.566820	0.567062
1	0.808845	0.862045	0.867317	0.867688
2	0.880945	0.938887	0.944629	0.945034
3	0.894152	0.952962	0.958791	0.959201

Table 4. Mission success probabilities under different amount of initial spares using perfect maintenance policy

number of X_2 spares / number of X_1 spares	0	1	2	3
0	0.153131	0.253415	0.276997	0.279930
1	0.364261	0.602812	0.658908	0.665884
2	0.477438	0.790094	0.863609	0.872749
3	0.511017	0.845644	0.924317	0.934097

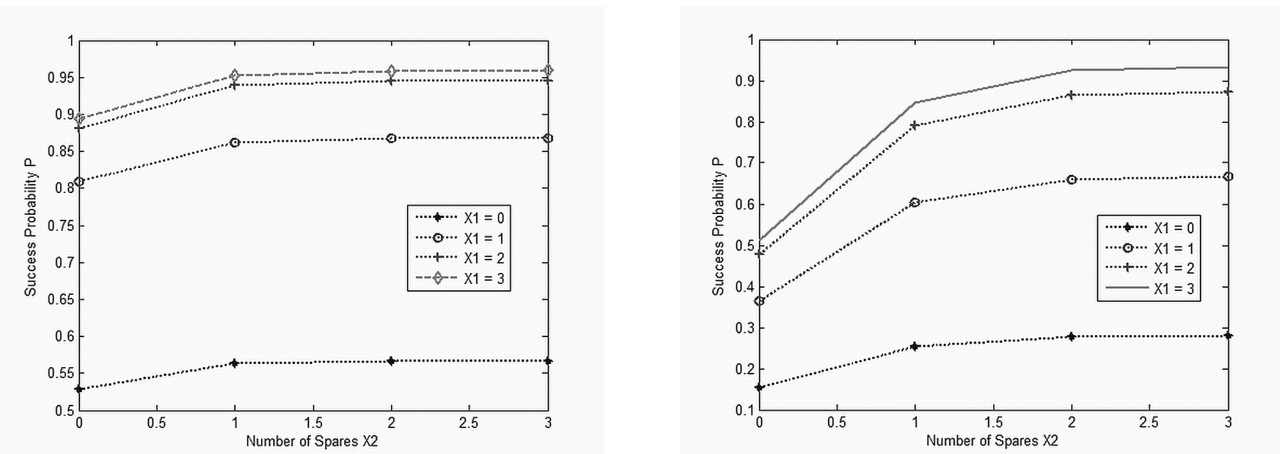

Fig. 4. Mission success probability curves under different initial spares scenario

6. Summary

This paper presents a success probability model of the phased missions under given limited spares. The spare replacement policy considered in this paper is MSRP and the spare replacement time is not neglected. In the interval of two phases, any maintenance policy can be considered by using different functions of $\bar{X}_D(S, F)$ and $S_D(S, F)$. The model advanced in this paper also can be used in the reliability analysis of PMS with cold standby components considering the switch time or not. In the practice application of this model, the system may be divided into several parts with independent models to reduce the size of the system state space, and the success probability of whole mission can be calculated by the combination of the probabilities of all parts.

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References

1. Mo Yu-chang, Siewiorek Daniel and Yang Xiao-zong. Mission reliability analysis of fault-tolerant multiple-phased systems. Reliability Engineering and System Safety, 2008; (93): 1036–1046.
2. Ma Y. and Trivedi K.S. An algorithm for reliability analysis of phased-mission systems. Reliability Engineering and System Safety, 1999; (66): 157–170.
3. Vaurio J.K. Fault tree analysis of phased mission systems with repairable and non-repairable components. Reliability Engineering and System Safety, 2001; (74): 169-180.
4. Vaurio J.K. Making systems with mutually exclusive events analyzable by standard fault tree analysis tools. Reliability Engineering and System Safety, 2001; (74): 75-80.
5. Andrews J.D. Birnbaum and criticality measures of component contribution to the failure of phased missions. Reliability Engineering and System Safety, 2008; (93): 1861-1866.
6. Mura I., Bondavalli A., Zang X. and Trivedi K.S. Dependability modeling and evaluation of phased mission systems: a DSPN approach. In: Proceedings of Dependable Computing for Critical Applications (DCCA), 1999, (7): 299-318.
7. Chew S.P., Dunnett S.J. and Andrews J.D. Phased mission modelling of systems with maintenance-free operating periods using simulated Petri nets. Reliability Engineering and System Safety, 2008; (93): 980–994.
8. Zang X., Sun H. and Trivedi K. S. A BDD-based algorithm for reliability analysis of phased-mission systems. IEEE Trans. Reliability, 1999; 48(1): 50-60.
9. Tang Z. and Dugan J. B. BDD-based reliability analysis of phasedmission systems with multimode failures. IEEE Trans. Reliability, 2006; 55(2): 350-360.
10. Zhang Tao, Guo Bo, Guo Ji-zhou and Zhang Jian-jun. A New BDD-Based Algorithm for Reliability Analysis of Phased-Mission Systems. ICRMS'2004, Xi'an, China, 2004; (8): 11-15.
11. Wang D., Trivedi K. S. Reliability Analysis of Phased-Mission System With Independent Component Repairs, IEEE TRANSACTIONS ON RELIABILITY, 2007; 56(3): 540-551.
12. Zhang Tao, Gao Da-hua, Guo Bo, Wu Xiao-yue and Tan Yue-jin. Spare availability model for phased mission systems(in Chinese). Journal of Systems Engineering, 2006; 21(1): 86-91.
13. Guo Bo, Zhang Tao, Zhang Quan and Tan Yue-jin. Phased-Mission availability assessment model for system given limited spares(in Chinese). Systems Engineering–Theory & Practice, 2005; 2: 94-100.

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