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NIEZAWODNOŚĆ SZEREGOWYCH UKŁADÓW F: TYPU "K Z N" PRZY USZKODZONYCH ELEMENTACH SKŁADOWYCH

RELIABILITY OF CONSECUTIVE *K* OUT *N*:F SYSTEMS WITH FAILED COMPONENTS

W artykule opracowano model analizy niezawodności szeregowych układów F: typu „k z n" uwarunkowany identyfikacją kilku uszkodzonych elementów w systemie. W przedstawionym modelu, system zostaje podzielony na kilka podsystemów według granic wyznaczanych przez następujące po sobie uszkodzone elementy składowe. Najpierw formułuje się niezawodność podsystemów, a model niezawodności systemu otrzymuje się analizując relację pomiędzy podsystemami a systemem jako całością. W artykule rozważono przypadki układów liniowych i kołowych. Zastosowanie proponowanego modelu zilustrowano przykładem.

Słowa kluczowe: *szeregowy układ F: typu „k z n" , niezawodność układu, uszkodzenie.*

In this paper, a model is developed for analysing the reliability of consecutive k out n:F systems under the condition that several failed components are identified in the system. The system then is partitioned into a number of subsystems by the consecutive failed components. The subsystem reliability is evaluated first and the model of system reliability is obtained by analysing the relationship between the subsystems and the system. An example is given to illustrate the operation of the proposed model.

Keywords: *consecutive k out of n F: system, system reliability, failure.*

Notation:

- $C(m,l,r,k)$ subsystem consisting of a number of components, in which there are $m(m\geq 1)$ consecutive functional components in the middle, and $l(l\geq 0)$ and $r(r\geq 0)$ consecutively defective components at its left and right ends, respectively, at time t
- p reliability of a component at time $t+\tau$ under the condition that it is working at time t
- q $1-p$
- $r(t)$ reliability of a subsystem at time t
- $H(j,x,z)$ the number of ways in which j identical balls can be placed in x distinct urns subject to the requirement that at most z balls are placed in any one urn.
- l_i number of consecutively failed components at left end of subsystem i at time t, where $0\leq l_i \leq k-1$.
- r_i number of consecutively failed components at right end of subsystem i at time t, where $0\leq r_i \leq k-1$ and $r_i \equiv l_{i+1}$ for $i=1,2,\cdots,s-1$
- v_i $min(k-1, m_i)$
- $R(m,l,r,k)$ reliability of the subsystem $C(m,l,r,k)$
- $R_{SL}(\tau,k|t)$, reliability of a linearly consecutive k out n:F system at time $t+\tau$ under the condition that it is reliable at time t and contains several failed components
- $A_j(l,r)$ event that the subsystem $C(m,l,r,k)$ works when j components fail out of the m components in the middle during $[t,t+\tau]$
- $B_j(l,r)$ event that the subsystem $C(m,0,0,k)$ works but $C(m,l,r,k)$ fails when there are j components fail out of m components in the middle during $[t, t+\tau]$, in which $C(m,0,0,k)$ and $C(m,l,r,k)$ share the same m components
- $n_{i,j}$ $\equiv \sum_{x=i}^{j}(l_x+m_x)+r_j$, for $i\leq j$; $\equiv r_i$, for $i>j$
- E_{sys} event of the system being working
- E_i event of the ith subsystem being working
- $S_{i,j}$ $\equiv\{i,i+1,\cdots,j\}$ for $i>j$, which is the set of subsystems involved in $D_{i,j}$ and is also referred to as the assembly $S_{i,j}$
- $S^0_{i,j}$ $\equiv\{i+1,\cdots,j-1\}$, which is the set of all the subsystems in $S_{i,j}$ except the first and the last ones
- $D_{i,j}$ event that all the subsystems in the system are working separately but there are at least k consecutive components failing in the assembly $S_{i,j}$, and the number of consecutive failed components is less than k in any assembly $S_{x,y}$ where $S_{x,y}\subset S_{i,j}$. $D_{i,j}$ is also refereed to as the D event.
- a_i, b_i number of consecutive components failing in the ith subsystem during $[t,t+\tau]$ next to the left bound and right bound of the subsystem, respectively
- $R_i(a_i,b_i)$ reliability of the ith subsystem with at least $a_i(0\leq a_i\leq m_i)$ and exactly $b_i(0\leq b_i\leq m_i)$ as defined earlier
- $R_l(a_i)$, $R_r(b_i)$ reliability of the ith subsystem with at least $a_i(1\leq a_i\leq m_i)$, and exactly $b_i(1\leq b_i\leq m_i)$ as defined earlier, respectively
- l_{bi}, u_{bi} minimum and maximum of b_i for $D_{i,j}$
- l_{aj}, u_{aj} minimum and maximum of a_j for $D_{i,j}$
- d_i binary variable where $d_i=1$ if all the components in subsystem i fail; otherwise, $d_i=0$
- $w(b_i,a_j)$ $\equiv b_i+a_j+n_{i+1,j-1}+d_il_i+d_jr_j$

1. Introduction

A consecutive-k-out-of-n:F system consists of n linearly or circularly ordered components where the system fails if and only if at least k consecutive components fail. It was first investigated by Kontoleon [5]. One speciality of the system is its tolerance to the dispersive failures of components. Thus, when several components have already failed in the operation of the system, one important issue for asset management is to know whether the system will work reliably for an additional period of time. If not, then maintenance work should be conducted to meet the requirement of reliability. For example, railway sleepers can be treated as a consecutive-k-out-of-n:F system in view of their reliable operation. Usually, sleepers are inspected periodically in order to identify the defects for the purpose of reliable and safe operation. After the inspection, what an asset manager needs to do is evaluating the reliability of components within a period of time. Then, based on the analysis of reliability one can make a choice between an immediate intervention and a deferred one.

In the area of consecutive k-out-of-n:F systems, comprehensive studies have been done on the reliability of the system (e.g. ref [1~9]). These include the subjects such as exact reliability models, approximate evaluation and bounds, lifetime distribution and statistic characteristics, importance of components, optimization of systems and maintenance [1]. However, not much work has been done in the analysis of a consecutive k-out-of-n:F system under the condition of several failed components being identified in it.

In this paper, we study the reliability of consecutive k out n:F systems with several failed components. A model is developed based on the analysis of reliability of subsystems. In addition, an approximate model is also presented for the case that the reliability of components is high. Finally, an example is given to illustrate the operation of the proposed model.

2. Development of reliability model

Consider a linear consecutive k out of n:F system, which consists of n linearly arranged components. The system fails if and only if at least k consecutive components fail in it. A component may either be good (working) or failed (defective), and failures of components are distributed identically and independently. Suppose at time t, the system works and there are several failed components in it. The failures of components can be identified as soon as they occur. In this situation, the system can be divided into a number of subsystems by failed components, as shown in figure 1. Each subsystem contains at least one functional component in the middle, and has a number of consecutively defective components at its one or two ends. The subsystem is denoted as $C(m,l,r,k)$, and the consecutive defective components are referred to as left and right bounds of the subsystem. It should be noted that the bound between two adjacent subsystems belongs to both of the two subsystems.

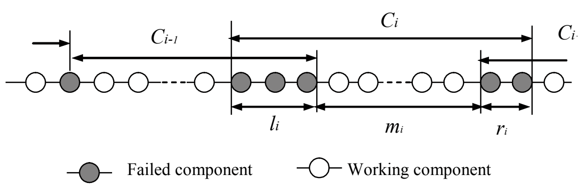

Fig. 1. Illustration of division of subsystems

The problem here is to evaluate the system reliability at time $t+\tau$. In the following sections, the reliability of a subsystem will be modelled first; and then an analysis is done in system reliability; at last, a discussion is given for a special case.

2.1. Reliability of a subsystem

Consider the ith subsystem $C(m_i,l_i,r_i,k_i)$ in the system. The subsystem fails if and only if at least k consecutive components fail in it. Obviously, if $n_{i,i}<k$, the subsystem will never fail. In the case that there is no failed component at its two ends, i.e. $l_i=0$, $r_i=0$, the number of ways of component failures for the event $A_j\,(0,0)$ is $H(j,m_i\text{-}j+1,k\text{-}1)$ [2]. If writing $x=m_i\text{-}j+1$, then we can calculate $H(j,x,z)$ recursively from z=1 to z=k-1. That is:

$$H(j,x,1)=\begin{cases}\binom{x}{j}, & 0\le j\le x\\ 0, & j>x \text{ or } j<0\end{cases} \quad (1a)$$

$$H(j,x,z)=\sum_{i=0}^{x}\binom{x}{i}H(j-zi,x-i,z-1), \qquad z\ge 2 \quad (1b)$$

where $\binom{x}{j}=1$, for x=0.

It should be noted that we add a limitation to the original definition of $H(j,x,1)$ [2] in that when $j<0$ it equals zero. This is necessary to enable the recursive calculation for all possible values of parameters j, x and z. For example, to obtain $H(2,3,2)$ from equation (1b), $H(-2,1,1)$ (for i=2) and $H(-4,0,1)$ (for i=3) are needed.

Hence, the probability of the event $A_j\,(0,0)$ is

$$\Pr\{A_j(0,0)\}=H(j,m_i-j+1,k-1)p^{m_i-j}q^j \quad (2)$$

If there are l_i and r_i ($l_i,r_i>0$) failed components at its left and right end respectively, the number of ways in the event $A_j\,(l_i,r_i)$ will be less than that of $A_j\,(0,0)$. The difference is caused by the event $B_j\,(l_i,r_i)$. It can be seen that

$$B_j(l_i,r_i)=B_j(l_i,0)\bigcup B_j(0,r_i) \quad (3)$$

Thus, we can first consider $B_j(l_i,0)$ and $B_j\,(0,r_i)$ separately. For the case that the total number of components in $C(m_i,l_i,0,k)$ is less than k, $B_j(l_i,0)$ will not occur. That is,

$$\Pr\{B_j(l_i,0)\}=0 \text{ for } m_i+l_i<k \quad (4a)$$

Otherwise for the case of $m_i+l_i\ge k$, when $B_j\,(l_i,0)$ happens, the possible number of consecutive components, i_l, failing during $[t,t+\tau]$ next to the left bound will satisfy $k\text{-}l_i\le i_l\le v_i$. For any i_l meeting this condition, since the one adjacent to the rightmost component in the left bound is functional, the possible ways for $B_j\,(l_i,0)$ to occur is that the rest $j\text{-}i_l$ ($j\text{-}i_l>0$) failed components lie in the rest $m_i\text{-}i_l\text{-}1$ ($m_i\text{-}i_l\text{-}1>0$) positions with no more than k consecutively defective components in the subsystem. Therefore,

$$\Pr\{B_j(l_i,0)\}=\sum_{i_l=k-l_i}^{v_i}H(j-i_l,m_i-j,k-1)p^{m_i-j}q^j$$

$$\text{for } m_i+l_i\ge k \quad (4b)$$

If $j=i_l$, then $j<k$ as $i_l<k$. Thus, $B_j\,(l_i,0)$ will occur with probability q^j. When $m_i\text{-}i_l\text{-}1<0$, i.e. $i_l=m_i$, then $j=i_l=m_i$ as $j\ge i_l$, and then $B_j\,(l_i,0)$ will happen with probability q^{m_i}. When $j>i_l$ and $m_i\text{-}i_l\text{-}1=0$, then $j=m_i$. In this case, if $m_i<k$, $B_j\,(l_i,0)$ will occur with probability q^j; and if $m_i=k$, the probability of $B_j\,(l_i,0)$ is zero. By

examining the above three scenarios, it can be seen that Eq.(4b) is still valid for the cases of j-i_l=0 and m_i-i_l-1≤0.

For the event $B_j(0,r_i)$, there could be $i_r(k\text{-}r_i \le i_r \le u_i)$ consecutive components next to the right end failing during $[t,t+\tau]$. Similar to the analysis for the event $B_j(l_i,0)$, we have

$$\Pr(B_j(0,r_i)) = \sum_{i_r=k-r_i}^{v_i} H(j-i_r, m_i-j, k-1)p^{m_i-j}q^j \quad \text{for} \quad m_i + r_i \ge k \qquad (5a)$$

$$\Pr\{B_j(0,r_i)\} = 0 \quad \text{for} \quad m_i + r_i < k \qquad (5b)$$

From the definition of $A_j(l_i,r_i)$ and $B_j(l_i,r_i)$, it follows that

$$\Pr\{A_j(l_i,r_i)\} = \Pr\{A_j(0,0)\} - \Pr\{B_j(l_i,0) \bigcup B_j(0,r_i)\} \qquad (6)$$

Then we can deduce that

$$\Pr\{A_j(l_i,r_i)\} = \Pr\{A_j(0,0)\} - [\Pr\{B_j(l_i,0)\} + \Pr\{B_j(0,r_i)\} - \Pr\{B_j(l_i,0) \bigcap B_j(0,r_i)\}] \qquad (7)$$

The possible ways of $B_j(l_i,0) \bigcap B_j(0,r_i)$ include that in the interval $[t,t+\tau]$, the rightmost and the leftmost one of the m_i-i_l-i_r components in the middle of the subsystem are functional and the rest j-i_l-i_r failed components occupy the rest m_i-i_l-i_r-2 positions with no more than k ones lying consecutively. Therefore, we have

$$\Pr\{B_j(l_i,0) \bigcap B_j(0,r_i)\} = \sum_{i_l=k-l_i}^{v_i} \sum_{i_r=k-r_i}^{v_i} H(j-i_l-i_r, m_i-j-1, k-1)p^{m_i-j}q^j \quad \text{for} \quad m_i + l_i + r_i > 2k \qquad (8a)$$

$$\Pr\{B_j(l_i,0) \bigcap B_j(0,r_i)\} = 0 \quad \text{for} \quad m_i + l_i + r_i \le 2k \qquad (8b)$$

It is noted that the case $j=m_i$ will not be considered for the event $B_j(l_i,0) \bigcap B_j(0,r_i)$. This is because if $m_i<k$ then the case has been considered in $B_j(l_i,0)$ or $B_j(0,r_i)$; if $m_i \ge k$, then $C(m_i,0,0,k)$ fails for $j=m_i$, and the event is impossible to happen.

From Eqs. (2)-(8), the reliability of the subsystem is given by

$$R(m_i,l_i,r_i,k) = \sum_{j=0}^{m_i} \Pr\{A_j(l_i,r_i)\}$$

$$= \sum_{j=0}^{m_i} [Pr\{A_j(0,0)\} - \Pr\{B_j(l_i,0)\} - \Pr\{B_j(0,r_i)\}] + \sum_{j=0}^{m_i-1} \Pr\{(B_j(l_i,0) \cap B_j(0,r_i)\} \qquad (9)$$

From the previous definitions of p and q, we know that they are conditional probabilities and can be given by

$$p = r(t+\tau)/r(t) \qquad (10)$$

$$q = [r(t) - r(t+\tau)]/r(t) \qquad (11)$$

2.2. System reliability

Although all the subsystems work, the system may fail. This can be seen by simply considering the ith and $(i+1)$th subsystems where $b_i(b_i \ge 1)$ additional consecutive components fail adjacent to the right end of the ith subsystem and $a_{i+1}(a_{i+1} \ge 1)$ additional consecutive components fail adjacent to the left bound of the $(i+1)$th subsystem within $[t,t+\tau]$. As shown in fig.2(a), when the two subsystems work, at the same time the whole system could be down if $b_i+r_i+a_{i+1} \ge k$. A more complex case is that the event $D_{i,j}$ may happen when $n_{i+1,j-1} \le k$-2, as shown in fig.2(b).

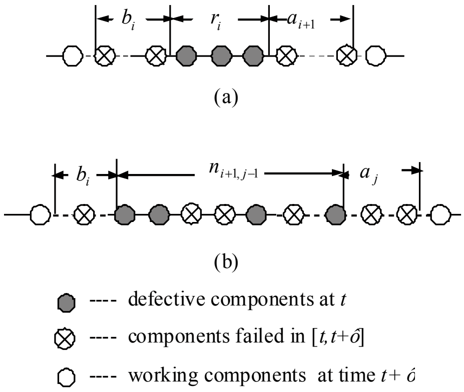

Fig. 2. Scenarios for the occurrence of event $D_{i,j}$

Consider a system consisting of s subsystems. If the reliabilities of subsystems are considered separately, then $\bigcap_{i=1}^{s} E_i$ gives the event of system being reliable. The difference between E_{sys} and $\bigcap_{i=1}^{s} E_i$ is the sum of all $D_{i,j}$. That is

$$E_{sys} = \bigcap_{i=1}^{s} E_i - \bigcup_{(i,j)\in\psi} D_{i,j} \qquad (13)$$

where ψ denotes the set of pairs (i, j) for which $D_{i,j}$ is a possible event.

Hence,

$$\Pr(E_{sys}) = \Pr\{\bigcap_{i=1}^{s} E_i\} - \Pr\{\bigcup_{(i,j)\in\psi} D_{i,j}\} \qquad (14)$$

From the previous definitions of events, it follows that

$$\Pr\{E_i\} = R(m_i,l_i,r_i,k) \qquad (15)$$

$$\Pr\{E_{sys}\} = R_{SL}(\mathrm{t}\,,k|t) \qquad (16)$$

From the assumption of independent components, it follows that E_i and $E_j(i \ne j)$ are independent. Therefore,

$$\Pr\{\bigcap_{i=1}^{s} E_i\} = \prod_{i=1}^{s} \Pr\{E_i\} \qquad (17)$$

Substituting (15)~(17) into (14) yields

$$R_{SL}(\tau,k|t) = \prod_{i=1}^{s} R(m_i,l_i,r_i,k) - \Pr\{\bigcup_{(i,j)\subset\psi} D_{i,j}\} \qquad (18)$$

where

$$\Pr\{\bigcup_{(i,j)\subset\psi} D_{i,j}\} = \sum_{(i,j)\subset\psi} \Pr\{D_{i,j}\} - \sum_{\substack{i<x,(i,j)\subset\psi \\ (x,y)\subset\psi}} \Pr\{D_{i,j} \bigcap D_{x,y}\} + \cdots + (-1)^{z+1} \Pr\{\bigcap_{(i,j)\subset\psi} D_{i,j}\} \qquad (19)$$

Since the reliability of a subsystem can be evaluated using Eq. (9), then a focus is given on the calculation of $\Pr\{\bigcup_{(i,j)\subset\psi} D_{i,j}\}$.

A. Calculation of Pr{$D_{i,j}$}

Consider the subsystems in assembly $S_{i,j}$, where $(i,j)\in\psi$, and at time $t+\tau$ there are exactly b_i and at least a_j consecutive components failing to the right bound of the ith subsystem and the left of the jth subsystem respectively.

For the ith subsystem, the component adjacent to the b_i failed ones in the left should be working with probability p, if $b_i \le m_i$-1. Thus, these b_i+1 components have no effect on the subsystem reliability if $b_i+r_i \le k$-1. Consequently, the subsystem is equivalent to that of $C(m_i-b_i-1,l_i,0,k)$ with respect to reliability, i.e.

$$R_r(b_i) = pR(m_i - b_i - 1, l_i, 0, k)$$
$$\text{for } b_i \le \min(k - r_i - 1, m_i - 1) \qquad (20a)$$

If $n_{i,j}<k$, the ith subsystem will never fail. Hence,

$$R_r(b_i) = 1 \text{ for } b_i = m_i \text{ and } n_{i,i} < k \qquad (20b)$$

Similarly, the reliability of the jth subsystem is equivalent to $C(m_j-a_j,l_j+a_j,r_j,k)$. Therefore

$$R_l(a_j) = R(m_j - a_j, l_j + a_j, r_j, k) \qquad (21)$$

For the xth subsystem in $S^0_{i,j}$, since $n_{x,x}$ must be less than k in order for $D_{i,j}$ to occur, then its reliability will be 100%. In addition, all the components in it will fail at time $t+\tau$, and the total number of components failing during $[t,t+\tau]$ in the subsystem is: $b_i + a_j + \sum_{x=i+1}^{j-1} m_x$.

Denote $\xi_{i,j}$ as the set of pairs (b_i,a_j), where $(i,j)\in\psi$ and b_i,a_j satisfy

$$w(b_i, a_j) = k \text{ for } b_i < m_i \text{ or } a_j < m_j \qquad (22a)$$
$$w(b_i, a_j) \ge k \text{ for } b_i = m_i \text{ and } a_j = m_j \qquad (22b)$$
$$1 \le b_i \le \min(k - r_i - 1, m_i) \qquad (22c)$$

It is seen that $\xi_{i,j}$ provides all the minimum of a_j given each b_i in order for $D_{i,j}$ to occur. Then, Pr{$D_{i,j}$} can be expressed as:

$$\Pr\{D_{i,j}\} = \sum_{(b_i,a_j)\in\xi_{i,j}} R_r(b_i)R_l(a_j)q^{b_i+a_j+\sum_{x=i+1}^{j-1} m_x} \prod_{x\notin S_{i,j}} R_x \qquad (23)$$

where R_x is the reliability of the xth subsystem, i.e.,

$$R_x = R(m_x, l_x, r_x, k) \qquad (24)$$

It is noted that using exactly b_i consecutively defective components in the analysis is to avoid duplicate count of the events.

B. Probability of the intersection of a number of D events

First, we consider the probability of $D_{i,j}\bigcap D_{x,y}$ ($i\le x$). There are four cases in terms of the relationship between assemblies $S_{i,j}$ and $S_{x,y}$, as illustrated in figure 3.

(1) $S_{i,j}\bigcap S_{x,y}=\phi$. Since $D_{i,j}\bigcap D_{x,y}$ means that there are at least k components failing in $S_{i,j}$ and $S_{x,y}$, but all the subsystems in the system are working and because the assemblies $S_{i,j}$ and $S_{x,y}$ are independent, then

$$\Pr\{D_{i,j}\bigcap D_{x,y}\} = f_{i,j} f_{x,y} \prod_{z\notin S_{i,j},S_{x,y}} R_z \qquad (25a)$$

where

$$f_{u,v} = \sum_{(b_u,a_v)\in\xi_{u,v}} R_r(b_u)R_l(a_v)q^{b_u+a_v+\sum_{z=u+1}^{v-1} m_z}$$

which is given similarly to the derivation for Eq. (23).

(2) $j\in S^0_{x,y}$ and $x\in S^0_{i,j}$. In this case, all the components involved in $S_{i,j}$ and $S_{x,y}$ except subsystems i and y will fail at time $t+\tau$. Thus, the probability of the intersection depends only on the ith and the yth subsystems, then.

$$\Pr\{D_{i,j}\bigcap D_{x,y}\} = R_r(b_i)R_l(a_y)q^{b_i+a_y+\sum_{z=i+1}^{y-1} m_z} \prod_{z\notin S_{i,j},S_{x,y}} R_z \quad (25b)$$

where b_i, a_y can be determined using conditions (22a), (22b) and (22c) for pairs (b_i,a_j) and (b_x,a_y) respectively given $a_j=m_j$ and $b_x=m_x$.

(3) $j\in S^0_{x,y}$ and i=x. Since range of b_i for the intersection of $D_{i,j}$ and $D_{x,y}$ will be $\max(l_{bi}, l_{bx}) \le b_i \le \min(u_{bi}, u_{bx})$, then

$$\Pr\{D_{i,j}\cap D_{x,y}\} = \sum_{\substack{b_i=\max(l_{bi},l_{bx})\\ (b_x,a_y)\in X_{x,y}}}^{\min(u_{bi},u_{bx})} R_r(b_i)R_l(a_y)q^{b_i+a_y+\sum_{x=i+1}^{y-1} m_x} \prod_{x\notin S_{i,j},S_{x,y}} R_x \quad (25c)$$

(4) $S_{i,j}\bigcap S_{x,y} = j = x$. A focus is given on the subsystems i, j or x, and y, as the reliabilities of all the other subsystems involved in $S_{i,j}$ and $S_{x,y}$ equal to 1. In addition, since $j = x$, then $a_j+b_j \le m_j$. Similar to the previous analysis,

$$\Pr\{D_{i,j}\cap D_{x,y}\} = \sum_{(b_i,a_j)\in X_{i,j}} R_r(b_i)$$
$$\sum_{\substack{(b_i,a_j)\in X_{i,j}\\ a_j+b_j\le m_j}} R_j(a_j,b_j)R_l(a_y)q^{b_i+a_j+b_j+a_y+\sum_{z\in S^0_{i,j},S^0_{x,y}} m_z} \prod_{z\notin S_{i,j},S_{x,y}} R_z \qquad (25d)$$

where $R_j(a_j,b_j)$ is obtained in a way similar to the derivation of Eq.(20a), i.e.,

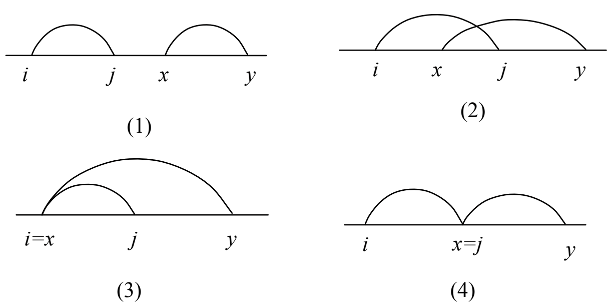

Fig. 3. Illustration of possible relationships between $S_{i,j}$ and $S_{x,y}$

$$R_j(a_j,b_j) = pR(m_j - a_j - b_j - 1, l_j + a_j, 0, k) \quad \text{for } a_j + b_j < m_j \tag{26a}$$

$$R_j(a_j,b_j) = 1 \text{ for } a_j + b_j = m_j \text{, and } n_{i,i} < k \tag{26b}$$

Now, based on the above analysis, a general situation is considered for the intersection of any number of D events, i.e., $D_{i_1,j_1} \bigcap D_{i_2,j_2} \cdots \bigcap D_{i_z,j_z}$. If $S_{i_x,jx} \cap S_{i_y,j_y} \neq \phi$, we say that the two events are connected. Hence, the subsystems involved can be partitioned into a number of disjoint sets, $V_i(i=1,2,\ldots,z_d)$, where each subsystem in a set must connect with at least one other subsystem in the set, or there is only one subsystem in the set which doesn't connect with any other subsystems involved in the intersection, as illustrated in fig. 4.

Denote J_i as the set of the subsystems j involved in V_i, where $S_{i_x,jx} \cap S_{i_y,j_y} = j = j_x = i_y$ and $j \notin S^0_{i_n,j_n}$ (n=1,2,…), and including the first and last subsystems of V_i. From the previous analysis, the probability of the intersection of these $D_{i,j}$ events depends only on the subsystems in J_i. For example, in fig. 4, $n_4 \in J_i$, and then a focus should be given on subsystems n_1, n_4 and n_8. For any of such subsystem, say x, suppose it belongs to S_{i_n,j_n} (n=1,2,…), where $x=i_n$ or $x=j_n$.

Denote θ_x as the set of pairs (a_x,b_x) for subsystem x where $x \in J_i$ such that $(b_x, a_{j_n}) \in \xi_{i_n,j_n}$, $(b_{i_m}, a_x) \in \xi_{i_m,j_m}$, and $l_{a\max} \le a_x \le u_{a\max}$, $l_{b\max} \le b_x \le u_{b\max}$, $b_x + a_x \le m_x$ where $l_{a\max} = \max_n(l_{an})$, $u_{a\max} = \min_n(u_{an})$, and $l_{b\max} = \max_m(l_{bm})$, $u_{b\max} = \min_m(u_{bm})$. Similar to the analysis for $D_{i,j} \bigcap D_{x,y}$ (case (4) discussed earlier), it can be seen that θ_x gives all the possible values of pairs (a_x,b_x). If letting g_i denote the probability of the intersection of these D events involved in V_i, then

$$g_i = \sum_{(b_1,a_1)\in\theta_1} R_l(b_1)\{\prod_{x=2}^{h-1} \sum_{(a_x,b_x)\in\theta_x} R_x(a_x,b_x) R_r(a_h) q^{\sum_{j\in V_i}(a_j+b_j)+\sum_z m_z}\} \tag{27}$$

where a_1=0 and h is the number of subsystems in J_i.

Let $V_s = \bigcup_{i=1}^{z_d} V_i$, and similar to the derivation of Eq.(25a) we have

$$P(D_{i_1,j_1} \bigcap D_{i_2,j_2} \cdots \bigcap D_{i_x,j_x}) = \prod_{i=1}^{z_d} g_i \prod_{y \notin V_s} R_y \tag{28}$$

Hence, the probability of the intersection of a number of D events can be calculated from Eqs. (27) and (28), and then system reliability can be evaluated using Eq.(18), and Eq.(19).

Since the reliability of a subsystem or a component is never greater than 1, it follows from Eq.(27) and Eq.(28) that the probability of intersection of D events has the form of $\lambda_1 q^{\alpha_1} + \lambda_2 q^{\alpha_2} + \cdots$, where $\lambda_1 \le 1$, $\lambda_2 \le 1, \cdots$. If the allowed error of the analysis is set at ε, and let n_e be the number of items with magnitude order higher than q^e, then

$$\varepsilon \le n_e q^e \tag{29a}$$

or

$$e \ge \frac{\ln(\varepsilon / n_e)}{\ln q} \tag{29b}$$

Thereby, all the items with $O(q^e)$ can be neglected to guarantee the maximum error of ε.

3. Example

A linear consecutive 6 out of 24 system is considered. At time t_0, 7 defective components have been identified by inspection with sequence numbers of 5, 8, 12, 13, 19, 20 and 21. The conditional reliability of component at $t_0+\tau$ is p=0.9. Then, the system reliability can be predicted using the proposed model.

In this case, the system can be partitioned by defective components into 5 subsystems, and their parameters are shown in table 1.

Tab. 1. Parameters and reliabilities of subsystems

Subsystem No	*m*	*l*	*r*	*R*
1	4	0	1	1.0
2	2	1	1	1.0
3	2	1	2	1.0
4	6	2	3	0.998901
5	3	3	0	0.999

Tab. 2. Parameters and probabilities of D events

$D_{i,j}$	$(b_i, a_j) \in \xi_{i,j}$	*Pr{ $D_{i,j}$ }*
$D_{1,2}$	(2,2), (4,1)	1.09×10^{-4}
$D_{1,3}$	(1,1)	9.98×10^{-5}
$D_{2,3}$	(1,2), (2,2)	9.98×10^{-4}
$D_{3,4}$	(1,3), (2,1)	1.078×10^{-3}
$D_{4,5}$	(1,2), (2,1)	1.701×10^{-3}

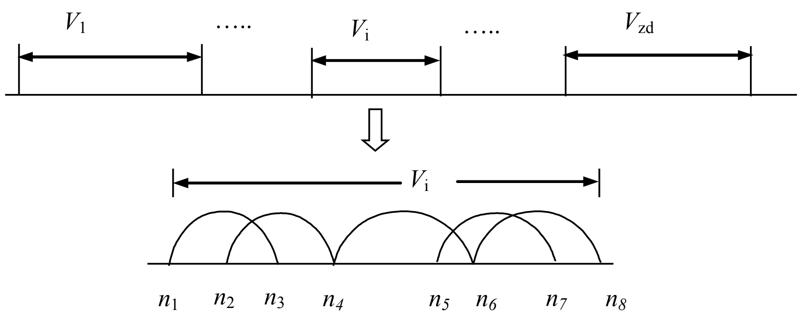

Fig. 4. Partition of subsystems for the intersection of several D events

The reliabilities of subsystems with defective boundary components are calculated using Eq. (9), as shown in table 1.

Then, an analysis of the intersection of the *D* events is conducted. For all the *D* events, (b_i, a_j) values can be obtained according to conditions (22a), (22b) and (22c), as shown in table 2. Thereby, $\Pr\{D_{i,j}\}$ can be calculated using Eq.(23). Taking $D_{3,4}$ as an example, we have

$$\Pr(D_{3,4}) = \{R_3(0,2)R_4(1,0)q^3 + R_3(0,1)R_4(3,0)q^4\}R_1R_2R_5$$

Since $R_1 = R_2 = R_3 = 1$, $R_5 = 0.999$, $R_3(0,2) = 1$, $R_3(0,1) = 0.9$, $R_4(1,0) = 0.998001$ and $R_4(3,0) = 0.9$, then it follows that, $\Pr(D_{3,4}) = 1.078\times10^{-3}$.

As an example of intersection of two *D* events, $\Pr(D_{2,3} \cap D_{3,4})$ can be obtained using Eq. (25d) by

$$\Pr(D_{2,3} \cap D_{3,4}) = \sum_{b_2=1}^{2} R_2(0,b_2)R_4(1,0)q^{b_2+3}R_5$$

$$= R_4(1,0)(pq^4 + q^5)R_5 = 9.97\times10^{-5}$$

Finally, the system reliability is obtained using the proposed approach, i.e.

$$R_{SL}(\tau, k|t) = 0.99404$$

In fact, in this example, the probabilities of $D_{1,2} \cap D_{1,3}$, $D_{3,4} \cap D_{4,5}$ and all the intersection of more than two *D* events have orders of magnitude higher than q^6. The number of these items is: $2 + \binom{3}{5} + \binom{4}{5} + \binom{5}{5} = 18$. If neglecting all these items, then the system reliability is estimated at 0.99403, and using Eq.(29a) the error is: $\varepsilon \le 18 \times q^6 = 1.8\times10^{-5}$.

4. Concluding remarks

This paper analyses the reliability of consecutive *k* out *n*: F systems with several failed components. The system reliability is modelled through analysing the relationship between the system and subsystems. The approach is valid for both linear and circular systems. To analyse the system reliability using the proposed model, considerable effort is needed to calculate the probabilities of the intersections of *D* event. However, the amount of the work can be reduced by neglecting the higher order terms.

As a further work of the study, the authors intend to apply the model to analyse a section of railway sleepers which form a consecutive *k* out of *n* system with respect to reliability.

5. References

1. Chao MT, Fu JC and Koutras MV. Survey of reliability studies of consecutive-k-out-of-n:F & related systems. IEEE T-Rel. R-44, 1995: 120-127.
2. Derman C. On the consecutive –k-out –of –n:F system. IEEE T-Rel. R -31, 1982: 57-63.
3. Dudewicz. E J. Introduction to Statistics and Probability. Holt, Rinehart and Winston. 1976. New York.
4. Flynn J, Chung C. A heuristic algorithm for determining replacement policies in consecutive k-out-of-n systems, Computers & Operations Research, 2004; 31(8): 1335-1348.
5. Kontoleon JM. Optimum allocation of components in a special 2-port network. IEEE T-Rel. R-27, 1978: 112-115.
6. Kuo W, Zuo M J. Optimal reliability modeling: principles and applications. John Wiley and Sons. New York, 2003.
7. Lambiris M and Papastavridis S. Exact reliability formulas for linear & circular consecutive-k-out-of-n: F system. IEEE T-Rel. R-34, 1985: 124-126.
8. Xiao D, Li Z and Li T. Ependability estimation for non-Markov consecutive-k-out-of-n: F repairable systems by fast simulation. Reliability Engineering & System Safety, 2007; 92(3): 293-299.
9. Zhao J., Chan AHC, Burrow MP. Reliability Analysis and Maintenance Decision for Railway Sleepers using track condition information. Journal of Operational Research Society, 2007; 8: 1047-1055.

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