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ON INFLUENCE OF FEEDBACK ON HARMONICS IN MILDLY NONLINEAR ANALOG CIRCUITS

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Summary: First, it is shown that applying the linear feedback increases the order of nonlinearity of the whole mildly nonlinear amplifier in comparison with that characterizing the amplifier without this feedback. Second, the impact of this fact on harmonics, which appear in the feedback amplifier driven by a single complex harmonic signal, is analyzed in detail. Finally, the associated model of the feedback amplifier is derived. It is shown that using this model, and only then, it is possible to interpret correctly the means of harmonic distortion calculations in weakly nonlinear amplifiers proposed by Palumbo and Pennisi in one of their recent papers.

Keywords: harmonic distortion analysis, mildly nonlinear circuits and systems, feedback, influence of feedback on harmonics, Volterra series

1. INTRODUCTION

The fact that feedback reduces nonlinear distortion is always pointed out in papers on mildly nonlinear circuits or systems, of which structure contains such a (linear) feedback. More precisely, and restricting ourselves here to consideration of only nonautonomous analog circuits like, for example, weakly nonlinear amplifiers, this is so expressed: the second and third order harmonic distortion factors or the second and third order intermodulation distortion factors of these amplifiers are considerably reduced by applying the linear negative feedback in their structures, in comparison to the structures without feedback. The amount of reduction of the harmonic or intermodulation distortion factors can be calculated from expressions, which can be found in the literature on these topics. The feedback return ratio and the feedback return difference, well known quantities in the linear feedback theory [1], play a prominent role in them.

With regard to the context sketched shortly above, the pioneering work done by Narayanan [2] in this area should be mentioned. The results of his investigations were published in 1970. Since then numerous papers, and even books, devoted to the analysis and reducing nonlinear distortion in weakly nonlinear analog circuits and systems, appeared. Because of their huge number, we will not list all of them here. Amongst them, we mention only the recent one. It is a paper by Palumbo and Pennisi [3] on the analysis of high-frequency harmonic distortion in weakly nonlinear feedback amplifiers. We refer here, in a course of presenting new results, to the method, some expressions, and results published in [3] (of course, we could do this also with regard to others, as for example, to those given by Narayanan [2]).

In all the papers mentioned above, only the advantageous effect of feedback is emphasized. That is the fact that it reduces nonlinear distortion in a circuit. Two disadvantageous effects of introducing feedback (which are of general nature and occur also when the negative type of feedback is applied) were not perceived at all. These are the following:

1. The order of nonlinearity of the whole circuit, that is of the weakly nonlinear circuit to which a linear feedback was applied, increases.
2. In consequence, new harmonics appear in the circuit containing feedback in comparison with this circuit without feedback when they are driven by a sinusoidal signal.

One of the main objectives of this paper is to explain in detail how and why the order of nonlinearity of a weakly nonlinear circuit increases after applying to it a linear feedback, and what is the mechanism of appearance of additional harmonics in the latter circuit.

2. INCREASE OF ORDER OF NONLINEARITY BY INTRODUCING LINEAR FEEDBACK

Consider a weakly nonlinear amplifier as shown in Fig. 1(a). Let its input-output characteristic be described by a nonlinear operator *H*.

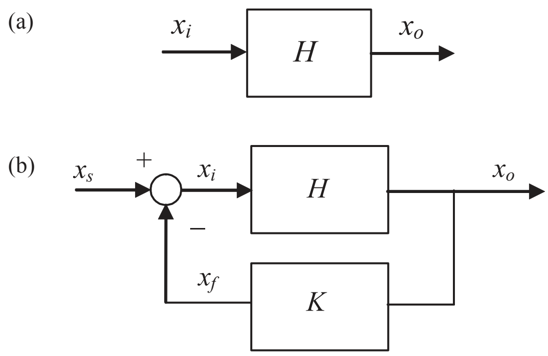

Fig. 1. Weakly nonlinear amplifier: (a) in configuration without feedback, (b) in closed-loop configuration containing linear feedback

Rys. 1. Nieliniowy wzmacniacz (z małymi nieliniowościami): (a) bez sprzężenia zwrotnego, (b) w pętli z liniowym sprzężeniem zwrotnym

When a linear feedback represented by a block *K* is applied to this amplifier, we get a closed-loop configuration illustrated in Fig. 1(b).

In Fig. 1(a), the input and output signals x_i and x_o, respectively, of a continuous time *t* are related to each other through the operator *H* as

$$x_o(t) = H(x_i(t)) \text{ or shortly } x_o = H(x_i). \tag{1}$$

In the feedback configuration of Fig. 1(b), we have two additional equations

$$x_i = x_s - x_f \tag{2a}$$

and

$$x_f = K\left(x_o\right) \tag{2b}$$

where now the input signal to the whole amplifier is denoted by x_s. Moreover, x_f means the feedback return signal supplied to the input of amplifier *H* by the linear feedback block *K*. The input-output characteristic of this block is described by a linear operator *K*.

Substituting (2a) and (2b) into (1) gives

$$x_o = H\left(x_s - K\left(x_o\right)\right). \tag{3}$$

Equation (3) is obviously an implicit form of the input-output characteristic of the whole (feedback) amplifier. The task is to find its explicit form that is an operator H_f defined as

$$x_o = H_f\left(x_s\right). \tag{4}$$

Now, to understand better the problem of increase of the order of nonlinearity in the closed-loop, we consider first a weakly nonlinear amplifier in Fig. 1(b) of which both the components: the open-loop amplifier *H* and the linear feedback block *K* are frequency independent. Furthermore, assume that the amplifier *H* is described exactly by a third order polynomial. That is its description (exact) is in the form

$$x_o = H\left(x_i\right) = a_1 x_i + a_2 x_i^2 + a_3 x_i^3 \tag{5}$$

where the coefficients a_1, a_2, and a_3 are real numbers. This means that the nonlinearity of the open-loop amplifier is of the third order.

Note that the above assumptions allow us to assume that the behavior of the operator H_f can be also described in form of a polynomial. However, we cannot assume a'priori that it will be a polynomial of the third degree (order), too. We must assume, generally, an infinite power series what will be evident from the course of further derivations. That is

$$x_o = H_f\left(x_s\right) = a_{1f} x_s + a_{2f} x_s^2 + a_{3f} x_s^3 + a_{4f} x_s^4 + a_{5f} x_s^5 + \ldots \tag{6}$$

where the coefficients a_{1f}, a_{2f}, a_{3f}, a_{4f}, a_{5f}, and so on are real numbers with the letter "*f*" in their subscripts standing for "feedback". Moreover, rewrite (2b) for a frequency independent linear feedback as

$$x_f = k \cdot x_o \tag{7}$$

where *k* is a real number.

Using the description (5) for *H* and substituting (6) and (7) into (3), we obtain

$$\begin{aligned}
&a_{1f}x_s + a_{2f}x_s^2 + a_{3f}x_s^3 + a_{4f}x_s^4 + a_{5f}x_s^5 + .. = \\
&= a_1\left[x_s - k\left(a_{1f}x_s + a_{2f}x_s^2 + a_{3f}x_s^3 + a_{4f}x_s^4 + a_{5f}x_s^5 + ..\right)\right] + \\
&+ a_2\left[x_s - k\left(a_{1f}x_s + a_{2f}x_s^2 + a_{3f}x_s^3 + a_{4f}x_s^4 + a_{5f}x_s^5 + ..\right)\right]^2 + \\
&+ a_3\left[x_s - k\left(a_{1f}x_s + a_{2f}x_s^2 + a_{3f}x_s^3 + a_{4f}x_s^4 + a_{5f}x_s^5 + ..\right)\right]^3 .
\end{aligned} \tag{8}$$

Then, equating to each other the expressions of the same order (degree) – that is such which contain the same power of the variable x_s – on both sides of (8), we get successively

$$a_{1f}x_s = a_1\left(x_s - ka_{1f}x_s\right) \tag{9a}$$

$$a_{2f}x_s^2 = -a_1ka_{2f}x_s^2 + a_2\left(x_s - ka_{1f}x_s\right)^2 \tag{9b}$$

$$a_{3f}x_s^3 = -a_1ka_{3f}x_s^3 - 2a_2\left(x_s - ka_{1f}x_s\right)ka_{2f}x_s^2 + a_3\left(x_s - ka_{1f}x_s\right)^3 \tag{9c}$$

$$\begin{aligned}
a_{4f}x_s^4 = &-a_1ka_{4f}x_s^4 - 2a_2\left(x_s - ka_{1f}x_s\right)ka_{3f}x_s^3 + \\
&+ a_2\left(ka_{2f}x_s^2\right)^2 - 3a_3\left(x_s - ka_{1f}x_s\right)^2 ka_{2f}x_s^2
\end{aligned} \tag{9d}$$

$$\begin{aligned}
a_{5f}x_s^5 = &-a_1ka_{5f}x_s^5 - 2a_2\left(x_s - ka_{1f}x_s\right)ka_{4f}x_s^4 + 2a_2k^2a_{2f}a_{3f}x_s^5 + \\
&+ 3a_3\left(x_s - ka_{1f}x_s\right)\left(ka_{2f}x_s^2\right)^2 - 3a_3\left(x_s - ka_{1f}x_s\right)^2 ka_{3f}x_s^3
\end{aligned} \tag{9e}$$

and so on. From these equations, after eliminating x_s and performing a number of algebraic manipulations, we obtain

$$a_{1f} = \frac{a_1}{1 + ka_1} \tag{10a}$$

$$a_{2f} = \frac{a_2}{\left(1 + ka_1\right)^3} \tag{10b}$$

$$a_{3f} = \frac{a_3}{\left(1 + ka_1\right)^4} - \frac{2{a_2}^2k}{\left(1 + ka_1\right)^5} \tag{10c}$$

$$a_{4f} = \frac{5{a_2}^3k^2}{\left(1 + ka_1\right)^7} - \frac{a_2a_3k\left(5 + 3ka_1\right)}{\left(1 + ka_1\right)^6} \tag{10d}$$

$$a_{5f} = -\frac{14{a_2}^4k^3}{\left(1 + ka_1\right)^9} + \frac{{a_2}^2a_3k^2\left(21 + 6ka_1\right)}{\left(1 + ka_1\right)^8} - \frac{3{a_3}^2k}{\left(1 + ka_1\right)^7} \tag{10e}$$

and so on.

Note from (10d) and (10e) that the coefficients a_{4f} and a_{5f} are, generally, nonzero. The same regards also the further coefficients a_{6f}, a_{7f}, and so on, in the expansion (6). So really the degree (order) of the polynomial describing an amplifier after adding to it a linear feedback is different from that describing the same amplifier but without feedback. It changed here from three in (5) to infinity in (6).

So the conclusion is that the linear feedback has an influence upon the order of nonlinearity that is incorporated in the amplifier's closed-loop description. Furthermore, observe that if the amplifier's open-loop description as (5) would be in form of an infinite power series we did not see the above effect. Probably of this reason, this effect was not perceived before.

At this point, we pay also the reader's attention to the fact that the radius of convergence of the (finite) power series (5) is infinite, and it does not mean that the same holds for the (infinite) power series (6), too. Probably, the latter has a finite radius of convergence and it must be found. From the form of expressions (10a-e), we see that it is not a simple task.

A more general case of a weakly nonlinear amplifier in Fig. 1(b), of which both the components: the open-loop amplifier H and the linear feedback block K are frequency dependent, is also, as we will see, more cumbersome. Assume then that the amplifier H is described (in the time domain) by a Volterra series [4, 5] consisting of only first three components. That is its description (exact) will be, analogously to (5), in the following form

$$x_o(t)=H\left(x_i(t)\right)=x_o^{(1)}(t)+x_o^{(2)}(t)+x_o^{(3)}(t)=\int_{-\infty}^{\infty} h^{(1)}(\tau)x_i(t-\tau)d\tau++\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^{(2)}(\tau_1,\tau_2)\cdot$$
$$\cdot x_i(t-\tau_1)x_i(t-\tau_2)d\tau_1 d\tau_2+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^{(3)}(\tau_1,\tau_2,\tau_3)x_i(t-\tau_1)x_i(t-\tau_2)x_i(t-\tau_3)d\tau_1 d\tau_2 d\tau_3 \; . \tag{11}$$

The terms $x_o^{(1)}(t)$, $x_o^{(2)}(t)$, and $x_o^{(3)}(t)$ in (11), which are the respective components of $x_o(t)$, we call the amplifier partial responses of the first, second, and third order, respectively. This order is with respect to a variable representing the signal (here, in (11), x_i). Furthermore, by $h^{(1)}(t)$, $h^{(2)}(t_1,t_2)$, and $h^{(2)}(t_1,t_2,t_3)$ we denote, respectively, the first (linear), second, and third order, nonlinear impulse responses of the amplifier without feedback.

The equivalent of (6) will be now an infinite Volterra series as

$$\begin{aligned}
x_o(t)&=H_f\left(x_s(t)\right)=x_o^{(1)}(t)+x_o^{(2)}(t)+x_o^{(3)}(t)+x_o^{(4)}(t)+x_o^{(5)}(t)+\ldots=\\
&=\int_{-\infty}^{\infty} h_f^{(1)}(\tau)x_s(t-\tau)d\tau+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_f^{(2)}(\tau_1,\tau_2)x_s(t-\tau_1)x_s(t-\tau_2)d\tau_1 d\tau_2+\\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_f^{(3)}(\tau_1,\tau_2,\tau_3)\left(\prod_{k=1}^{3} x_s(t-\tau_k)d\tau_k\right)+\\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_f^{(4)}(\tau_1,\tau_2,\tau_3,\tau_4)\left(\prod_{k=1}^{4} x_s(t-\tau_k)d\tau_k\right)+\\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_f^{(5)}(\tau_1,\tau_2,\tau_3,\tau_4,\tau_5)\left(\prod_{k=1}^{5} x_s(t-\tau_k)d\tau_k\right)+\ldots\;.
\end{aligned} \tag{12}$$

And the equivalent of (7) will be the linear convolution having the form

$$x_f(t) = K(x_o(t)) = \int_{-\infty}^{\infty} k(\tau) x_o(t-\tau) d\tau \ . \tag{13}$$

In (12) and (13), the letter "*f*" in subscripts stands, as before, for "feedback". Furthermore, $h_f^{(1)}(t)$, $h_f^{(2)}(t_1,t_2)$, $h_f^{(3)}(t_1,t_2,t_3)$, $h_f^{(4)}(t_1,t_2,t_3,t_4)$, $h_f^{(5)}(t_1,t_2,t_3,t_4,t_5)$, and so on are, respectively, the first (linear), second, third, fourth, fifth order, and next orders, nonlinear impulse responses of the feedback amplifier. Moreover, the function $k(t)$ in (13) is the (linear) impulse response of the now frequency dependent block *K* in Fig. 1(b).

Substituting x_o and x_f given by (12) and (13), respectively, into (3), and using also in (3) the formula (11) for the operator *H*, we get an equivalent of (8) for this far more complicated case with the amplifier and feedback in Fig. 1(b) being frequency dependent. In the next step, we proceed with the resulting equation similarly as before. That is we equate to each other expressions of the same order (degree) occurring on its both sides. More precisely, we equate expressions in which the number of appearances of the variable x_s is the same. As a result, we get the equivalent of equations (9a-e).

Further procedure leading to getting the equivalents of (10a-e) is quite involved because of occurrence of a huge number of multidimensional integrals in equivalents of (9a-e). In such situations, the method usually used, which is dated back to the appearance of the work [2], is carrying out the transformation of equations to the frequency domain by the use of the multidimensional Fourier (or Laplace) transforms. In consequence, one obtains the nonlinear transfer functions of a mildly nonlinear circuit. In the case of calculation of nonlinear transfer functions of the fourth or fifth order, and of next orders, for a circuit of Fig. 1(b), it is needed to carry out a huge number of algebraic manipulations to get the final results. The higher the order, the larger is the number of manipulations.

We will not do this here (details of these calculations will be eventually published later). The important for us here is one result that follows from these calculations. That is, similarly as in the case of (10a-e), we get (generally) nonzero nonlinear transfer functions of the fourth, fifth, and of higher orders for the circuit in Fig. 1(b). And finally, it follows from the latter that the nonlinear impulse responses related with them (through the inverse transforms) are nonzero functions, too.

So the conclusions, which can be drawn from the above, are similar to those presented a while before for the structure of Fig. 1(b) considered to be purely frequency independent. First, the Volterra series describing a mildly nonlinear amplifier after adding to it a linear feedback (with memory) has a different number of components from that describing the same amplifier but without feedback. This number changed here from three in (11) to infinity in (12).

Second, which is related to the previous conclusion, we observe that the linear feedback (with memory) has an influence upon the order of nonlinearity that is represented in the amplifier's closed-loop description by a Volterra series through the highest partial response in it. Furthermore, observe that if the amplifier's open-loop description as (11) would be in form of a Volterra series containing the infinite number of components, we did not see the above effect. Probably of this reason, this effect was not reported in the literature up to now.

3. FEEDBACK AND POWER SERIES-LIKE MODEL OF WEAKLY NONLINEAR AMPLIFIER WITH MEMORY

In [3], Palumbo and Pennisi have developed a power series-like model for weakly nonlinear amplifiers with memory in configuration (without feedback) of Fig. 1(a). It has the following form

$$x_o = a_1(j\omega)x_i + a_2(j\omega)x_i^2 + a_3(j\omega)x_i^3 \tag{14}$$

in the notation of [3]. In (14), $\omega = 2\pi f$ means the angular frequency with f denoting the usual frequency variable. Moreover, $j = \sqrt{-1}$ and the frequency-dependent coefficients $a_1(j\omega)$, $a_2(j\omega)$, and $a_3(j\omega)$ are the coefficients in this (truncated) power series-like description of a mildly nonlinear amplifier with memory.

One thing is very important of which we must be aware when applying the model given by (14). This is the fact that it is valid only for input signals of the form

$$x_i(t) = A_i \exp(j\omega t) \tag{15}$$

where A_i is a real number and means the amplitude of this complex harmonic signal. In other words, (14) does not represent a Volterra series valid for any input signal, but only for such that having the form $x_i(t) = A_i \exp(j\omega t)$.

The above is evident from the derivation of (14) which can be done by substituting $x_i(t)$ given by (15) into the finite Volterra series consisting of only first three components (such as that in (11)) and performing next the needed operations in it, as shown in the following

$$\begin{aligned}
x_o(t) &= \int_{-\infty}^{\infty} h^{(1)}(\tau) A_i \exp(j\omega(t-\tau))\, d\tau + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^{(2)}(\tau_1,\tau_2) A_i^2 \exp(j\omega(t-\tau_1)) \cdot \\
&\cdot \exp(j\omega(t-\tau_2))\, d\tau_1 d\tau_2 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^{(3)}(\tau_1,\tau_2,\tau_3) A_i^3 \exp(j\omega(t-\tau_1)) \exp(j\omega(t-\tau_2)) \cdot \\
&\cdot \exp(j\omega(t-\tau_3))\, d\tau_1 d\tau_2 d\tau_3 = A_i \exp(j\omega t) \int_{-\infty}^{\infty} h^{(1)}(\tau) \exp(-j\omega\tau)\, d\tau + A_i^2 \exp(j2\omega t) \cdot \\
&\cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^{(2)}(\tau_1,\tau_2) \exp(-j\omega(\tau_1+\tau_2))\, d\tau_1 d\tau_2 + A_i^3 \exp(j3\omega t) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^{(3)}(\tau_1,\tau_2,\tau_3) \cdot \\
&\cdot \exp(-j3\omega(\tau_1+\tau_2+\tau_3))\, d\tau_1 d\tau_2 d\tau_3 \, .
\end{aligned} \tag{16}$$

Applying in (16) the notion of n-dimensional Fourier transforms [5], here for n = 1, 2 or 3, and choosing the same frequency point f at each dimension, we arrive finally at

$$x_o(t) = H^{(1)}(f)\,A_i \exp(j2\pi ft) + \\ + H^{(2)}(f,f)\,A_i^2 \exp(j2\pi 2ft) + \\ + H^{(2)}(f,f,f)\,A_i^3 \exp(j2\pi 3ft) \tag{17}$$

where $H^{(1)}(f)$, $H^{(2)}(f,f)$, and $H^{(3)}(f,f,f)$ denote the Fourier transforms (mentioned above) of $h^{(1)}(t)$, $h^{(2)}(t_1,t_2)$, and $h^{(3)}(t_1,t_2,t_3)$, respectively, at the same frequency point f chosen at each dimension. At this point, note also that $H^{(1)}(f)$, $H^{(2)}(f,f)$, and $H^{(3)}(f,f,f)$ are called the circuit nonlinear transfer functions of the first (linear one), of the second, and of the third order, accordingly. Furthermore, $f = \omega/(2\pi)$ in (17).

Observe that (17) is identical with (14) when we identify $a_1(j\omega)$ with $H^{(1)}(f)$, $a_2(j\omega)$ with $H^{(2)}(f,f)$, and $a_3(j\omega)$ with $H^{(3)}(f,f,f)$.

In this paper, we call shortly the signal of the form given by (15) a harmonic at frequency ω (or equivalently f). From (17), we see that an amplifier in the structure of Fig. 1(a), having the description given by (11), and driven by a single harmonic at frequency f has three (and only three) harmonics, at frequencies f, $2f$, and $3f$, at its output. Observe that this number is identical with the number of components in the finite Volterra series (11). Moreover, the number three staying at $3f$ (at the highest frequency of these harmonics) is equal to the order of nonlinearity incorporated in the amplifier description (identical with the highest order among partial responses in the Volterra series).

Note now that we have a quite different situation when the linear feedback (with memory) as in Fig. 1(b) is applied to the amplifier characterized above. Then, we must use for the whole amplifier, as shown in the previous section, an infinite Volterra series of the form (12). And it is clear from the derivation underlying (17) that, in this case, after substituting into (12) the input signal denoted now as

$$x_s(t) = A_s \exp(j\omega t) \tag{18}$$

where A_s is a real number and means the amplitude of this harmonic signal, and performing afterwards the needed operations in it, we get finally an equivalent of (17) as

$$x_o(t) = H_f^{(1)}(f)\,A_s \exp(j2\pi ft) + H_f^{(2)}(f,f)\,A_s^2 \exp(j2\pi 2ft) + \\ + H_f^{(3)}(f,f,f)\,A_s^3 \exp(j2\pi 3ft) + H_f^{(4)}(f,f,f,f)\,A_s^4 \exp(j2\pi 4ft) + \\ + H_f^{(5)}(f,f,f,f,f)\,A_s^5 \exp(j2\pi 5ft) + \dots \tag{19}$$

In (19), similarly as before, $H_f^{(1)}(f)$, $H_f^{(2)}(f,f)$, $H_f^{(3)}(f,f,f)$, $H_f^{(4)}(f,f,f,f)$, $H_f^{(5)}(f,f,f,f,f)$, and so on are, respectively, the Fourier transforms of $h_f^{(1)}(t)$, $h_f^{(2)}(t_1,t_2)$, $h_f^{(3)}(t_1,t_2,t_3)$, $h_f^{(4)}(t_1,t_2,t_3,t_4)$, $h_f^{(5)}(t_1,t_2,t_3,t_4,t_5)$, and of the next nonlinear impulse responses. They are calculated here at the same frequency point f chosen at each dimension. Furthermore, these functions are called the nonlinear transfer functions of the corresponding orders of the feedback amplifier.

Observe from (19) that, opposite to the previous case, we have at the output of the same amplifier, which is put now into the feedback structure of Fig. 1(b), an infinite number of harmonics, having frequencies *f*, 2*f*, 3*f*, 4*f*, 5*f*, and so on. Note further that this infinite number of harmonics corresponds with the infinite number of components in the Volterra series (12). This (infinite) number is equal to the order of nonlinearity incorporated in the description (12) (i.e. identical with the highest order of the partial response in the Volterra series which is here infinite).

4. RESCUE FOR PALUMBO AND PENNISI'S MEANS OF MODELING WEAKLY NONLINEAR FEEDBACK AMPLIFIERS

It follows evidently from the results of the previous two sections that the power series-like model for weakly nonlinear amplifiers expressed by (14) cannot be applied to study the feedback structure of Fig. 1(b). More precisely:

1. It is not a proper model for the whole circuit with feedback as represented by Fig. 1(b). Then, as we have shown, expression (19) must be used instead of (17) (which is identical with (14)).
2. It is not also a proper model for modeling the input-output behavior of the weakly nonlinear amplifier *H* in Fig. 1(b) because the input signal at its input is now a sum of an infinite number of harmonics (not a single harmonic of the form $x_i(t) = A_i \exp(j2\pi ft)$).

Nevertheless, it has been used in the above context in an approach developed by Palumbo and Pennisi in [3] for calculation of harmonic distortion in weakly nonlinear feedback amplifiers. See equations: (15) in [3] and (10) in [6] – with regard to point 1, and equations: (17) in [3] and (A3) in [6] – with regard to point 2, for example.

We derive here a model that enables to obtain correctly the results presented by Palumbo and Pennisi in [3]. It is an associated model incorporating some simplifications with regard to the original formulation. So, for that reason, it can lead to results that are not necessarily identical with those one gets with the use of the original model. And, concluding, the approach from [3] for feedback amplifier should be perceived similarly.

To achieve our goal mentioned above, we proceed now in the following way: First, we postulate the form as shown in Fig. 2 for an associated model we look for. Then, we check whether it really describes correctly the results presented by Palumbo and Pennisi in [3].

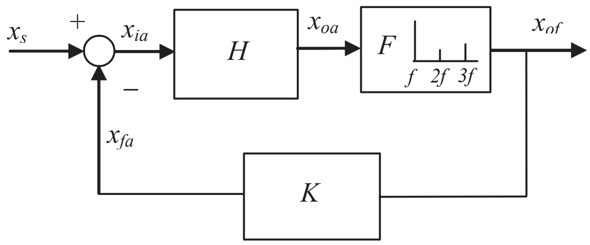

Fig. 2. Associated model of weakly nonlinear amplifier in closed-loop configuration containing linear feedback

Rys. 2. Stowarzyszony model nieliniowego wzmacniacza w pętli z liniowym sprzężeniem zwrotnym

In Fig. 2, an ideal filter *F* plays a role of a filter that allows only the harmonics at frequencies *f*, 2*f*, and 3*f* at the output of *H* to pass to the output of the whole amplifier. It fully rejects all the other harmonics. In consequence, under the assumption of the input signal to the whole amplifier x_s being given by (18), we get such a situation in the loop of Fig. 2 that the signals x_{ia}, x_{of}, and x_{fa} contain exclusively harmonics of the frequencies *f*, 2*f*, and 3*f*. Other harmonics occur in the circuit of Fig. 2 only at the output of the amplifier *H*. That is they are components of the signal x_{oa} (besides the harmonics at frequencies *f*, 2*f*, and 3*f*).

It follows from the above and the results of previous sections that the signals x_{ia}, x_{oa}, and x_{fa} in the associated model of Fig. 2 are not identical with the corresponding signals x_i, x_o, and x_f of the original model of Fig. 1(b). For that reason, we use an additional letter "*a*" in their subscript notation. Of course, this additional letter should be also put at the subscript of the symbol of filter *F* output signal occurring in Fig. 2. However, that more consistent notation x_{ofa} has been abbreviated in this case to x_{of}, to avoid too long subscripts, and in such a form is used in the paper.

Observe now that the signal x_{ia}, as containing three harmonics, can be expressed in the following way

$$x_{ia}(t) = A_{1i}\exp(j2\pi ft) + A_{2i}\exp(j2\pi 2ft) + A_{3i}\exp(j2\pi 3ft) \tag{20}$$

where A_{1i}, A_{2i}, and A_{3i} are real numbers and mean the amplitudes of the corresponding harmonics at frequencies *f*, 2*f*, and 3*f*, respectively.

As we already know the form given by (14) for a power series-like model cannot be used to model the amplifier *H* in Fig. 1(b) or in Fig. 2. Remember that it is so because (14) is valid only for input signals being single harmonics. Extension of applicability of the power series-like model is however possible. We do this by finding each time its specific form, for a particular input signal. So, for *H* in the original model of Fig. 1(b), a counterpart of (14) must be derived assuming the input signal in form of an infinite sum of harmonics at frequencies *f*, 2*f*, 3*f*, 4*f*, 5*f*, and so on. Further, in the case of *H* in the associated model of Fig. 2, the situation is simpler in comparison with the latter. Then, we have to consider in the calculations the input signal consisting of only three harmonics (at frequencies *f*, 2*f*, and 3*f*) as in (20).

Now, we will derive a power series-like formula regarding the latter case. To this end, we use a specific form of the Volterra series that enables to express its components (partial responses) through the circuit nonlinear transfer functions – for more details see [5]). So applying it to the series (11), we get the following expressions for its components:

$$x_o^{(1)}(t) = \int_{-\infty}^{\infty} H^{(1)}(f_1) X_i(f_1) \exp(j2\pi f_1 t) df_1 \tag{21a}$$

$$x_o^{(2)}(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H^{(2)}(f_1, f_2) X_i(f_1) X_i(f_2) \exp(j2\pi(f_1 t + f_2 t)) df_1 df_2 \tag{21b}$$

$$\begin{aligned} x_o^{(3)}(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} & H^{(3)}(f_1, f_2, f_3) X_i(f_1) X_i(f_2) \cdot \\ & \cdot X_i(f_3) \exp(j2\pi(f_1 t + f_2 t + f_3 t)) df_1 df_2 df_3 \end{aligned} \tag{21c}$$

where $X_i(f_x)$, $x = 1, 2, 3$, means the Fourier transform of the input signal, and the sets of frequency variables: $\{f_1\}$, $\{f_1, f_2\}$, and $\{f_1, f_2, f_3\}$ occurring in (21a), (21b), and (21c), respectively, form the corresponding one-, two-, and three-dimensional frequency spaces.

The Fourier transform of the input signal to the amplifier H in Fig. 2 is given by

$$X_{ia}(f_x) = A_{1i}\delta(f_x - f) + A_{2i}\delta(f_x - 2f) + A_{3i}\delta(f_x - 3f) \tag{22}$$

where δ means the Dirac impulse and f_x is the current frequency in the Fourier transform.

Specializing the general expressions (21a), (21b), and (21c) to the case of H in the structure of Fig. 2 means introduction in them $X_{ia}(f_x)$ instead of $X_i(f_x)$, and adding the letter "a" at the subscripts by $x_o^{(1)}(t)$, $x_o^{(2)}(t)$, and $x_o^{(3)}(t)$. Carrying out afterwards the standard algebraic manipulations, exploiting the sifting property of the Dirac impulse and using the symmetric nonlinear transfer functions in them, we get finally

$$\begin{aligned} x_{oa}^{(1)}(t) &= H^{(1)}(f) A_{1i} \exp(j2\pi ft) + H^{(1)}(2f) \cdot \\ &\cdot A_{2i} \exp(j2\pi 2ft) + H^{(1)}(3f) A_{3i} \exp(j2\pi 3ft) \end{aligned} \tag{23a}$$

$$\begin{aligned} x_{oa}^{(2)}(t) &= H^{(2)}(f, f) A_{1i}^2 \exp(j2\pi 2ft) + 2H^{(2)}(f, 2f) \cdot \\ &\cdot A_{1i} A_{2i} \exp(j2\pi 3ft) + \Big[2H^{(2)}(f, 3f) A_{1i} A_{3i} \exp(j2\pi 4ft) + \\ &+ H^{(2)}(2f, 2f) A_{2i}^2 \exp(j2\pi 4ft) \Big] + 2H^{(2)}(2f, 3f) A_{2i} A_{3i} \exp(j2\pi 5ft) + \\ &+ H^{(2)}(3f, 3f) A_{3i}^2 \exp(j2\pi 6ft) \end{aligned} \tag{23b}$$

$$\begin{aligned}
x_{oa}^{(3)}(t) &= H^{(3)}(f,f,f)\,A_{1i}^{3}\exp(j2\pi 3ft) + 3H^{(3)}(f,f,2f)\,A_{1i}^{2}A_{2i}\exp(j2\pi 4ft) + \\
&+ \Big[3H^{(3)}(f,f,3f)\,A_{1i}^{2}A_{3i}\exp(j2\pi 5ft) + 3H^{(3)}(f,2f,2f)\cdot \\
&\cdot A_{1i}A_{2i}^{2}\exp(j2\pi 5ft)\Big] + \Big[6H^{(3)}(f,2f,3f)\,A_{1i}A_{2i}A_{3i}\exp(j2\pi 6ft) + \\
&+ H^{(3)}(2f,2f,2f)\,A_{2i}^{3}\exp(j2\pi 6ft)\Big] + \Big[3H^{(3)}(f,3f,3f)\,A_{1i}A_{3i}^{2}\exp(j2\pi 7ft) + \\
&+ 3H^{(3)}(2f,2f,3f)\,A_{2i}^{2}A_{3i}\exp(j2\pi 7ft)\Big] + 3H^{(3)}(2f,3f,3f)\cdot \\
&\cdot A_{2i}A_{3i}^{2}\exp(j2\pi 8ft) + H^{(3)}(3f,3f,3f)\,A_{3i}^{3}\exp(j2\pi 9ft)\;.
\end{aligned} \tag{23c}$$

Denoting the components of $x_{ia}(t)$ in (20) as $x_{1i}(t) = A_{1i}\exp(j2\pi ft)$, $x_{2i}(t) = A_{2i}\exp(j2\pi 2ft)$, and $x_{3i}(t) = A_{3i}\exp(j2\pi 3ft)$, respectively, applying them afterwards in (23a), (23b), and (23c), and summing the partial responses, we obtain the following

$$\begin{aligned}
x_{oa} &= a_1(j\omega)x_{1i} + a_2(j\omega)x_{2i} + a_3(j\omega)x_{3i} + a_{11}(j\omega)x_{1i}^2 + 2a_{12}(j\omega)x_{1i}x_{2i} \\
&+ \Big[2a_{13}(j\omega)x_{1i}x_{3i} + a_{22}(j\omega)x_{2i}^2\Big] + 2a_{23}(j\omega)x_{2i}x_{3i} + a_{33}(j\omega)x_{3i}^2 + \\
&+ a_{111}(j\omega)x_{1i}^3 + 3a_{112}(j\omega)x_{1i}^2x_{2i} + \ldots\ldots + a_{333}(j\omega)x_{1i}^3
\end{aligned} \tag{24}$$

where the expressions describing the coefficients in the resulting multivariate polynomial can be easily determined by comparison of (24) with (23a), (23b) or (23c). So we have $a_1(j\omega) = H^{(1)}(f)$, $a_2(j\omega) = H^{(1)}(2f)$, $a_3(j\omega) = H^{(1)}(3f)$, $a_{11}(j\omega) = H^{(2)}(f,f)$, $a_{12}(j\omega) = 2H^{(2)}(f,2f)$, $a_{13}(j\omega) = 2H^{(2)}(f,3f)$, ... , $a_{33}(j\omega) = H^{(2)}(3f,3f)$, $a_{111}(j\omega) = H^{(3)}(f,f,f)$, $a_{112}(j\omega) = 3H^{(3)}(f,f,2f)$, ... , $a_{333}(j\omega) = H^{(3)}(3f,3f,3f)$. (By the way, note that the principle of indexing the coefficients of the multivariate polynomial in (24) is a little bit different than that used in (14).)

Expression (24) is a direct counterpart of (14) for correct modeling the input-output behavior of the amplifier H in the associated model of Fig. 2. We see however that it is much more complicated than (14), and thereby very heavy to use. So we resign from this form in further derivations. In what follows, we prefer to use the other form of the power series-like model that is summarized in equations (23a-c).

Note now that for the structure of Fig. 2 we can write the following two equations (in the operator form)

$$x_{fa} = x_s - x_{ia} \tag{25a}$$

and

$$x_{fa} = KFHx_{ia} \tag{25b}$$

where F stands for the mapping which is carried out by the ideal filter F of Fig. 2, according to the rule described beneath this figure.

Using (25a) in (25b), we get

$$x_s - x_{ia} = KFHx_{ia} = KFx_{oa} \ . \tag{26}$$

Further, knowing that the signal $x_{oa}(t) = (Hx_{ia})(t)$ at the output of the amplifier *H* in Fig. 2 is equal to the sum $x_{oa}^{(1)}(t) + x_{oa}^{(2)}(t) + x_{oa}^{(3)}(t)$ with its components given by (23a-c), and applying to it the filtering rule of the filter *F*, we arrive for the signal $(Fx_{oa})(t)$ at

$$\begin{aligned}(Fx_{oa})(t) &= H^{(1)}(f)A_{1i}\exp(j2\pi ft) + H^{(1)}(2f)A_{2i}\exp(j2\pi 2ft) + \\ &+ H^{(1)}(3f)A_{3i}\exp(j2\pi 3ft) + H^{(2)}(f,f)A_{1i}^2\exp(j2\pi 2ft) + 2H^{(2)}(f,2f)\cdot \\ &\cdot A_{1i}A_{2i}\exp(j2\pi 3ft) + H^{(3)}(f,f,f)A_{1i}^3\exp(j2\pi 3ft) \ .\end{aligned} \tag{27}$$

The linear feedback block *K* (as a linear circuit with memory) transfers the signal (27) to its output according to the following formula

$$\begin{aligned}(KFx_{oa})(t) &= K(f)H^{(1)}(f)A_{1i}\exp(j2\pi ft) + K(2f)H^{(1)}(2f) \\ &\cdot A_{2i}\exp(j2\pi 2ft) + K(3f)H^{(1)}(3f)A_{3i}\exp(j2\pi 3ft) + K(2f)\cdot \\ &\cdot H^{(2)}(f,f)A_{1i}^2\exp(j2\pi 2ft) + 2K(3f)H^{(2)}(f,2f)A_{1i}A_{2i}\cdot \\ &\cdot \exp(j2\pi 3ft) + K(3f)H^{(3)}(f,f,f)A_{1i}^3\exp(j2\pi 3ft) \ .\end{aligned} \tag{28}$$

$K(f_x)$ on the right-hand side of (28) means the transfer function of the linear feedback block *K*, calculated at the corresponding frequencies $f_x = f,\ 2f$ or $3f$.

Introducing (18), (20), and (28) into (26) gives

$$\begin{aligned}&A_s\exp(j2\pi ft) - A_{1i}\exp(j2\pi ft) - A_{2i}\exp(j2\pi 2ft) - A_{3i}\exp(j2\pi 3ft) = \\ &= K(f)H^{(1)}(f)A_{1i}\exp(j2\pi ft) + K(2f)H^{(1)}(2f)A_{2i}\exp(j2\pi 2ft) + K(3f)\cdot \\ &\cdot H^{(1)}(3f)A_{3i}\exp(j2\pi 3ft) + K(2f)H^{(2)}(f,f)\cdot A_{1i}^2\exp(j2\pi 2ft) + \\ &+ 2K(3f)H^{(2)}(f,2f)A_{1i}A_{2i}\exp(j2\pi 3ft) + K(3f)H^{(3)}(f,f,f)A_{1i}^3\exp(j2\pi 3ft) \ .\end{aligned} \tag{29}$$

The next step is to equate to each other the expressions staying by the exponents of the same frequency on both sides of (29). As a result, we get

$$A_s - A_{1i} = K(f)H^{(1)}(f)A_{1i} \tag{30a}$$

$$-A_{2i} = K(2f)H^{(1)}(2f)A_{2i} + K(2f)H^{(2)}(f,f)A_{1i}^2 \tag{30b}$$

$$\begin{aligned}-A_{3i} &= K(3f)H^{(1)}(3f)A_{3i} + 2K(3f)\cdot \\ &\cdot H^{(2)}(f,2f)A_{1i}A_{2i} + K(3f)H^{(3)}(f,f,f)A_{1i}^3 \ .\end{aligned} \tag{30c}$$

Now, solving (30a) for A_{1i}, afterwards solving (30b) for A_{2i} and using A_{1i} from the previous step, and finally solving (30c) for A_{3i} and using A_{1i} and A_{2i} from the previous two steps, we obtain successively

$$A_{1i} = \frac{A_s}{1+K(f)H^{(1)}(f)} \tag{31a}$$

$$A_{2i} = \frac{-K(2f)H^{(2)}(f,f)A_s^2}{\left[1+K(2f)H^{(1)}(2f)\right]\left[1+K(f)H^{(1)}(f)\right]^2} \tag{31b}$$

$$\begin{aligned} A_{3i} &= \frac{K(3f)A_s^3}{\left[1+K(3f)H^{(1)}(3f)\right]\left[1+K(f)H^{(1)}(f)\right]^2}\cdot \\ &\cdot\left\{-H^{(3)}(f,f,f)+\frac{2H^{(2)}(f,f)K(2f)H^{(2)}(f,2f)}{\left[1+K(2f)H^{(1)}(2f)\right]}\right\}. \end{aligned} \tag{31c}$$

Having the expressions determining the amplitudes A_{1i}, A_{2i}, and A_{3i} as the functions of the amplitude A_s, we can eliminate them from (27). This leads to

$$\begin{aligned} x_{of}(t) &= (Fx_{oa})(t) = a_{1f}(j2\pi f)A_s\exp(j2\pi ft)+ \\ &+a_{2f}(j2\pi 2f)A_s^2\exp(j2\pi 2ft)+a_{3f}(j2\pi 3f)A_s^3\exp(j2\pi 3ft) \end{aligned} \tag{32}$$

with the functions $a_{1f}(j2\pi f)$, $a_{2f}(j2\pi f)$, and $a_{3f}(j2\pi f)$ given by

$$a_{1f}(j2\pi f) = \frac{H^{(1)}(f)}{1+K(f)H^{(1)}(f)} \tag{33a}$$

$$a_{2f}(j2\pi f) = \frac{H^{(2)}(f,f)}{\left[1+K(f)H^{(1)}(f)\right]^2\left[1+K(2f)H^{(1)}(2f)\right]} \tag{33b}$$

$$\begin{aligned} a_{3f}(j2\pi f) &= \frac{H^{(3)}(f,f,f)}{\left[1+K(f)H^{(1)}(f)\right]^3\left[1+K(3f)H^{(1)}(3f)\right]}\cdot \\ &\cdot\left\{1-\frac{2H^{(2)}(f,f)H^{(2)}(f,2f)}{H^{(1)}(2f)H^{(3)}(f,f,f)}\cdot\frac{K(2f)H^{(1)}(2f)}{\left[1+K(2f)H^{(1)}(2f)\right]}\right\}. \end{aligned} \tag{33c}$$

Note now that introducing in (33a-c) the following notation for the nonlinear transfer functions: $H^{(1)}(f)=a_1(j2\pi f)$, $H^{(2)}(f,f)=a_2(j2\pi f)$, and $H^{(3)}(f,f,f)=$ $=a_3(j2\pi f)$ we get the expressions identical with those given in [3] for the so-called *closed-loop nonlinear coefficients*. Therefore, we conclude that (32) is identical with the corresponding power series-like description for the whole amplifier that has been used in [3] (see equation (15) in [3]). And this ends the proof of the statement that the proper model for the results presented by Palumbo and Pennisi [3] for the feedback amplifier is the associated model in Fig. 2.

(On the occasion, we put the reader's attention to the fact that a small correction is needed in equation (19) in [3] because, generally, $H^{(2)}(f,f)\neq H^{(2)}(f,2f)$.)

In summary, having in mind the results of this and of the previous section, we can say that the means of calculation of harmonic distortion in weakly nonlinear feedback amplifies developed in [3] relies upon the approximation of the original model in Fig. 1(b) by the associated model in Fig.2. It is assumed that writing

$$x_{of}(t)=(Fx_{oa})(t)\cong x_o(t) \tag{34}$$

makes sense. The validity of (34) has been checked in [3] by carrying out calculations for some CMOS amplifiers and their comparison with the results of exact simulations.

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O WPŁYWIE SPRZĘŻENIA ZWROTNEGO NA SKŁADOWE HARMONICZNE W UKŁADACH ANALOGOWYCH Z MAŁYMI NIELINIOWOŚCIAMI

Streszczenie

W pracy pokazano, że zastosowanie liniowego sprzężenia zwrotnego w układzie wzmacniacza analogowego pracującego w zakresie tzw. małych nieliniowości powoduje zwiększenie rzędu nieliniowości wykazywanej przez ten wzmacniacz w porównaniu ze wzmacniaczem bez sprzężenia. Przeanalizowano wpływ powyższego zjawiska na składowe harmoniczne wyższych rzędów, powstające przy pobudzeniu wzmacniacza pojedynczym sygnałem harmonicznym. Ponadto wyprowadzono model stowarzyszony wzmacniacza ze sprzężeniem zwrotnym. Pokazano, że wykorzystując ten model, i tylko wtedy, można zinterpretować w sposób prawidłowy metodę obliczeń zniekształceń harmonicznych we wzmacniaczach analogowych pracujących w zakresie tzw. małych nieliniowości, która to została zaproponowana przez Palumbo i Pennisi w jednym z ich ostatnio opublikowanych artykułów.

Słowa kluczowe: analiza zniekształceń harmonicznych, układy i systemy analogowe pracujące w zakresie tzw. małych nieliniowości, sprzężenie zwrotne, wpływ sprzężenia zwrotnego na składowe harmoniczne wyższych rzędów, szereg Volterry