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Powers of Asymmetrically Supplied Loads in Terms of the CPC Power Theory

Leszek S. CZARNECKI
Louisiana State University, USA

***Summary*: Power related phenomena in three-phase, three-wire circuits with linear time-invariant (LTI) loads supplied with asymmetrical sinusoidal voltage are investigated in this paper. The study is based on the concept of the Currents' Physical Component (CPC) power theory. It is shown that the supply current of LTI loads with sinusoidal asymmetrical voltage, as in the case of symmetrical voltage, is composed of only three physical components, the active, reactive and unbalanced currents. Consequently, loads in such conditions can be characterized by the active, reactive and unbalanced powers. The equivalent and unbalanced admittances of three-phase loads at symmetrical supply voltage are constant, independent on the supply voltage, parameters. The paper shows that these parameters at asymmetrical supply depend on the voltage asymmetry.**

1. INTRODUCTION

The development of the power theory of circuits with nonsinusoidal voltages and currents was focused in the XX$^{\rm th}$ century almost entirely on single-phase circuits with linear, time-invariant (LTI) loads and on three-phase circuits with balanced and unbalanced loads supplied with a symmetrical voltage. Now, when the Currents' Physical Components (CPC) based power theory has explained the electric power related phenomena in such circuits, is the time to extend the power theory to circuits with asymmetrical supply voltage. Still however the area of study has to be confined. It is confined in this paper to three-phase, three-wire circuits with LTI loads, meaning, four-wire circuits or circuits with harmonic generating loads are not the subject of interest in this paper.

The voltage asymmetry in distribution systems is usually a by-product of three-phase load imbalance which results in line currents asymmetry. It occurs mainly at a junction of single-phase and three-phase distribution systems as a result of different loading of individual phases by single-phase loads. In particular, high power loads that by their nature, like electric trains, have to be single-phase loads, contribute to distribution voltage asymmetry. Also, impedance asymmetry of distribution line contributes to the asymmetry of the distribution voltage. Voltage asymmetry occurs also when a three-phase load is supplied from two single-phase transformers, not connected in full Δ configuration.

It is well known that the voltage asymmetry in distribution systems harmfully affects customer loads, in particular, induction motors. The energy delivered to induction motors by the negative sequence voltage is not converted to mechanical energy, but to motor heat. This asymmetry also affects ac/dc converters. In presence of such asymmetry, non-characteristic harmonics occur in the converter supply current. Therefore, the acceptable level of the distribution voltage asymmetry is specified by various national standards and confined usually to a few percent.

Power theory of such circuits should provide accurate results independently on the level of the voltage asymmetry. Therefore, in spite of the fact that the voltage asymmetry in distribution systems is usually not higher than a few percents, the study in this paper on such systems shall be carried out without any limitations on the level of this asymmetry. Just the opposite, it will be assumed in the numerical illustration presented in Section 4 that it is much higher than the voltage asymmetry in real systems. This shall enable emphasizing various effects of the voltage asymmetry on the power related phenomena in the circuits. At the level of voltage asymmetry that can occur in real circuits these effects could not be visible clearly enough.

The study in this paper is based on the concept of symmetrical components, developed by Fortescue [1] in 1918, and on the CPC based power theory of three-phase unbalanced circuits [2, 3]. The study will be focused on power related phenomena in three-phase, three-wire circuits with asymmetrical but sinusoidal supply voltage. Comprehension of power properties of circuits under sinusoidal conditions is a necessity for any successful investigation of these properties when voltages and currents are nonsinusoidal. Moreover, it will be assumed that the supply voltage is independent of the load current.

2. MAIN SYMBOLS

Three phase quantities such as voltages, $u_{\rm R}$, $u_{\rm S}$ and $u_{\rm T}$ or line currents $i_{\rm R}$, $i_{\rm S}$ and $i_{\rm T}$, denoted by $x_{\rm R}$, $x_{\rm S}$ and $x_{\rm T}$, can be arranged into three-phase vectors:

$$\boldsymbol{x} \triangleq \begin{bmatrix} x_{\rm R} \\ x_{\rm S} \\ x_{\rm T} \end{bmatrix} = \sqrt{2}\,{\rm Re}\begin{bmatrix} \boldsymbol{X}_{\rm R} \\ \boldsymbol{X}_{\rm S} \\ \boldsymbol{X}_{\rm T} \end{bmatrix} e^{j\omega_1 t} \triangleq \sqrt{2}\,{\rm Re}\,\boldsymbol{X}\, e^{j\omega_1 t} \qquad (1)$$

The scalar product of three-phase quantities $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ of the same period T is defined as:

$$(\boldsymbol{x},\boldsymbol{y}) \triangleq \frac{1}{T}\int_0^T \boldsymbol{x}^{\rm T}(t)\,\boldsymbol{y}(t)\,dt = \frac{1}{T}\int_0^T (x_{\rm R}\,y_{\rm R} + x_{\rm S}\,y_{\rm S} + x_{\rm T}\,y_{\rm T})\,dt \quad (2)$$

where superscript T denotes a transposed matrix $\boldsymbol{x}(t)$. This scalar product can be calculated in the frequency-domain as:

$$(\boldsymbol{x},\boldsymbol{y}) = \mathrm{Re}\{\boldsymbol{X}^{\mathrm{T}}\boldsymbol{Y}^{*}\} \qquad (3)$$

where the asterisk denotes a conjugate number.

The rms value of a three-phase vector is defined as:

$$\|\boldsymbol{x}\| \triangleq \sqrt{(\boldsymbol{x},\boldsymbol{x})} = \sqrt{\frac{1}{T}\int_0^T \boldsymbol{x}^{\mathrm{T}}(t)\cdot\boldsymbol{x}(t)\,dt} = \sqrt{\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}^{*}} \qquad (4)$$

The last formula could be rearranged to the form:

$$\|\boldsymbol{x}\| = \sqrt{\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}^{*}} = \sqrt{X_{\mathrm{R}}^{2}+X_{\mathrm{S}}^{2}+X_{\mathrm{T}}^{2}} = \sqrt{\|x_{\mathrm{R}}\|^{2}+\|x_{\mathrm{S}}\|^{2}+\|x_{\mathrm{T}}\|^{2}} \qquad (5)$$

thus, the rms value of a three-phase quantity is equal to the root of the sum of squares of the rms value of phase quantities.

Three-phase sinusoidal quantities can be expressed as a sum of the zero-sequence, $\boldsymbol{x}^{\mathrm{z}}$, positive sequence, $\boldsymbol{x}^{\mathrm{p}}$, and the negative sequence, $\boldsymbol{x}^{\mathrm{n}}$, symmetrical components, namely:

$$\boldsymbol{x} = \boldsymbol{x}^{\mathrm{z}} + \boldsymbol{x}^{\mathrm{p}} + \boldsymbol{x}^{\mathrm{n}} \qquad (6)$$

The zero sequence component of voltages and currents in three-wire systems does not contribute to energy flow and consequently, will not occur in the following studies. The positive sequence symmetrical components will be presented in this paper in the form:

$$\boldsymbol{x}^{\mathrm{p}} \triangleq \begin{bmatrix} x_{\mathrm{R}}^{\mathrm{p}} \\ x_{\mathrm{S}}^{\mathrm{p}} \\ x_{\mathrm{T}}^{\mathrm{p}} \end{bmatrix} = \sqrt{2}\,\mathrm{Re}\begin{bmatrix} \boldsymbol{X}_{\mathrm{R}}^{\mathrm{p}} \\ \alpha^{*}\boldsymbol{X}_{\mathrm{R}}^{\mathrm{p}} \\ \alpha\,\boldsymbol{X}_{\mathrm{R}}^{\mathrm{p}} \end{bmatrix} e^{j\omega_1 t} \triangleq \sqrt{2}\,\mathrm{Re}\{\boldsymbol{X}^{\mathrm{p}}\, e^{j\omega_1 t}\} \qquad (7)$$

where:

$$\boldsymbol{X}^{\mathrm{p}} \triangleq \begin{bmatrix} \boldsymbol{X}_{\mathrm{R}}^{\mathrm{p}} \\ \alpha^{*}\boldsymbol{X}_{\mathrm{R}}^{\mathrm{p}} \\ \alpha\,\boldsymbol{X}_{\mathrm{R}}^{\mathrm{p}} \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha^{*} \\ \alpha \end{bmatrix} \boldsymbol{X}_{\mathrm{R}}^{\mathrm{p}} \triangleq \boldsymbol{T}^{\mathrm{p}}\boldsymbol{X}^{\mathrm{p}}, \qquad \alpha \triangleq 1e^{j120^{0}} \qquad (8)$$

Similarly, the negative sequence symmetrical component will be expressed in the form:

$$\boldsymbol{x}^{\mathrm{n}} \triangleq \begin{bmatrix} x_{\mathrm{R}}^{\mathrm{n}} \\ x_{\mathrm{S}}^{\mathrm{n}} \\ x_{\mathrm{T}}^{\mathrm{n}} \end{bmatrix} = \sqrt{2}\,\mathrm{Re}\begin{bmatrix} \boldsymbol{X}_{\mathrm{R}}^{\mathrm{n}} \\ \alpha\,\boldsymbol{X}_{\mathrm{R}}^{\mathrm{n}} \\ \alpha^{*}\boldsymbol{X}_{\mathrm{R}}^{\mathrm{n}} \end{bmatrix} e^{j\omega_1 t} \triangleq \sqrt{2}\,\mathrm{Re}\{\boldsymbol{X}^{\mathrm{n}}\, e^{j\omega_1 t}\} \qquad (9)$$

where:

$$\boldsymbol{X}^{\mathrm{n}} \triangleq \begin{bmatrix} \boldsymbol{X}_{\mathrm{R}}^{\mathrm{n}} \\ \alpha\,\boldsymbol{X}_{\mathrm{R}}^{\mathrm{n}} \\ \alpha^{*}\boldsymbol{X}_{\mathrm{R}}^{\mathrm{n}} \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha \\ \alpha^{*} \end{bmatrix} \boldsymbol{X}^{\mathrm{n}} \triangleq \boldsymbol{T}^{\mathrm{n}}\boldsymbol{X}^{\mathrm{n}} \qquad (10)$$

The complex rms (CRMS) values of the symmetrical components, $\boldsymbol{X}^{\mathrm{z}}$, $\boldsymbol{X}^{\mathrm{p}}$ and $\boldsymbol{X}^{\mathrm{n}}$ are defined as:

$$\begin{bmatrix} \boldsymbol{X}^{\mathrm{z}} \\ \boldsymbol{X}^{\mathrm{p}} \\ \boldsymbol{X}^{\mathrm{n}} \end{bmatrix} \triangleq \begin{bmatrix} \boldsymbol{X}_{\mathrm{R}}^{\mathrm{z}} \\ \boldsymbol{X}_{\mathrm{R}}^{\mathrm{p}} \\ \boldsymbol{X}_{\mathrm{R}}^{\mathrm{n}} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^{*} \\ 1 & \alpha^{*} & \alpha \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_{\mathrm{R}} \\ \boldsymbol{X}_{\mathrm{S}} \\ \boldsymbol{X}_{\mathrm{T}} \end{bmatrix} \triangleq \boldsymbol{S}\,\boldsymbol{X} \qquad (11)$$

The scalar product of symmetrical components of the opposite sequence, $\boldsymbol{x}^{\mathrm{p}}$ and $\boldsymbol{x}^{\mathrm{n}}$, is equal to zero, since:

$$(\boldsymbol{x}^{\mathrm{p}}, \boldsymbol{x}^{\mathrm{n}}) = \mathrm{Re}\{\boldsymbol{X}^{\mathrm{pT}}\boldsymbol{X}^{\mathrm{n}*}\} = \mathrm{Re}\{\boldsymbol{T}^{\mathrm{pT}}X^{\mathrm{p}}\,\boldsymbol{T}^{\mathrm{n}*}X^{\mathrm{n}}\} = \mathrm{Re}\{[1, \alpha^{*}, \alpha]\begin{bmatrix} 1 \\ \alpha^{*} \\ \alpha \end{bmatrix} \boldsymbol{X}^{\mathrm{p}}\boldsymbol{X}^{\mathrm{n}}\} = \mathrm{Re}\{(1+\alpha+\alpha^{*})\,\boldsymbol{X}^{\mathrm{p}}\boldsymbol{X}^{\mathrm{n}}\} = 0 \qquad (12)$$

thus such components are mutually orthogonal and consequently, the rms value of the sum of the positive and negative sequence components has the relationship:

$$\|\boldsymbol{x}^{\mathrm{p}}+\boldsymbol{x}^{\mathrm{n}}\|^{2} = \|\boldsymbol{x}^{\mathrm{p}}\|^{2} + \|\boldsymbol{x}^{\mathrm{n}}\|^{2} \qquad (13)$$

3. CIRCUITS WITH BALANCED LOADS

Although the distribution voltage, $\boldsymbol{e}$, can contain a zero sequence component, $\boldsymbol{e}^{\mathrm{z}}$, this component cannot contribute to energy flow in three-wire systems. It means that the line-to-artificial zero voltage, $\boldsymbol{u}$, shown in Figure 1, but not the line-to-ground voltage, $\boldsymbol{e}$, should be considered as the load supply voltage. Thus the apparent power of the load should be defined as:

$$S = \|\boldsymbol{u}\|\,\|\boldsymbol{i}\| \qquad (14)$$

but not as $S \triangleq \|\boldsymbol{e}\|\,\|\boldsymbol{i}\|$. At such definition the power factor, $\lambda = P/S$, would be lower than its true value.

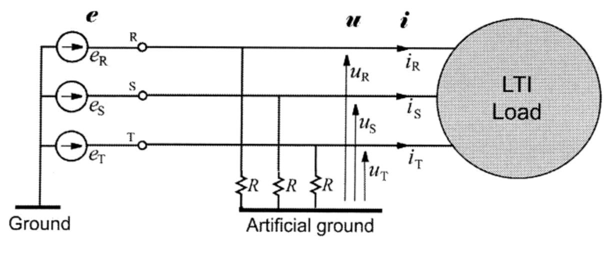

Fig. 1. Three-phase circuit

For the circuit shown in Figure 1 it holds:

$$\boldsymbol{u}^{\mathrm{z}} \equiv 0, \qquad \boldsymbol{u}^{\mathrm{p}} \equiv \boldsymbol{e}^{\mathrm{p}}, \qquad \boldsymbol{u}^{\mathrm{n}} \equiv \boldsymbol{e}^{\mathrm{n}} \qquad (15)$$

Let us suppose that the LTI load in the circuit shown in Figure 1 is a balanced load. Usually it is a motor. In the USA power system approximately 2/3 of electric energy produced by power plants is used by motors.

Let us denote the motor phase admittance for the positive and negative sequence voltage by:

$$Y^p \triangleq G^p + jB^p \qquad Y^n \triangleq G^n + jB^n \tag{16}$$

respectively, then, the load current is:

$$\boldsymbol{i} = \boldsymbol{i}^p + \boldsymbol{i}^n = \sqrt{2}\,\mathrm{Re}\{\boldsymbol{T}^p \boldsymbol{I}^p e^{j\omega_1 t}\} + \sqrt{2}\,\mathrm{Re}\{\boldsymbol{T}^n \boldsymbol{I}^n e^{j\omega_1 t}\} \tag{17}$$

with:

$$\boldsymbol{I}^p = \boldsymbol{Y}^p \boldsymbol{U}^p, \qquad \boldsymbol{I}^n = \boldsymbol{Y}^n \boldsymbol{U}^n \tag{18}$$

The load active power can be expressed, due to orthogonality of the opposite sequence components, in the form:

$$P = (\boldsymbol{u}^p + \boldsymbol{u}^n, \boldsymbol{i}^p + \boldsymbol{i}^n) = (\boldsymbol{u}^p, \boldsymbol{i}^p) + (\boldsymbol{u}^n, \boldsymbol{i}^n) = P^p + P^n \tag{19}$$

where:

$$P^p \triangleq G^p \,||\boldsymbol{u}^p||^2, \qquad P^n \triangleq G^n \,||\boldsymbol{u}^n||^2 \tag{20}$$

This result is rather trivial. It only shows that the load active power is a sum of active powers associated with the positive and negative sequence components of the load voltages and currents. One should notice only that, in the case of motors, the active power P^n is not a useful power. The energy transferred by the negative sequence voltage and current component is entirely dissipated in the motor, thus it only increases its temperature. Therefore, it would not be fair to charge a customer for this energy. Just the opposite, the energy provider should reimburse the customer for negative effects of the supply voltage asymmetry.

Illustration 1. To have an idea of the level of these two powers, let us observe that a common, 220 V, 100 kW, induction motor at the rated speed has the phase admittance for the positive and negative sequence voltage equal approximately to:

$$\boldsymbol{Y}^p = 0.6 - j0.4 = 0.7\,e^{j33^0}\ \mathrm{S} \tag{21}$$

$$\boldsymbol{Y}^n = 0.4 - j1.3 = 1.4\,e^{j73^0}\ \mathrm{S} \tag{22}$$

Taking into account that $G^n < G^p$ and that the active power is proportional to the square of the supply voltage rms value, thus at the level of the voltage asymmetry that is allowed by standards, the power P^n is only a minute part of the motor active power. Observe however, that most of the power P^p is converted to mechanical power on the motor shaft, while only a few percents of this power contribute to the motor heating. Consequently, the share of the positive and negative sequence voltages in the motor heating can be comparable.

Observe that the motor admittance for the negative sequence voltage is higher than this admittance for the positive sequence. Consequently, the motor current asymmetry has to be higher than the supply voltage asymmetry. Thus, due to the voltage drop on the distribution system internal impedance, induction motors contribute to an increase of the voltage asymmetry in such a system.

4. CIRCUITS WITH UNBALANCED LTI LOADS

Let us investigate power phenomena in the circuit with an unbalanced load shown in Figure 2. This circuit could be considered as an equivalent circuit of the supply of a residential area, with mainly linear, single-phase loads.

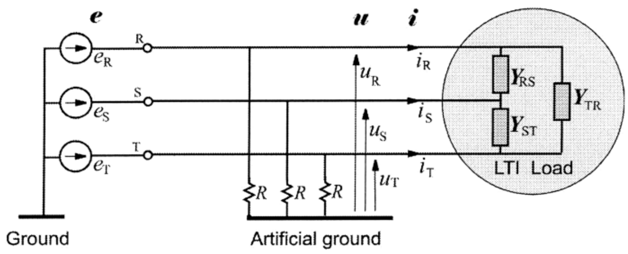

Fig. 2. Three-phase circuit with unbalanced LTI load

The vector of the line currents can be presented in the form:

$$\boldsymbol{i} = \sqrt{2}\,\mathrm{Re}\{\boldsymbol{I}\, e^{j\omega t}\} \tag{23}$$

where the CRMS values of the line currents, i.e., the elements of the vector $\boldsymbol{I}$ can be expressed as follows. The CRMS value of the line R current in such a circuit is equal to:

$$\boldsymbol{I}_R = \boldsymbol{Y}_{RS}(\boldsymbol{U}_R - \boldsymbol{U}_S) - \boldsymbol{Y}_{TR}(\boldsymbol{U}_T - \boldsymbol{U}_R) \tag{24}$$

and can be rearranged to the form:

$$\boldsymbol{I}_R = \boldsymbol{Y}_e \boldsymbol{U}_R - (\boldsymbol{Y}_{ST}\boldsymbol{U}_R + \boldsymbol{Y}_{TR}\boldsymbol{U}_T + \boldsymbol{Y}_{RS}\boldsymbol{U}_S) \tag{25}$$

where:

$$\boldsymbol{Y}_e = G_e + jB_e = \boldsymbol{Y}_{ST} + \boldsymbol{Y}_{TR} + \boldsymbol{Y}_{RS}, \tag{26}$$

is the equivalent admittance of the load, introduced to power theory of three-phase systems in [2]. Similarly, the CRMS value of the remaining line currents can be rearranged to the form:

$$\boldsymbol{I}_S = \boldsymbol{Y}_e \boldsymbol{U}_S - (\boldsymbol{Y}_{ST}\boldsymbol{U}_T + \boldsymbol{Y}_{TR}\boldsymbol{U}_S + \boldsymbol{Y}_{RS}\boldsymbol{U}_R) \tag{27}$$

$$\boldsymbol{I}_T = \boldsymbol{Y}_e \boldsymbol{U}_T - (\boldsymbol{Y}_{ST}\boldsymbol{U}_S + \boldsymbol{Y}_{TR}\boldsymbol{U}_R + \boldsymbol{Y}_{RS}\boldsymbol{U}_T) \tag{28}$$

If the following vectors of the voltage CRMS values are denoted as:

$$\begin{bmatrix}\boldsymbol{U}_R\\ \boldsymbol{U}_S\\ \boldsymbol{U}_T\end{bmatrix} \triangleq \boldsymbol{U}, \quad \begin{bmatrix}\boldsymbol{U}_R\\ \boldsymbol{U}_T\\ \boldsymbol{U}_S\end{bmatrix} \triangleq {}^{R}\boldsymbol{U}, \quad \begin{bmatrix}\boldsymbol{U}_T\\ \boldsymbol{U}_S\\ \boldsymbol{U}_R\end{bmatrix} \triangleq {}^{T}\boldsymbol{U}, \quad \begin{bmatrix}\boldsymbol{U}_S\\ \boldsymbol{U}_R\\ \boldsymbol{U}_T\end{bmatrix} \triangleq {}^{S}\boldsymbol{U} \tag{29}$$

then the vector of line current CRMS values can be expressed in the form:

$$\boldsymbol{I} \triangleq \begin{bmatrix}\boldsymbol{I}_R\\ \boldsymbol{I}_S\\ \boldsymbol{I}_T\end{bmatrix} = \boldsymbol{Y}_e \boldsymbol{U} - (\boldsymbol{Y}_{ST}{}^{R}\boldsymbol{U} + \boldsymbol{Y}_{TR}{}^{T}\boldsymbol{U} + \boldsymbol{Y}_{RS}{}^{S}\boldsymbol{U}) \tag{30}$$

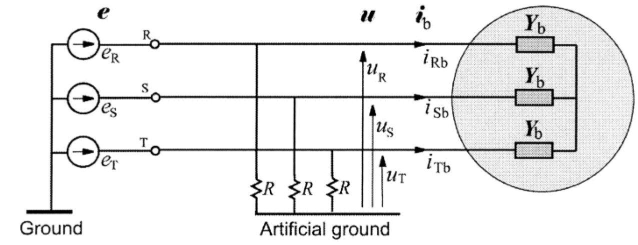

Fig. 3. Circuit with the equivalent balanced load

The upper front index in matrices ${}^{\mathrm{R}}\boldsymbol{U}$, ${}^{\mathrm{S}}\boldsymbol{U}$ and ${}^{\mathrm{T}}\boldsymbol{U}$, indicates the first coordinate of the vector, it means the CRMS value $\boldsymbol{U}_{\mathrm{R}}$, $\boldsymbol{U}_{\mathrm{T}}$ and $\boldsymbol{U}_{\mathrm{S}}$, respectively.

With respect to the active and reactive powers, P and Q, the original load can be equivalent to a balanced load shown in Figure 3. The vector of CRMS values of the line current is equal to:

$$\boldsymbol{I} = \boldsymbol{Y}_{\mathrm{b}}\boldsymbol{U} \tag{31}$$

The load in Figure 3 is equivalent to the load in Figure 2 with respect to the active and reactive power, if the complex power at its terminal is equal to:

$$\boldsymbol{S} \triangleq \boldsymbol{U}^{\mathrm{T}}\boldsymbol{I}_{\mathrm{b}}^{*} = \boldsymbol{U}^{\mathrm{T}}(\boldsymbol{Y}_{\mathrm{b}}\boldsymbol{U})^{*} = \boldsymbol{Y}_{\mathrm{b}}^{*}\,||\boldsymbol{u}||^{2} = P + jQ \tag{32}$$

hence:

$$\boldsymbol{Y}_{\mathrm{b}} \triangleq \frac{P - jQ}{||\boldsymbol{u}||^{2}} = \frac{\boldsymbol{S}^{*}}{||\boldsymbol{u}||^{2}} \tag{33}$$

Since $\boldsymbol{Y}_{\mathrm{b}}$ is admittance of a balanced load, equivalent with respect to the active and reactive powers, it will be referred to as the *equivalent balanced admittance*.

Let us express the equivalent balanced admittance $\boldsymbol{Y}_{\mathrm{b}}$ in terms of the load line-to-line admittances, $\boldsymbol{Y}_{\mathrm{RS}}$, $\boldsymbol{Y}_{\mathrm{ST}}$ and $\boldsymbol{Y}_{\mathrm{TR}}$. Since the active and reactive powers satisfy the balance principle, thus, the conjugate value of the complex apparent power of the load can be expressed as:

$$\boldsymbol{S}^{*} \triangleq P - jQ = \boldsymbol{S}_{\mathrm{RS}}^{*} + \boldsymbol{S}_{\mathrm{ST}}^{*} + \boldsymbol{S}_{\mathrm{TR}}^{*} \tag{34}$$

The conjugate apparent power of the admittance $\boldsymbol{Y}_{\mathrm{RS}}$ can be expressed as:

$$\begin{aligned}\boldsymbol{S}_{\mathrm{RS}}^{*} &= \boldsymbol{U}_{\mathrm{RS}}^{*}\boldsymbol{Y}_{\mathrm{RS}}\boldsymbol{U}_{\mathrm{RS}} = (\boldsymbol{U}_{\mathrm{R}}^{*} - \boldsymbol{U}_{\mathrm{S}}^{*})\,\boldsymbol{Y}_{\mathrm{RS}}(\boldsymbol{U}_{\mathrm{R}} - \boldsymbol{U}_{\mathrm{S}}) = \\ &= \boldsymbol{Y}_{\mathrm{RS}}(\boldsymbol{U}_{\mathrm{R}}^{2} + \boldsymbol{U}_{\mathrm{S}}^{2} - 2\,\mathrm{Re}\{\boldsymbol{U}_{\mathrm{R}}\boldsymbol{U}_{\mathrm{S}}^{*}\})\end{aligned} \tag{35}$$

Since:

$$U_{\mathrm{T}}^{2} = (-\boldsymbol{U}_{\mathrm{R}} - \boldsymbol{U}_{\mathrm{S}})(-\boldsymbol{U}_{\mathrm{R}} - \boldsymbol{U}_{\mathrm{S}})^{*} = U_{\mathrm{R}}^{2} + U_{\mathrm{S}}^{2} + 2\mathrm{Re}\{\boldsymbol{U}_{\mathrm{R}}\boldsymbol{U}_{\mathrm{S}}^{*}\}, \tag{36}$$

thus:

$$\boldsymbol{S}_{\mathrm{RS}}^{*} = \boldsymbol{Y}_{\mathrm{RS}}(2U_{\mathrm{R}}^{2} + 2U_{\mathrm{S}}^{2} - U_{\mathrm{T}}^{2}) = \boldsymbol{Y}_{\mathrm{RS}}(2||\boldsymbol{u}||^{2} - 3U_{\mathrm{T}}^{2}) \tag{37}$$

Similarly:

$$\boldsymbol{S}_{\mathrm{ST}}^{*} = \boldsymbol{U}_{\mathrm{ST}}^{*}\boldsymbol{Y}_{\mathrm{ST}}\boldsymbol{U}_{\mathrm{ST}} = \boldsymbol{Y}_{\mathrm{ST}}(2||\boldsymbol{u}||^{2} - 3U_{\mathrm{R}}^{2}) \tag{38}$$

$$\boldsymbol{S}_{\mathrm{TR}}^{*} = \boldsymbol{U}_{\mathrm{TR}}^{*}\boldsymbol{Y}_{\mathrm{TR}}\boldsymbol{U}_{\mathrm{TR}} = \boldsymbol{Y}_{\mathrm{TR}}(2||\boldsymbol{u}||^{2} - 3U_{\mathrm{S}}^{2}) \tag{39}$$

Thus the equivalent balanced admittance of the load can be expressed in the form:

$$\begin{aligned}\boldsymbol{Y}_{\mathrm{b}} &\triangleq \frac{\boldsymbol{S}^{*}}{||\boldsymbol{u}||^{2}} = \frac{\boldsymbol{S}_{\mathrm{RS}}^{*} + \boldsymbol{S}_{\mathrm{ST}}^{*} + \boldsymbol{S}_{\mathrm{TR}}^{*}}{||\boldsymbol{u}||^{2}} = \\ &= 2\boldsymbol{Y}_{\mathrm{e}} - \frac{3}{||\boldsymbol{u}||^{2}}(\boldsymbol{Y}_{\mathrm{ST}}U_{\mathrm{R}}^{2} + \boldsymbol{Y}_{\mathrm{TR}}U_{\mathrm{S}}^{2} + \boldsymbol{Y}_{\mathrm{RS}}U_{\mathrm{T}}^{2}) \triangleq \boldsymbol{Y}_{\mathrm{e}} + \boldsymbol{Y}_{\mathrm{d}}\end{aligned} \tag{40}$$

This result shows that the equivalent balanced admittance $\boldsymbol{Y}_{\mathrm{b}}$ differs from the equivalent admittance $\boldsymbol{Y}_{\mathrm{e}}$ of the load by term:

$$\boldsymbol{Y}_{\mathrm{d}} \triangleq \boldsymbol{Y}_{\mathrm{e}} - \frac{3}{||\boldsymbol{u}||^{2}}(\boldsymbol{Y}_{\mathrm{ST}}U_{\mathrm{R}}^{2} + \boldsymbol{Y}_{\mathrm{TR}}U_{\mathrm{S}}^{2} + \boldsymbol{Y}_{\mathrm{RS}}U_{\mathrm{T}}^{2}) = G_{\mathrm{d}} + jB_{\mathrm{d}} \tag{41}$$

Admittance $\boldsymbol{Y}_{\mathrm{d}}$ can have a non-zero value only if, at the same time, the load is unbalanced and the supply voltage is asymmetrical. When the load is balanced, i.e.:

$$\boldsymbol{Y}_{\mathrm{RS}} = \boldsymbol{Y}_{\mathrm{TR}} = \boldsymbol{Y}_{\mathrm{RS}} = \boldsymbol{Y}_{\mathrm{e}}/3, \text{ then } \boldsymbol{Y}_{\mathrm{d}} = 0 \tag{42}$$

independently on the supply voltage asymmetry. Similarly, when the supply voltage is symmet-rical and consequently, the rms values of phase voltages are mutually equal:

$$U_{\mathrm{R}} = U_{\mathrm{S}} = U_{\mathrm{T}}, \quad \text{then } \boldsymbol{Y}_{\mathrm{d}} = 0, \tag{43}$$

independently on the load imbalance. Therefore, the admittance $\boldsymbol{Y}_{\mathrm{d}}$ will be referred to as the *asymmetry dependent admittance* of three-phase loads.

Let us assume that the supply current $\boldsymbol{i}$ of the original load contains component $\boldsymbol{i}_{\mathrm{a}}$ proportional to the supply voltage $\boldsymbol{u}$ and, at the same time, a minimum current of a balanced load of the same active power P as the original load. It is a component of the $\boldsymbol{i}_{\mathrm{b}}$ current that is in-phase with the supply voltage, i.e.:

$$\boldsymbol{i}_{\mathrm{a}} \triangleq \sqrt{2}\,\mathrm{Re}\{\boldsymbol{I}_{\mathrm{b}}\,\mathrm{e}^{j\omega t}\} = \sqrt{2}\,\mathrm{Re}\{\boldsymbol{Y}_{\mathrm{b}}\boldsymbol{U}\,\mathrm{e}^{j\omega t}\} \tag{44}$$

If:

$$\boldsymbol{Y}_{\mathrm{b}} \triangleq G_{\mathrm{b}} + jB_{\mathrm{b}} \tag{45}$$

then:

$$\boldsymbol{i}_{\mathrm{a}} = \sqrt{2}\,G_{\mathrm{b}}\,\mathrm{Re}\{\boldsymbol{U}\,\mathrm{e}^{j\omega t}\} = G_{\mathrm{b}}\,\boldsymbol{u} \tag{46}$$

The current $\boldsymbol{i}_{\mathrm{a}}$ is, of course, the active current of the original load, since the scalar product of the supply voltage and this current is equal to:

$$(\boldsymbol{u}, \boldsymbol{i}_{\mathrm{a}}) = G_{\mathrm{b}}\,||\boldsymbol{u}||^{2} = \frac{P}{||\boldsymbol{u}||^{2}}\,||\boldsymbol{u}||^{2} = P \tag{47}$$

The remaining component of the unbalanced current:

$$\boldsymbol{i}_{\rm r} = \sqrt{2}\, B_{\rm b}\, \mathrm{Re}\{j\boldsymbol{U}\mathrm{e}^{j\omega t}\} = B_{\rm b}\, \boldsymbol{u}(t+T/4) = -B_{\rm b}\, \boldsymbol{u}(t-T/4) \tag{48}$$

meaning, the current component shifted by $T/4$ with respect to the supply voltage, is the reactive current.

The active and reactive currents in circuits with asymmetrical supply voltage, although formally identical with these currents in circuits with loads supplied symmetrically, have components dependent on the voltage asymmetry and the load imbalance, since:

$$\boldsymbol{i}_{\rm a} = G_{\rm b}\, \boldsymbol{u} = (G_{\rm e} + G_{\rm d})\, \boldsymbol{u} \tag{49}$$

$$\boldsymbol{i}_{\rm r} = -(B_{\rm e} + B_{\rm d})\, \boldsymbol{u}(t-T/4) \tag{50}$$

Components proportional to the conductance $G_{\rm d}$ and susceptance $B_{\rm d}$ disappear when the supply voltage is symmetrical or the load is balanced.

It can be assumed that supply current $\boldsymbol{i}$ of the original load contains the balanced current $\boldsymbol{i}_{\rm b}$ and a residual current:

$$\boldsymbol{i} - \boldsymbol{i}_{\rm b} = \sqrt{2}\, \mathrm{Re}\{(\boldsymbol{I} - \boldsymbol{I}_{\rm b})\, e^{j\omega t}\} \tag{51}$$

This residual component does not exist of course, when the load is balanced thus, it can be associated with the load imbalance and consequently, this component can be called an *unbalanced current*, $\boldsymbol{i}_{\rm u}$. Thus:

$$\boldsymbol{i}_{\rm u} \triangleq \boldsymbol{i} - \boldsymbol{i}_{\rm b} = \sqrt{2}\, \mathrm{Re}\{\boldsymbol{I}_{\rm u}\, e^{j\omega t}\} \tag{52}$$

where:

$$\boldsymbol{I}_{\rm u} \triangleq \boldsymbol{I} - \boldsymbol{I}_{\rm b} = (Y_{\rm e} - Y_{\rm b})\boldsymbol{U} - (Y_{\rm ST}{}^{\rm R}\boldsymbol{U} + Y_{\rm TR}{}^{\rm T}\boldsymbol{U} + Y_{\rm RS}{}^{\rm S}\boldsymbol{U}) = -(\boldsymbol{Y}_{\rm ST}{}^{\rm R}\boldsymbol{U} + \boldsymbol{Y}_{\rm TR}{}^{\rm T}\boldsymbol{U} + \boldsymbol{Y}_{\rm RS}{}^{\rm S}\boldsymbol{U}) - Y_{\rm d}\boldsymbol{U} \tag{53}$$

The vectors ${}^{\rm R}\boldsymbol{U}$, ${}^{\rm S}\boldsymbol{U}$ and ${}^{\rm T}\boldsymbol{U}$ can be expressed in terms of vectors of symmetrical components, $\boldsymbol{U}^{\rm p\#}$ and $\boldsymbol{U}^{\rm n\#}$, where index # denotes matrices with switched CRMS values $\boldsymbol{U}_{\rm S}$ and $\boldsymbol{U}_{\rm T}$, as follows:

$${}^{\rm R}\boldsymbol{U} \triangleq \begin{bmatrix} \boldsymbol{U}_{\rm R} \\ \boldsymbol{U}_{\rm T} \\ \boldsymbol{U}_{\rm S} \end{bmatrix} = \begin{bmatrix} \boldsymbol{U}^{\rm p} + \boldsymbol{U}^{\rm n} \\ \alpha\boldsymbol{U}^{\rm p} + \alpha^*\boldsymbol{U}^{\rm n} \\ \alpha^*\boldsymbol{U}^{\rm p} + \alpha\boldsymbol{U}^{\rm n} \end{bmatrix} = \boldsymbol{U}^{\rm p\#} + \boldsymbol{U}^{\rm n\#} \tag{54}$$

$${}^{\rm T}\boldsymbol{U} \triangleq \begin{bmatrix} \boldsymbol{U}_{\rm T} \\ \boldsymbol{U}_{\rm S} \\ \boldsymbol{U}_{\rm R} \end{bmatrix} = \begin{bmatrix} \alpha\boldsymbol{U}^{\rm p} + \alpha^*\boldsymbol{U}^{\rm n} \\ \alpha^*\boldsymbol{U}^{\rm p} + \alpha\boldsymbol{U}^{\rm n} \\ \boldsymbol{U}^{\rm p} + \boldsymbol{U}^{\rm n} \end{bmatrix} = \alpha\, \boldsymbol{U}^{\rm p\#} + \alpha^*\, \boldsymbol{U}^{\rm n\#} \tag{55}$$

$${}^{\rm S}\boldsymbol{U} \triangleq \begin{bmatrix} \boldsymbol{U}_{\rm S} \\ \boldsymbol{U}_{\rm R} \\ \boldsymbol{U}_{\rm T} \end{bmatrix} = \begin{bmatrix} \alpha^*\boldsymbol{U}^{\rm p} + \alpha\boldsymbol{U}^{\rm n} \\ \boldsymbol{U}^{\rm p} + \boldsymbol{U}^{\rm n} \\ \alpha\boldsymbol{U}^{\rm p} + \alpha^*\boldsymbol{U}^{\rm n} \end{bmatrix} = \alpha^*\, \boldsymbol{U}^{\rm p\#} + \alpha\, \boldsymbol{U}^{\rm n\#} \tag{56}$$

thus, the vector of the unbalanced current CRMS values can be rearranged to the form:

$$\boldsymbol{I}_{\rm u} \triangleq -(\boldsymbol{Y}_{\rm ST}{}^{\rm R}\boldsymbol{U} + \boldsymbol{Y}_{\rm TR}{}^{\rm T}\boldsymbol{U} + \boldsymbol{Y}_{\rm RS}{}^{\rm S}\boldsymbol{U}) - Y_{\rm d}\boldsymbol{U} = \boldsymbol{A}^{\rm p}\boldsymbol{U}^{\rm p\#} + \boldsymbol{A}^{\rm n}\boldsymbol{U}^{\rm n\#} - Y_{\rm d}(\boldsymbol{U}^{\rm p} + \boldsymbol{U}^{\rm n}) \tag{57}$$

where:

$$\boldsymbol{A}^{\rm p} \triangleq -(\boldsymbol{Y}_{\rm ST} + \alpha\, \boldsymbol{Y}_{\rm TR} + \alpha^*\, \boldsymbol{Y}_{\rm RS}) \tag{58}$$

$$\boldsymbol{A}^{\rm n} \triangleq -(\boldsymbol{Y}_{\rm ST} + \alpha^*\, \boldsymbol{Y}_{\rm TR} + \alpha\, \boldsymbol{Y}_{\rm RS}) \tag{59}$$

are *unbalanced admittances* of the load for the positive and negative sequence voltages. Observe, that the unbalanced current contains both positive and negative sequence components, since the vector:

$$\boldsymbol{A}^{\rm p}\boldsymbol{U}^{\rm p\#} - Y_{\rm d}\boldsymbol{U}^{\rm n} \triangleq \boldsymbol{I}_{\rm u}^{\rm n} \tag{60}$$

is a vector of CRMS values of the supply currents of the negative sequence, while the vector:

$$\boldsymbol{A}^{\rm n}\boldsymbol{U}^{\rm n\#} - Y_{\rm d}\boldsymbol{U}^{\rm p} \triangleq \boldsymbol{I}_{\rm u}^{\rm p} \tag{61}$$

is a vector of CRMS values of the positive sequence currents. Thus, the unbalanced current can be expressed in the form:

$$\boldsymbol{i}_{\rm u} = \sqrt{2}\mathrm{Re}\{\boldsymbol{I}_{\rm u}\, e^{j\omega t}\} = \sqrt{2}\mathrm{Re}\{(\boldsymbol{I}_{\rm u}^{\rm p} + \boldsymbol{I}_{\rm u}^{\rm n})\, e^{j\omega t}\} = \boldsymbol{i}_{\rm u}^{\rm p} + \boldsymbol{i}_{\rm u}^{\rm n} \tag{62}$$

Taking into account that the balanced current $\boldsymbol{i}_{\rm b}$ is composed of the active and reactive currents, the load current of the original unbalanced load supplied from a source of asymmetrical voltage can be decomposed into three components:

$$\boldsymbol{i} = \boldsymbol{i}_{\rm a} + \boldsymbol{i}_{\rm r} + \boldsymbol{i}_{\rm u} \tag{63}$$

This decomposition is formally identical with the decomposition of the supply current of unbalanced loads supplied with a symmetrical voltage. Also, definitions of the current components are identical. However, mathematical forms of these currents are much more complex now. Also, it is not evident that these components are mutually orthogonal. Let us verify orthogonality of these currents.

The active and reactive currents are orthogonal, because according to their definitions, vectors of these two currents are shifted mutually by $\pi/2$. Thus:

$$(\boldsymbol{i}_{\rm a}, \boldsymbol{i}_{\rm r}) = \mathrm{Re}\{\boldsymbol{I}_{\rm a}^{\rm T}\, \boldsymbol{I}_{\rm r}^{*}\} = \mathrm{Re}\{G_{\rm b}\, \boldsymbol{U}^{\rm T}(\, jB_{\rm b}\, \boldsymbol{U})^{*}\} = 0 \tag{64}$$

Unfortunately, the proof of orthogonality of the unbalanced current with the active and reactive ones is more complex. To prove orthogonality of the balanced and unbalanced currents, let us calculate the scalar product:

$$(\boldsymbol{i}_b, \boldsymbol{i}_u) = \mathrm{Re}\{\boldsymbol{I}_b^T \boldsymbol{I}_u^*\} = \mathrm{Re}\{\boldsymbol{I}_b^{T*} \boldsymbol{I}_u\} =$$

$$= -\mathrm{Re}\{\boldsymbol{Y}_b^* \boldsymbol{U}^{T*} (\boldsymbol{Y}_{ST}{}^R\boldsymbol{U} + \boldsymbol{Y}_{TR}{}^T\boldsymbol{U} + \boldsymbol{Y}_{RS}{}^S\boldsymbol{U} + \boldsymbol{Y}_d \boldsymbol{U})\} =$$

$$= -\mathrm{Re}\{\boldsymbol{Y}_b^* [Y_d ||\boldsymbol{u}||^2 + \boldsymbol{Y}_{ST}(U_R^2 + 2\mathrm{Re}\boldsymbol{U}_S^* \boldsymbol{U}_T) + \tag{65}$$

$$+ Y_{TR}(U_S^2 + 2\mathrm{Re}\boldsymbol{U}_R^* \boldsymbol{U}_T) + Y_{RS}(U_T^2 + 2\mathrm{Re}\boldsymbol{U}_R^* \boldsymbol{U}_S)]\} =$$

$$= -\mathrm{Re}\{\boldsymbol{Y}_b^* ||\boldsymbol{u}||^2 (\boldsymbol{Y}_d - \boldsymbol{Y}_d)\} = 0$$

thus, the balanced and unbalanced currents are mutually orthogonal. Because the balanced current $\boldsymbol{i}_b$ is composed of the active and reactive currents, thus if the reactive current is assumed to be zero, then we can conclude that the active and unbalanced currents are orthogonal. When it is assumed that the active current is zero, then we can conclude that the reactive and unbalanced currents are orthogonal. Thus:

$$(\boldsymbol{i}_a, \boldsymbol{i}_r) = (\boldsymbol{i}_a, \boldsymbol{i}_u) = (\boldsymbol{i}_r, \boldsymbol{i}_u) = 0 \tag{66}$$

and consequently, the rms value of the active, reactive and unbalanced currents fulfills the relationship:

$$||\boldsymbol{i}||^2 = ||\boldsymbol{i}_a||^2 + ||\boldsymbol{i}_r||^2 + ||\boldsymbol{i}_u||^2 \tag{67}$$

In this equation:

$$||\boldsymbol{i}_a|| = \sqrt{3}\, I_{Ra} = G_b ||\boldsymbol{u}|| \tag{68}$$

$$||\boldsymbol{i}_r|| = \sqrt{3}\, I_{Rr} = |B_b|\, ||\boldsymbol{u}|| \tag{69}$$

$$||\boldsymbol{i}_u|| = \sqrt{3}\, I_{Ru} = \sqrt{3}\sqrt{(I_{Ru}^p)^2 + (I_{Ru}^n)^2} \tag{70}$$

Multiplying this equation by the square of the supply voltage vector rms value, $||\boldsymbol{u}||^2$, we obtain the power equation of unbalanced loads supplied with asymmetrical sinusoidal voltage:

$$S^2 = P^2 + Q^2 + D^2 \tag{71}$$

with:

$$P = ||\boldsymbol{u}||\, ||\boldsymbol{i}_a|| = G_b ||\boldsymbol{u}||^2 \tag{72}$$

$$Q = ||\boldsymbol{u}||\, ||\boldsymbol{i}_r|| = -B_b ||\boldsymbol{u}||^2 \tag{73}$$

$$D = ||\boldsymbol{u}||\, ||\boldsymbol{i}_u|| \tag{74}$$

This power equation is identical with the power equation of three-phase unbalanced loads with a symmetrical supply voltage. Thus one might conclude that the supply voltage asymmetry does not affect power properties of the three-phase loads. These properties are still specified in terms of three powers, it means the active, reactive and the unbalanced power. However, when the load is unbalanced, each of these three powers is affected by the voltage asymmetry. In particular, the active power is:

$$P = G_b ||\boldsymbol{u}||^2 = (G_e + G_d)||\boldsymbol{u}||^2 = P_s + P_d \tag{75}$$

where P_s denotes the load active power at a symmetrical supply voltage, but with the same rms value as the asymmetrical supply voltage. The power P_d occurs because of the supply voltage asymmetry, but it disappears, independently of this asymmetry, when the load is balanced. Similarly, the reactive power:

$$Q = -B_b ||\boldsymbol{u}||^2 = -(B_e + B_d)||\boldsymbol{u}||^2 = Q_s + Q_d \tag{76}$$

where Q_s denotes the reactive power at a symmetrical supply voltage, while the power Q_d occurs because of the supply voltage asymmetry in presence of the load imbalance.

Illustration 2. Let us apply the presented analysis to the three-phase circuit shown in Figure 4, with a single phase RL load supplied through an ideal Δ/Y transformer, with the turn ratio 1:1, from a source of a positive sequence voltage, but with zero line-to-ground voltage at terminal S, assuming that the CRMS value of the line-to-ground voltage at terminal R and T are $\boldsymbol{E}_R = 100$ V and $\boldsymbol{E}_T = 100\, e^{j120}$ V, respectively.

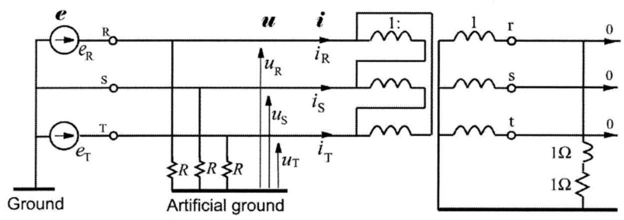

Fig. 4. Example of a three-phase circuit

The load as observed from the supply source is equivalent to a load in Δ configuration, with line-to-line admittances:

$$\boldsymbol{Y}_{RS} = \frac{1}{1 + j1} = 0.5 - j0.5 \text{ S} \qquad \boldsymbol{Y}_{ST} = \boldsymbol{Y}_{TR} = 0$$

hence the equivalent admittance of symmetrically supplied load is:

$$\boldsymbol{Y}_e = G_e + jB_e = \boldsymbol{Y}_{ST} + \boldsymbol{Y}_{TR} + \boldsymbol{Y}_{RS} = 0.5 - j0.5 \text{ S}$$

The artificial ground-to-ground voltage is:

$$\boldsymbol{V}_0 = \frac{1}{3}(\boldsymbol{E}_R + \boldsymbol{E}_S) = \frac{1}{3}(100 + 100\,e^{j120^0}) = 33.33\,e^{j60^0} \text{ V}$$

thus, the CRMS values of the line-to-artificial ground voltages at the transformer terminals are:

$$\boldsymbol{U}_R = \boldsymbol{E}_R - \boldsymbol{V}_0 = 88.192\,e^{-j19.107^0} \text{ V}$$

$$\boldsymbol{U}_S = \boldsymbol{E}_S - \boldsymbol{V}_0 = 33.333\,e^{-j120^0} \text{ V}$$

$$\boldsymbol{U}_T = \boldsymbol{E}_T - \boldsymbol{V}_0 = 88.192\,e^{j139.107^0} \text{ V}$$

The rms value of the load voltage vector $\boldsymbol{u}$ is equal to:

$$||\boldsymbol{u}|| = \sqrt{U_\mathrm{R}^2 + U_\mathrm{S}^2 + U_\mathrm{T}^2} = 129.10 \text{ V}$$

The equivalent admittance of the load at asymmetrical supply voltage is:

$$\boldsymbol{Y}_\mathrm{d} = G_\mathrm{d} + jB_\mathrm{d} = \boldsymbol{Y}_\mathrm{e} - \frac{3}{||\boldsymbol{u}||^2}(\boldsymbol{Y}_\mathrm{ST}\, U_\mathrm{R}^2 + \boldsymbol{Y}_\mathrm{TR}\, U_\mathrm{S}^2 + \boldsymbol{Y}_\mathrm{RS}\, U_\mathrm{T}^2) =$$

$$= 0.5 - j0.5 - \frac{3}{129.1^2}(0.5 - j\,0.5)\, 88.191^2 =$$

$$= -\,0.20 + j\,0.20 \text{ S}$$

The equivalent admittance of a balanced load is:

$$\boldsymbol{Y}_\mathrm{b} = G_\mathrm{b} + jB_\mathrm{b} = \boldsymbol{Y}_\mathrm{e} + \boldsymbol{Y}_\mathrm{d} =$$

$$= (0.5 - j0.5) + (-\,0.2 + j0.2) = 0.3 - j0.3 \text{ S}$$

hence, the rms value of the active and reactive current vectors:

$$||\boldsymbol{i}_\mathrm{a}|| = G_\mathrm{b}\, ||\boldsymbol{u}|| = 0.3 \times 129.1 = 38.73 \text{ A}$$

$$||\boldsymbol{i}_\mathrm{r}|| = |B_\mathrm{b}|\, ||\boldsymbol{u}|| = 0.3 \times 129.1 = 38.73 \text{ A}$$

To calculate the rms value of the unbalanced current vector, $\boldsymbol{i}_\mathrm{u}$, the load unbalanced admittances for the positive and the negative sequence voltages, as well as the CRMS values of these voltages have to be calculated. The admittances are equal to:

$$\boldsymbol{A}^\mathrm{p} = -(\boldsymbol{Y}_\mathrm{ST} + \alpha \boldsymbol{Y}_\mathrm{TR} + \alpha^* \boldsymbol{Y}_\mathrm{RS}) = -\,\alpha^* \boldsymbol{Y}_\mathrm{RS} = 0.707\, e^{j15^0} \text{ S}$$

$$\boldsymbol{A}^\mathrm{n} = -(\boldsymbol{Y}_\mathrm{ST} + \alpha^* \boldsymbol{Y}_\mathrm{TR} + \alpha \boldsymbol{Y}_\mathrm{RS}) = -\,\alpha \boldsymbol{Y}_\mathrm{RS} = 0.707\, e^{-j105^0} \text{ S}$$

while the load voltage symmetrical components have the following CRMS values:

$$\begin{bmatrix} \boldsymbol{U}^\mathrm{z} \\ \boldsymbol{U}^\mathrm{p} \\ \boldsymbol{U}^\mathrm{n} \end{bmatrix} = \boldsymbol{S} \begin{bmatrix} 88.192\, e^{-j19.107^0} \\ 33.333\, e^{-j120^0} \\ 88.192\, e^{j139.107^0} \end{bmatrix} = \begin{bmatrix} 0 \\ 66.66 \\ 33.33\, e^{-j60^0} \end{bmatrix} \text{V}$$

With these values, the CRMS values of the unbalanced current of the positive and negative sequence in line R can be calculated:

$$\boldsymbol{I}_\mathrm{Ru}^\mathrm{p} = \boldsymbol{A}^\mathrm{n} \boldsymbol{U}^\mathrm{n} - \boldsymbol{Y}_\mathrm{d} \boldsymbol{U}^\mathrm{p} = 21.60\, e^{-j115.86^0} \text{ A}$$

$$\boldsymbol{I}_\mathrm{Ru}^\mathrm{n} = \boldsymbol{A}^\mathrm{p} \boldsymbol{U}^\mathrm{p} - \boldsymbol{Y}_\mathrm{d} \boldsymbol{U}^\mathrm{n} = 43.20\, e^{j4.1^0} \text{ A}$$

hence, the rms value of the unbalanced current vector is:

$$||\boldsymbol{i}_\mathrm{u}|| = \sqrt{3}\,\sqrt{(I_\mathrm{Ru}^\mathrm{p})^2 + (I_\mathrm{Ru}^\mathrm{n})^2} = \sqrt{3}\,\sqrt{21.60^2 + 43.20^2} = 83.85 \text{ A}$$

To verify this decomposition, let us observe that the root of the sum of squares of the active, reactive and unbalanced current vectors should be equal to the rms value of the supply current vector, $||\boldsymbol{i}||$. This value can be calculated directly when the supply currents are known. Since the line current CRMS values are:

$$\boldsymbol{I}_\mathrm{R} = \boldsymbol{E}_\mathrm{R} \boldsymbol{Y}_\mathrm{RS} = 100 \frac{1}{1 + j1} = 70.71 e^{-j45^0} \text{ A}$$

$$\boldsymbol{I}_\mathrm{S} = -\boldsymbol{I}_\mathrm{R} = 70.71 e^{j135^0} \text{ A}, \qquad \boldsymbol{I}_\mathrm{T} = 0$$

thus, the rms value of the supply current vector is:

$$||\boldsymbol{i}|| = \sqrt{I_\mathrm{R}^2 + I_\mathrm{S}^2 + I_\mathrm{T}^2} = 100.0 \text{ A}$$

At the same time:

$$||\boldsymbol{i}|| = \sqrt{||\boldsymbol{i}_\mathrm{a}||^2 + ||\boldsymbol{i}_\mathrm{r}||^2 + ||\boldsymbol{i}_\mathrm{u}||^2} = \sqrt{38.73^2 + 38.73^2 + 83.65^2} = 100.0 \text{ A.}$$

This result verifies the correctness of decomposition of the supply current into physical components, meaning, active, reactive and unbalanced currents.
The active power:

$$P = G_\mathrm{b}\, ||\boldsymbol{u}||^2 = P_\mathrm{s} + P_\mathrm{d}$$

has a component independent of the supply voltage asymmetry:

$$P_\mathrm{s} = G_\mathrm{e}\, ||\boldsymbol{u}||^2 = 0.5 \times 129.1^2 = 8.3 \text{ kW}$$

but also a component dependent on this asymmetry:

$$P_\mathrm{d} = G_\mathrm{d}\, ||\boldsymbol{u}||^2 = -\,0.2 \times 129.1^2 = -3.3 \text{ kW}$$

and consequently, the load active power is P = 5 kW. The same applies to the reactive power, Q. At a symmetrical supply:

$$Q_\mathrm{s} = -B_\mathrm{e}\, ||\boldsymbol{u}||^2 = 0.5 \times 129.1^2 = 8.3 \text{ kVAr}$$

The reactive power associated with the supply voltage asymmetry:

$$Q_\mathrm{d} = -B_\mathrm{d}\, ||\boldsymbol{u}||^2 = -\,0.2 \times 129.1^2 = -3.3 \text{ kVAr}$$

thus, the reactive power of the load has the value:

$$Q = Q_\mathrm{s} + Q_\mathrm{d} = 5.0 \text{ kVAr}$$

The load unbalanced power is:

$$D = ||\boldsymbol{u}||\, ||\boldsymbol{i}_\mathrm{u}|| = 129.1 \times 83.65 = 10.8 \text{ kVA}$$

and hence, the apparent power S of the load is equal to:

$$S = \sqrt{P^2 + Q^2 + D^2} = \sqrt{5^2 + 5^2 + 10.8^2} = 12.9 \text{ kVA}$$

The same value results from the apparent power definition:

$$S = \|\boldsymbol{u}\|\,\|\boldsymbol{i}\| = 12.9 \text{ kVA}$$

and this verifies the correctness of the apparent power S decomposition into the active, reactive und unbalanced powers, P, Q and D.

5. CONCLUSIONS

The analysis of power phenomena in three-phase circuits with unbalanced LTI loads supplied with an asymmetrical sinusoidal voltage shows that the supply current of such loads, as in the case of symmetrical voltage, is composed of only three physical components, the active, reactive and unbalanced currents. Consequently, loads in such conditions can be characterized by the active, reactive and unbalanced powers. The equivalent and unbalanced admittances of three-phase loads at symmetrical supply voltage are constant, independent of the supply voltage, parameters. The paper shows that these parameters at asymmetrical supply depend on the voltage asymmetry, however.

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Leszek S. Czarnecki (Fellow IEEE, MIEE)

Alfredo M. Lopez Distinguished Professor, received the M.Sc. and Ph.D. degrees in electrical engineering and Habil. Ph.D. degree from the Silesian University of Technology, Poland, in 1963, 1969 and 1984, respectively, where he was employed as an Assistant Professor. Beginning in 1984 he worked for two years at the Power Engineering Section, Division of Electrical Engineering, National Research Council of Canada as a Research Officer. In 1987 he joined the Electrical Engineering Dept. at Zielona Gora University of Technology. In 1989 Dr. Czarnecki joined the Electrical and Computer Engineering Dept. of Louisiana State University, Baton Rouge, where he is a Professor of Electrical Engineering now. For developing a power theory of three-phase nonsinusoidal unbalanced systems and methods of compensation of such systems he was elected to the grade of Fellow IEEE. He climbed the main ridge of Ruwenzori, Kilimanjaro and Mt. Kenya in Africa, Lhotse in Himalaya to 8350m, Cordilliera Huayashi in Andas, climbed solo McKinley in Alaska and traversed (on ski) Spitsbergen.

Address:
ECE, Louisiana State University
Baton Rouge, LA 70803, USA
phone: 225 767 6528
www.lsczar.info, lsczar@cox.net