Electrical Power Quality and Utilisation, Journal *Vol. XIII, No. 1, 2007*

Generalized Symmetrical Components for Periodic Non-Sinusoidal Three-Phase Signals

Paolo TENTI[1], Jacques L. WILLEMS[2], Paolo MATTAVELLI[1] and Elisabetta TEDESCHI[1]

1) University of Padova, ITALY; 2) Ghent University, BELGIUM

***Summary*: This paper deals with the generalization of the symmetrical components technique to periodic non-sinusoidal three-phase currents and voltages. The generalization of the concept of symmetrical components is discussed together with their derivation in both time- and frequency domain. The main conclusion of the paper is that an orthogonal decomposition of periodic non-sinusoidal three-phase signals into positive sequence, negative sequence and zero-sequence components is not possible, but that an additional current and voltage component should be introduced which is called the residual component.**

1. INTRODUCTION

The use of symmetrical components is a well known classical technique for the analysis of symmetrical three-phase power systems with unbalanced sinusoidal currents and/or voltages. The decomposition of sinusoidal three-phase signals into positive sequence, negative sequence and zero-sequence components is used to simplify the analysis of a three-phase electrical power system. It makes it possible to reduce the analysis of a balanced linear three-phase system to the analysis of three decoupled single-phase systems. Linearity is required to be able to use superposition of the currents produced by the three sequence components of the voltages; the network should be balanced so that a symmetrical set of voltages (positive sequence, negative sequence, or zero-sequence) yields a symmetrical set of currents and the three-phase network can be analysed as a single-phase network.

Due to the presence of nonlinear loads, such as power electronic equipment, the currents and/or voltages in power systems are not purely sinusoidal. The question that arises is whether a technique, similar to the symmetrical components technique, can be developed for three-phase power systems with non-sinusoidal periodic currents and voltages. This question is dealt with in the present paper. First the generalization of the definition of positive sequence, negative sequence, and zero-sequence quantities is discussed. Then the decomposition of a periodic three-phase signal into such components is analysed. It is shown that a straightforward general extension of the symmetrical components method to the periodic non-sinusoidal situation is not possible, but that an additional current and voltage component should be introduced which is called the residual component.

2. PROBLEM STATEMENT

Consider a periodic three-phase signal (a current or a voltage, in a three-phase system with or without neutral conductor). To avoid some specific special cases, we assume that there is no d.c. component. The phases are denoted by the subscripts a, b, and c. The period is denoted by T, the frequency by $f = 1/T$, and the angular frequency by $\omega = 2\pi f = 2\pi / T$. This signal can be decomposed in the following way: in each phase the periodic signal is decomposed in a Fourier series of sinusoidal functions (in the three phases) of frequencies k/T with k any positive integer. Considering the signal components at each frequency separately, the corresponding (sinusoidal) components in the three phases can be decomposed in the positive sequence, the negative sequence, and the zero-sequence components, as in classical theory for sinusoidal currents and voltages. These components can then be dealt with by techniques for single-phase circuits if the network is balanced with respect to the three phases. The orthogonality of the components enables one to calculate powers, rms values, etc, for the components separately and adding them afterwards in the appropriate way.

The question however arises whether a simpler single-phase decomposition can be obtained without needing a decomposition with respect to the frequency components. In other words, can the concept of symmetrical components be directly applied to the three-phase periodic non-sinusoidal voltages and currents? A generalization of the concept of symmetrical components would be as follows, respectively for the positive sequence, the negative sequence and the zero-sequence components:

1. a component which is such that the current (or voltage) in phase b is the same as in phase a, but lagging over $T/3$, and the current (or voltage) in phase c is the same as in a, but lagging over $2T/3$;
2. a component which is such that the current (or voltage) in phase b is the same as in phase a, but leading over $T/3$, and the current (or voltage) in phase c is the same as in a, but leading over $2T/3$;
3. a component which is such that the currents (or voltages) in phase b and in phase c are the same as in a.

These three components can respectively be seen as the generalized positive sequence, generalized negative sequence, and generalized zero-sequence components.

In addition to the symmetry properies discussed above, a requirement of orthogonality between the components should be introduced since this is interesting (in fact necessary) for the simplification of the analysis as is known from the sinusoidal case. One of the essential features of the symmetrical components decomposition of sinusoidal three-phase time functions is indeed the orthogonality of the components. Explicitly two three-phase periodic time vectors $\boldsymbol{f}(t)$ and $\boldsymbol{g}(t)$ are orthogonal if the internal product, defined as:

$$\boldsymbol{f} * \boldsymbol{g} = \tfrac{1}{T}\int_0^T \left(f_a(t)g_a(t) + f_b(t)g_b(t) + f_c(t)g_c(t) \right) dt \quad (1)$$

vanishes. This orthogonality property implies that the active power and the square of the rms value can be computed as the sum of the corresponding values for the sequence components. This is a very interesting feature which considerably simplifies the analysis of three-phase systems.

In the Appendix an algorithm is derived to compute these generalized sequence components. It is shown that some part of the general periodic three-phase signal cannot be taken care of by the above framework, and that therefore an additional component is to be considered. Also we will notice some other differences with the classical case of sinusoidal three-phase quantities.

3. DERIVATION OF THE GENERALIZED COMPONENTS

The derivation of the generalized components and the underlying mathematical concepts are discussed in the Appendix. It is shown that the components can be derived in a rigorous mathematical way by finding currents (or voltages) in the three phases which have the property of a positive sequence, a negative sequence or a zero-sequence component, which are orthogonal to the other sequence components and are as close as possible (in the least squares sense) to the given three-phase currents (or voltages). Denoting the current (or voltage) in phase *a* by $f_a(t)$, the current (or voltage) in phase *b* by $f_b(t)$, and the current (or voltage) in phase *c* by $f_c(t)$, the mathematical analysis leads to the following expressions for the decomposition of the three-phase quantities:

1. the components with zero-sequence symmetry:

$$\begin{bmatrix} f_{oa}(t) \\ f_{ob}(t) \\ f_{oc}(t) \end{bmatrix} = \begin{bmatrix} f_o(t) \\ f_o(t) \\ f_o(t) \end{bmatrix} \quad (2)$$

with the generalized zero-sequence component:

$$f_o(t) = \tfrac{1}{3}\left(f_a(t) + f_b(t) + f_c(t) \right) \quad (3)$$

2. the components with positive sequence symmetry:

$$\begin{bmatrix} f_{pa}(t) \\ f_{pb}(t) \\ f_{pc}(t) \end{bmatrix} = \begin{bmatrix} f_p(t) \\ f_p(t-T/3) \\ f_p(t-2T/3) \end{bmatrix} \quad (4)$$

with the generalized positive sequence component:

$$f_p(t) = \tfrac{1}{3}\begin{pmatrix} f_a(t) - f_o(t) + f_b(t+T/3) - f_o(t+T/3) \\ + f_c(t+2T/3) - f_o(t+2T/3) \end{pmatrix} \quad (5)$$

3. the components with negative sequence symmetry:

$$\begin{bmatrix} f_{na}(t) \\ f_{nb}(t) \\ f_{nc}(t) \end{bmatrix} = \begin{bmatrix} f_n(t) \\ f_n(t+T/3) \\ f_n(t+2T/3) \end{bmatrix} \quad (6)$$

with the generalized negative sequence component:

$$f_n(t) = \tfrac{1}{3}\begin{pmatrix} f_a(t) - f_o(t) + f_b(t-T/3) - f_o(t-T/3) \\ + f_c(t-2T/3) - f_o(t-2T/3) \end{pmatrix} \quad (7)$$

Since the expression of the generalized zero-sequence component is exactly the same as for the sinusoidal case, the adjective 'generalized' could be dropped. It is clear from the above that there is a high degree of similarity between the expressions obtained here and the classical expressions used in the sinusoidal situation. An essential difference is that it is necessary to subtract the zero-sequence component from the phase quantities to calculate the positive sequence and the negative sequence components. This is not necessary for the sinusoidal case where the same result is obtained with or without subtracting the zero-sequence component first.

Expressions (5) and (7), of f_p and f_n respectively, can easily be understood from algebraic considerations. Consider, in fact, the positive sequence component: it is obtained by shifting the second waveform ahead by 1/3 of period T and the third by 2/3 of period T, then computing the mean value of the three superimposed variables. It is a known property that the mean value of a number of quantities has the minimum rms distance from this set of quantities. Thus the variables f_{pa}, f_{pb}, f_{pc} are the positive sequence components which are closest to f_a, f_b, f_c . Similarly for the negative sequence components. A more comprehensive demonstration, in the linear space domain, is given in the Appendix.

4. ALTERNATIVE EXPRESSIONS

The analysis in the previous section leads to a slightly different approach to the decomposition. First the phase quantities are decomposed in zero-sequence or homopolar quantities, which are the same in the three phases, and heteropolar quantities (denoted by a tilde) which are such that the sum over the three phases vanishes for all times. Explicitly:

$$\begin{bmatrix} f_a(t) \\ f_b(t) \\ f_c(t) \end{bmatrix} = \begin{bmatrix} f_o(t) + \tilde{f}_a(t) \\ f_o(t) + \tilde{f}_b(t) \\ f_o(t) + \tilde{f}_c(t) \end{bmatrix} \quad (8)$$

The heteropolar component is the difference between the phase quantity and the zero-sequence component; it is clear that the sum of the heteropolar quantities in the three phases at any time vanishes. The positive and the negative sequence components can then be expressed in a somewhat simpler way using the heteropolar quantities:

$$f_p(t) = \tfrac{1}{3}\left(\tilde{f}_a(t) + \tilde{f}_b(t+T/3) + \tilde{f}_c(t+2T/3)\right) \quad (9)$$

and:

$$f_n(t) = \tfrac{1}{3}\left(\tilde{f}_a(t) + \tilde{f}_b(t-T/3) + \tilde{f}_c(t-2T/3)\right) \quad (10)$$

These expressions are formally the same as in the sinusoidal case.

5. THE RESIDUAL COMPONENTS

An important question is whether the zero-sequence, the positive sequence and the negative sequence components of the phase quantities add up to the phase quantity. The analysis of the Appendix leads to the conclusion that, in contrast with the sinusoidal situation, this is not the case. Therefore it is necessary to define a residual component which is the remaining part; its physical meaning is discussed in Section 9. The explicit expression of the residual component is:

$$f_{ra}(t) = \tfrac{1}{3}\left(\tilde{f}_a(t) + \tilde{f}_a(t+T/3) + \tilde{f}_a(t+2T/3)\right) \quad (11)$$

$$f_{rb}(t) = \tfrac{1}{3}\left(\tilde{f}_b(t) + \tilde{f}_b(t+T/3) + \tilde{f}_b(t+2T/3)\right) \quad (12)$$

$$f_{rc}(t) = \tfrac{1}{3}\left(\tilde{f}_c(t) + \tilde{f}_c(t+T/3) + \tilde{f}_c(t+2T/3)\right) \quad (13)$$

This expression shows that the residual components constitute periodic functions with period T/3. The residual components vanish if and only if:

$$\begin{aligned} & f_a(t) + f_a(t+T/3) + f_a(t+2T/3) = \\ &= f_b(t) + f_b(t+T/3) + f_b(t+2T/3) = \\ &= f_c(t) + f_c(t+T/3) + f_c(t+2T/3) = \\ &= f_a(t) + f_b(t) + f_c(t) \end{aligned} \quad (14)$$

hold for all time in the period. It is readily checked that the residual component vanishes for sinusoidal time functions.

Note finally that all symmetrical components are easily derived in the time domain on an instantaneous basis.

6. DECOMPOSITION INTO GENERALIZED SYMMETRICAL COMPONENTS

The periodic non-sinusoidal three-phase quantities can be exactly decomposed into the generalized zero-sequence components, the generalized positive sequence components, the generalized negative sequence components and the residual components.

$$\begin{bmatrix} f_a(t) \\ f_b(t) \\ f_c(t) \end{bmatrix} = \begin{bmatrix} f_p(t) + f_n(t) + f_o(t) + f_{ra}(t) \\ f_p(t-T/3) + f_n(t+T/3) + f_o(t) + f_{rb}(t) \\ f_p(t-2T/3) + f_n(t+2T/3) + f_o(t) + f_{rc}(t) \end{bmatrix} \quad (15)$$

This is called the decomposition into generalized symmetrical components. The important feature is that, except for the residual components, the analysis of a blanced three-phase system can be reduced to the analysis of a single-phase system.

The essential differences with respect to the well known sinusoidal case are

— the existence of the residual components,
— the fact that the zero-sequence components should be subtracted for the computation of the positive and negative sequence components

Even if the sum of the three-phase quantities is identically zero (as is the case for line-to-line voltages or for the currents in a three-wire three-phase system), such that the zero-sequence component is identically zero, the three-phase currents and/or voltages cannot necessarily be represented by generalized positive and negative sequence components, with the symmetry properties set forth in Section 3. The residual component is necessary.

7. ORTHOGONALITY OF THE COMPONENTS

As pointed out in Section 3, the orthogonality of the components obtained for the symmetrical components in the sinusoidal case is a very interesting property. The decomposition of the periodic three-phase quantities into the generalized zero-sequence components, the generalized positive sequence components, the generalized negative sequence components and the residual components, is a decomposition into mutually orthogonal components, as follows from the analysis in the Appendix.

This orthogonality property implies:

— The active power corresponding to a three-phase periodic non-sinusoidal voltage and a three-phase periodic non-sinusoidal current is the sum of the active powers corresponding to the zero-sequence components of current and voltage, the generalized positive sequence components of current and voltage, the generalized negative sequence components of current and voltage, and the residual components of current and voltage.
— The rms value of a three-phase periodic sinusoidal quantity can be computed from the rms values of the generalized symmetrical components by means of the expression

$$\|\mathbf{f}\|^2 = \mathbf{f} * \mathbf{f} = \|\mathbf{f}_o\|^2 + \|\mathbf{f}_p\|^2 + \|\mathbf{f}_n\|^2 + \|\mathbf{f}_r\|^2 \quad (16)$$

where f_o, f_p, f_n and f_r denote the three-phase rms quantities corresponding to the generalized symmetrical and the residual components.

It also follows from the analysis of the Appendix that even stronger properties hold than the orthogonality property with respect to the internal product defined in Section 3.

- The dot product, as defined in the Appendix, of the generalized zero-sequence component and any of the other components (heteropolar, generalized positive sequence, generalized negative sequence, residual) is zero, which is a strong form of orthogonality, much stronger than the orthogonality corresponding to a zero internal product.
- The generalized dot product, as defined in the Appendix, of any of the pairs of components corresponding to the generalized positive sequence components, the generalized negative sequence components, or the residual components, is zero. This is a weaker form of orthogonality than the one corresponding to a zero dot product, but still a much stronger form than the one corresponding to a zero internal product.

These stronger orthogonality properties imply features with respect to the instantaneous powers and the instantaneous rms values, similar to the features with respect to the average powers and rms values referred to above.

8. THE RELATION TO THE FOURIER SERIES EXPANSION

It was pointed out in Section 3 that another approach to the analysis of three-phase non-sinusoidal three-phase quantities [1] is to decompose the periodic functions into Fourier series and to use a decomposition into symmetrical components for the sinusoidal three-phase time functions at each harmonic frequency. It is interesting to see how this decomposition relates to the decomposition into generalized symmetrical components obtained in the previous sections. The following results are readily obtained:

- Consider a harmonic of frequency order ($3k$+1), with k a nonnegative integer. The positive sequence, the negative sequence and the zero-sequence components of the corresponding sinusoidal three-phase function have respectively the symmetry properties of the generalized positive sequence, the generalized negative sequence and the generalized zero-sequence components.
- Consider a harmonic of frequency order ($3k$+2), with k a nonnegative integer. The positive sequence, the negative sequence and the zero-sequence components of the corresponding sinusoidal three-phase function have respectively the symmetry properties of the generalized negative sequence, the generalized positive sequence and the generalized zero-sequence components.
- Consider a harmonic of frequency order $3k$, with k a positive integer. The positive sequence and the negative sequence components of the corresponding sinusoidal three-phase function do not have the symmetry properties of any of the generalized positive sequence, the generalized negative sequence and the generalized zero-sequence components. On the other hand the zero-sequence components (it is because of these terms that the generalized zero-sequence component has to be subtracted for computing the generalized positive and negative sequence components) of the corresponding sinusoidal three-phase function have the symmetry properties of the generalized zero-sequence components.

This shows that the generalized positive sequence, negative sequence, and zero-sequence components are not sufficient to describe all periodic time three-phase functions. A component should be added, the residual component, which corresponds in the Fourier expansion approach to the positive and negative sequence components of the harmonics of orders which are multiples of 3. Note that the residual components completely correspond to the harmonics of orders which are a multiple of 3 if the sum of the three-phase quantities are zero, as is the case for line-to-line voltages or for the currents in a three-wire three-phase system.

The mutual orthogonality of the generalized positive sequence, the generalized negative sequence, the zero sequence, and the residual components, in the sense of a zero internal product, can obviously also be derived from the mutual orthogonality of the symmetrical components in the sinusoidal case and from the mutual orthogonality of sinusoidal functions of different harmonic orders.

It is interesting to notice that for a harmonic of an order k which is not a multiple of 3, the negative and positive sequence symmetrical components for this sinusoidal function of period T/k is different from the generalized symmetrical components for this periodic function with fundamental period T.

9. RELATION TO DEPENBROCK'S ANALYSIS

In a contribution to the previous workshop [3] M. Depenbrock proposes an algorithm to derive generalized symmetrical components. For the case of zero-sum three-phase quantities he claims that three-phase distorted quantities can be decomposed into generalized positive and negative sequence components. Examples below show the necessity of the residual components. We consider two simple examples:

1. A three-phase voltage with a positive sequence fundamental and a positive sequence third harmonic:

$$v_a(t) = V_1\sqrt{2}\,sin(2\pi 50t) + V_3\sqrt{2}\,sin(2\pi 150t)$$

$$v_b(t) = V_1\sqrt{2}\,sin(2\pi 50t - \frac{2\pi}{3}) + V_3\sqrt{2}\,sin(2\pi 150t - \frac{2\pi}{3})$$

$$v_c(t) = V_1\sqrt{2}\,sin(2\pi 50t + \frac{2\pi}{3}) + V_3\sqrt{2}\,sin(2\pi 150t + \frac{2\pi}{3})$$

2. A three-phase voltage with a positive sequence fundamental and a negative sequence third harmonic:

$$v_a(t) = V_1\sqrt{2}\,sin(2\pi 50t) + V_3\sqrt{2}\,sin(2\pi 150t)$$

$$v_b(t) = V_1\sqrt{2}\,sin(2\pi 50t - \frac{2\pi}{3}) + V_3\sqrt{2}\,sin(2\pi 150t + \frac{2\pi}{3})$$

$$v_c(t) = V_1\sqrt{2}\,sin(2\pi 50t + \frac{2\pi}{3}) + V_3\sqrt{2}\,sin(2\pi 150t - \frac{2\pi}{3})$$

In the first example the analysis of the present paper yields the fundamental component as the generalized positive sequence component, the third harmonic as the residual component, and no negative sequence or zero-sequence components. The same conclusion is obtained for the second example.

For both examples Depenbrock's theory yields the complete three-phase signal as the generalized positive sequence component, and no negative sequence or zero-sequence component. However it is clear that the three-phase signals do not have positive sequence symmetry, since the voltages in the phases *b* and *c* cannot be obtained from the voltage in phase *a* by lagging over respectively $T/3$ and $2T/3$.

10. FURTHER OBSERVATIONS

In the literature [2, 4] one often sees the statement that for three-phase periodic distorted currents (or voltages) the harmonics of order 1, 4, 7, ... or in general (3k+1), correspond to a positive sequence current, the harmonics of order 2, 5, 8, ... or in general (3k–1), correspond to a negative sequence current, the harmonics of order 3, 6, 9, or in general 3k, correspond to a zero-sequence current. The analysis of Section 9 shows that this statement should be used carefully. It is indeed only valid if the three-phase current (or voltage) in phase b is the same as in phase a, but lagging over T/3, and the current (or voltage) in phase c is the same as in phase a, but lagging over 2T/3. If this symmetry property does not hold, then the statement is not valid.

11. FINAL REMARKS

Summarizing the following conclusions follow from the analysis in this paper:

- The three-phase current (or voltage) cannot always be derived from generalized positive sequence, generalized negative sequence and generalized zero-sequence components. A residual component may be required.
- To compute the generalized positive and negative sequence components, the zero-sequence components should first be subtracted from the phase quantities, in contrast with the sinusoidal case where this is not necessary.
- In the sinusoidal case the residual component is absent and the other components reduce to the classical symmetrical components.
- The generalized positive sequence, negative sequence, and zero-sequence components have complete phase symmetry (up to displacement over a third of a period). This implies that the three-phase analysis can be reduced to a single-phase analysis for a balanced three-phase power system.
- The residual component does not have the same symmetry, and the corresponding three-phase analysis cannot be reduced to a single-phase analysis. It corresponds to a periodic time function in each of the three phases with a period which is one third of the period of the currents and voltages; this also may simplify the analysis.

APPENDIX

Derivation of the sequence components

To derive the explicit expressions for the generalized symmetrical components we first formulate the classical symmetrical components in a linear algebraic setting.

A general three-phase sinusoidal quantity (current or voltage) can be represented either as a three-dimensional vector of phasors:

$$\boldsymbol{f} = \begin{bmatrix} f_a \\ f_b \\ f_c \end{bmatrix} \tag{A.1}$$

where the entries are complex numbers, or as a three-dimensional vector of sinusoidal time functions:

$$\boldsymbol{f}(t) = \begin{bmatrix} F_a\sqrt{2}\,cos(\omega t + \alpha_a) \\ F_b\sqrt{2}\,cos(\omega t + \alpha_b) \\ F_c\sqrt{2}\,cos(\omega t + \alpha_c) \end{bmatrix} \tag{A.2}$$

The decomposition into symmetrical components is equivalent to writing the phasor vector as a sum of three vectors, respectively from the linear subspaces defined by vectors of the forms:

$$\boldsymbol{f}_o = \begin{bmatrix} f_o \\ f_o \\ f_o \end{bmatrix},\quad \boldsymbol{f}_p = \begin{bmatrix} f_p \\ f_p e^{-j2\pi/3} \\ f_p e^{j2\pi/3} \end{bmatrix},\quad \boldsymbol{f}_n = \begin{bmatrix} f_n \\ f_n e^{j2\pi/3} \\ f_n e^{-j2\pi/3} \end{bmatrix} \tag{A.3}$$

and similarly for the corresponding vectors of sinusoidal time functions.

The symmetrical components are derived by expressing that the sum of three component vectors, one from each subspace, should be equal to the given three-dimensional vector. Another approach is to express that the component vector from each subspace is the projection of the given vector onto that subspace. Equivalently, we can obtain the components by considering the component vector in the subspace as the best approximation of the given vector in the least squares sense.

To generalize this concept to periodic three-phase quantities, we consider a three-dimensional vector in the linear space:

$$\boldsymbol{f}(t) = \begin{bmatrix} f_a(t) \\ f_b(t) \\ f_c(t) \end{bmatrix} \tag{A.4}$$

where $f_a(t)$, $f_b(t)$ and $f_c(t)$ are periodic functions with period T. The question is to write this vector as the sum of components from the linear subspaces $\mathbf{s}_o$, $\mathbf{s}_p$ and $\mathbf{s}_n$ of the vectors, respectively of the forms:

$$\boldsymbol{f}_o(t) = \begin{bmatrix} f_o(t) \\ f_o(t) \\ f_o(t) \end{bmatrix}, \quad \boldsymbol{f}_p(t) = \begin{bmatrix} f_p(t) \\ f_p(t-T/3) \\ f_p(t-2T/3) \end{bmatrix}, \quad \boldsymbol{f}_n(t) = \begin{bmatrix} f_n(t) \\ f_n(t+T/3) \\ f_n(t+2T/3) \end{bmatrix} \quad (A.5)$$

where $f_o(t)$, $f_p(t)$ and $f_n(t)$ are periodic functions with period T.

The following questions are essential for the analysis of the possibility and the unicity of the decomposition:

- are the three subspaces mutually independent?
- are the three subspaces mutually orthogonal?
- do the three subspaces span the whole linear space?

The answer to these questions is affirmative in the sinusoidal case. This is the fundamental reason why symmetrical components are so useful. In the general periodic case the situation is not so favorable.

The three subspaces are indeed not independent (and hence not mutually orthogonal). This can be seen by considering the example of the following vector which lies in each of the three subspaces:

$$\boldsymbol{f}(t) = \begin{bmatrix} A\sqrt{2}\cos(3\omega t) \\ A\sqrt{2}\cos(3\omega t) \\ A\sqrt{2}\cos(3\omega t) \end{bmatrix} \quad (A.6)$$

It can also be shown that the subspaces do not span the total linear space of periodic vectors. Below is an example of a periodic vector which cannot be composed by vectors from the three subspaces:

$$\boldsymbol{f}(t) = \begin{bmatrix} A\sqrt{2}\cos(3\omega t) \\ A\sqrt{2}\cos(3\omega t - 2\pi/3) \\ A\sqrt{2}\cos(3\omega t - 4\pi/3) \end{bmatrix} \quad (A.7)$$

The required orthogonality corresponds to the condition that the internal product, defined by (1), is zero. For the further analysis we define two other products for two three-dimensional periodic vectors $\boldsymbol{f}(t)$ and $\boldsymbol{g}(t)$:

- The dot product is defined as:

$$\boldsymbol{f}(t) \bullet \boldsymbol{g}(t) = f_a(t)g_a(t) + f_b(t)g_b(t) + f_c(t)g_c(t) \quad (A.8)$$

- The generalized dot product is defined as:

$$\boldsymbol{f}(t) \circ \boldsymbol{g}(t) = \tfrac{1}{3}\left[\begin{matrix} \boldsymbol{f}(t) \bullet \boldsymbol{g}(t) + \boldsymbol{f}(t+T/3) \bullet \boldsymbol{g}(t+T/3) \\ + \boldsymbol{f}(t+2T/3) \bullet \boldsymbol{g}(t+2T/3) \end{matrix}\right] \quad (A.9)$$

The problem that the three linear subspaces $\mathbf{s}_o$, $\mathbf{s}_p$ and $\mathbf{s}_n$ are not independent and hence not orthogonal, is taken care of by restricting the subspaces defined by the vectors $\boldsymbol{f}_p$(t) and $\boldsymbol{f}_n(t)$ to the vectors of which the sum of the components is zero $\boldsymbol{f}_n(t)$. Mathematically speaking the subspaces of the vectors $\boldsymbol{f}_p(t)$ and by the vectors $\boldsymbol{f}_n(t)$ are restricted to their subspaces orthogonal to the subspace of the vectors $\boldsymbol{f}_o(t)$ (or to their projections on the orthogonal complement). In this way we obtain three subspaces linear subspaces $\mathbf{s}_o$, $\mathbf{s}_p^r$ and $\mathbf{s}_n^r$ which are linearly independent. Moreover the subspaces are mutually orthogonal in the sense of the internal product defined by (1). Even more, the dot product of a vector in $\mathbf{s}_o$ and a vector in $\mathbf{s}_p^r$ or $\mathbf{s}_n^r$ is zero. Also the generalized dot product of a vector in $\mathbf{s}_p^r$ and a vector in $\mathbf{s}_n^r$ vanishes.

Since the subspaces $\mathbf{s}_o$, $\mathbf{s}_p^r$ and $\mathbf{s}_n^r$ do not span the complete space, the generalized symmetrical components cannot be obtained, as in the sinusoidal situation, by expressing equality of the periodic vector and the sum of the three generalized symmetrical components. They should therefore be derived by computing the projection on the subspaces or by considering the best approximation in the subspaces. For example, the generalized positive sequence component is obtained by expressing that $\boldsymbol{f}_p(t)$ is in $\mathbf{s}_p^r$ and that:

$$\int_0^T (\boldsymbol{f} - \boldsymbol{f}_p) \bullet (\boldsymbol{f} - \boldsymbol{f}_p)\, dt = \int_0^T \left\| (\boldsymbol{f} - \boldsymbol{f}_p) \right\|^2 dt \quad (A.10)$$

is as small as possible. This can be formulated as an optimization problem that can be solved by means of the Lagrange multiplier technique. In this way the expressions shown in Section 4 are readily obtained.

REFERENCES

1. Chen C.C. and Hsu Y.Y.: *A novel approach to the design of a shunt active filter for an unbalanced three-phase four-wire system under nonsinusoidal conditions*. 2000, IEEE Transactions on Power Delivery, vol. 15, 4, pp. 1258–1264.
2. Czarnecki L.S.: *Reactive and unbalanced currents compensation in three-phase asymmetrical circuits under nonsinusoidal conditions*. 1989, IEEE Transactions on Measurement and Instrumentation, vol. 38, 3, pp. 754–759.
3. Depenbrock M.: *Generalized symmetrical components of 4-wire voltages*. 2004, L'Energia Elettrica, vol. 81, 5–6 (supplemento), pp. 46–50.
4. Stankovic A.M., Sanders S.S. and Aydin T.: *Dynamic phasors in modeling and analysis of unbalanced polyphase AC machines*. 2002, IEEE Transactions on Energy Conversion, vol. 17, 1, pp. 107–113.

Paolo Tenti (Fellow IEEE)
professor of Power Electronics and head of the Department of Information Engineering at the University of Padova, Italy. His main interests are industrial and power electronics and electromagnetic compatibility. In these areas he holds national and international patents and published more than 100 scientific and technical papers. His research focuses on application of modern control methods to power electronics and EMC analysis of electronic equipment.

In 1996 Paolo Tenti was elected Vice-President of the IEEE Industry Applications Society (IAS), in 1997 he served as IAS President. In 2000 he chaired the IEEE World Conference on Industrial Applications of Electrical Energy (Rome). For the years 2000-2001 he was appointed IEEE-IAS Distinguished Lecturer on "Electromagnetic compatibility in industrial equipment".

Paolo Tenti is a Fellow of the IEEE. He is also President of CREIVen, an industrial research consortium for tecnological advancements in industrial electronics, with special emphasis on electromagnetic compatibility.

Address:
Department of Information Engineering (DEI),
University of Padova,
Via Gradenigo 6/B, 35131 Padova, Italy,
e-mail tenti@dei.unipd.it,
tel.: +39-049-8277503, fax: +39-049-8277699.

Jacques L Willems (Fellow IEEE)
was born in Bruges, Belgium, on September 18, 1939. He graduated in electromechanical engineering from Ghent University, Belgium, and received the Master of Science degree from M.I.T. and the doctoral degree from Ghent University.
Since 1964 he has been with the Engineering Faculty of Ghent University, teaching and doing research in the areas of power system analysis and control theory. He was a Visiting Research Fellow at Harvard University in 1970 and Professeur Visiteur at the Université Nationale du Rwanda in 1982. He was Dean of the Engineering Faculty from 1988 tot 1992 and Rector of Ghent University from 1993 tot 2001.
The research of Prof. Willems is partially supported by the Belgian Programme of Interuniversity Attraction Poles, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with the author.
Address:
Department EESA,
Ghent University,
Technologiepark-Zwijnaarde, 914, B-9052 Gent, Belgium,
e-mail: Jacques.Willems@UGent.be,
tel.: +32-9-2645648, fax: +32-9-2645840.

Paolo Mattavelli
received the Dr. Ing. Degree (with honors) and the Ph. D. degree, both in electrical engineering, from the University of Padova (Italy) in 1992 and in 1995, respectively. From 1995 to 2001, he was a researcher at the University of Padova. In 2001 he joined the Department of Electrical, Mechanical and Management Engineering (DIEGM) of the University of Udine, where he was an Associate Professor of Electronics from 2002 to 2005. From 2001-2005 he was leading the Power Electronics Laboratory of the DIEGM at the University of Udine, which he founded in 2001. Since 2005 he has been with the DTG at the University of Padova with the same duties.
His major fields of interest include analysis, modeling and control of power converters, digital control techniques for power electronic circuits, and power quality issues.
Paolo Mattavelli is a member of IEEE Power Electronics, IEEE Industry Applications, IEEE Industrial Electronics Societies and the Italian Association of Electrical and Electronic Engineers (AEI). He also serves as an Associate Editor for IEEE Transactions on Power Electronics and Member-at-Large of PELS Adcom for terms 2004-2006.
Address:
Department of Technology and Management of Industrial Systems (DTG),
University of Padova,
Stradella S. Nicola, 3, 36100 Vicenza, Italy,
e-mail: paolo.mattavelli@unipd.it,
tel.: +39-0444-998819, fax: +39-0444-998888.

Elisabetta Tedeschi
was born in Noventa Vicentina, Italy, in 1980. She received the Dr. Ing. Degree (with honors) in electrical engineering from the University of Padova, Italy in 2005. She is currently pursuing the PhD degree in Mechatronics and Industrial Systems at the University of Padova in Vicenza.
Her research interests are in the field of digital control of power converters and in power quality issues.
Address:
Department of Technology and Management of Industrial Systems (DTG),
University of Padova,
Stradella S. Nicola, 3, 36100 Vicenza, Italy,
e-mail: elisabetta.tedeschi@unipd.it,
tel.: +39-0444-998855, fax: +39-0444-998888.