NONLINEAR ANALYSIS OF THE STABILITY OF HYDRODYNAMIC BEARINGS

Amira AMAMOU*, Mnaouar CHOUCHANE**, Sami NAIMI***

* Laboratory of Mechanical Engineering, National Engineering School of Monastir, Tunisia
Avenue Ibn Eljazzar, 5019 Monastir, Tunisia
e-mail: amira.amamou@yahoo.fr
** Laboratory of Mechanical engineering, National Engineering School of Monastir, Tunisia
Avenue Ibn Eljazzar, 5019 Monastir, Tunisia
e-mail: mnaouar.chouchane@enim.rnu.tn
*** Laboratory of Mechanical Engineering, National Engineering School of Monastir, Tunisia
Avenue Ibn Eljazzar, 5019 Monastir, Tunisia
e-mail: naîmisami@yahoo.fr

Summary

The linear analysis of the stability of a hydrodynamic bearing is used to determine the stability boundaries and to predict if the steady state is stable or not. A nonlinear or weakly nonlinear model is used to determine the behaviour of the system near the critical stability boundaries. By applying the Hopf bifurcation theory, the existence of stable or unstable limit cycles in the neighbourhood of the stability boundaries can be predicted depending on the characteristics of the bearing.A numerical integration of the nonlinear equations of motion is then carried out in order to verify the results obtained analytically.

Keywords: hydrodynamic bearing, stability analysis, nonlinear analysis, Hopf bifurcation, limit cycles.

INTRODUCTION

Hopf bifurcation theory is used in the analysis of the stability of a rigid rotor symmetrically supported by two identical journal bearings to determine the nature of instability and the existence of stable or unstable limit cycles near the stability boundaries.

Meyers [4] applied, in 1984, this theory to determine the stability of an infinitely long journal bearing. He identified the existence of three different regions in the parameters space of the steady state eccentricity ratio.

The same analysis is carried out by Hollis and Taylor [3] to identify the stability of a rigid rotor supported symmetrically by two infinitely short journal bearings.

All this work is based on the assumption of lubrication under the laminar flow condition. However, Khonsari and Wang [7] have applied the Hopf bifurcation theory for the analysis of the stability of short journal bearings considering the turbulence effect.

The purpose of this paper is to use nonlinear models for both infinitely short and long journal bearings to analyse the stability of a hydrodynamic bearing using Hopf bifurcation theory in order to determine the nature of instability according to the bearing's parameters. Analytical results are then verified numerically.

The purpose of this paper is to use nonlinear models for both infinitely short and long journal bearings to analyse the stability of a hydrodynamic bearing using Hopf bifurcation theory in order to determine the nature of instability according to the bearing's parameters. Analytical results are then verified numerically.

The outline of this paper is as follows. First, the equations of motion of the rotor bearing system are presented. Then, a linear analysis of the stability for short and long journal bearings is performed to determine the stability boundaries.

A nonlinear analysis of the stability is then carried out by applying the Hopf bifurcation theory to determine the shape, the size and stability of the periodic solutions of the journal orbit.

A numerical integration of the equations of motion is used to verify the analytical results.

The paper is concluded with the presentation of the implications of these results on the stability characteristics of journal bearing.

1. EQUATIONS OF MOTION

Consider a system of a rigid and perfectly balanced and symmetrical rotor supported by two identical hydrodynamic bearings, figure 1.

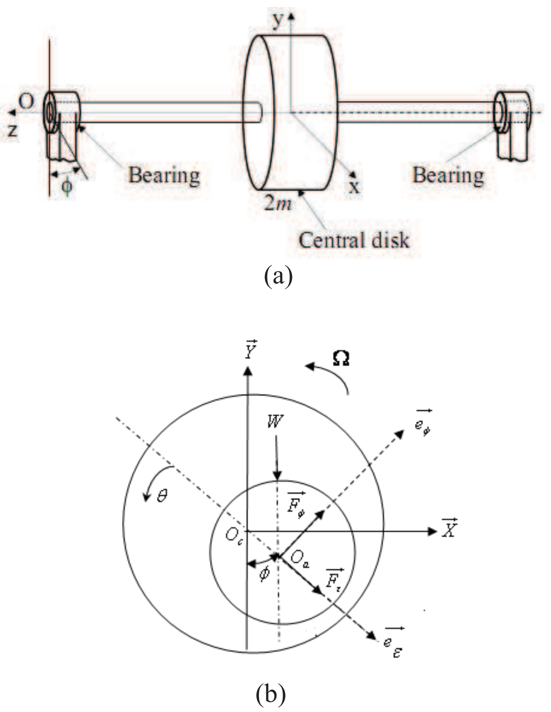

Fig. 1. (a) The model of the symmetric rigid rotor
(b) A section of the hydrodynamic bearing

The equations of motion of this system are:

$$\begin{cases} M\ddot{x} = F_x \\ M\ddot{y} = -W + F_y \end{cases} \Rightarrow \begin{cases} M\ddot{x} = F_\varepsilon \sin\phi + F_y \cos\phi \\ M\ddot{y} = -W - F_\varepsilon \cos\phi + F_\phi \sin\phi \end{cases} \quad (1)$$

where
x, y : Cartesian coordinates of the rotor centre
F_ε, F_ϕ : Radial and tangential components of the fluid force applied on the journal
W : Static load applied on the bearing

These equations may be written in polar coordinates as follow:

$$\begin{cases} M\ddot{\varepsilon} - Me\dot{\phi}^2 = F_\varepsilon + W\cos\phi \\ Me\ddot{\phi} + 2M\dot{e}\dot{\phi} = F_\varepsilon - W\sin\phi \end{cases} \quad (2)$$

In a non dimensional form, these equations become

$$\begin{cases} \ddot{\varepsilon} - \varepsilon\dot{\phi}^2 = \dfrac{F_\varepsilon}{Mc} + \dfrac{W}{Mc}\cos\phi \\ \varepsilon\ddot{\phi} + 2\dot{\varepsilon}\dot{\phi} = \dfrac{F_\phi}{Mc} - \dfrac{W}{Mc}\sin\phi \end{cases} \quad (3)$$

where

- $\dot{(\ )} = \dfrac{1}{\Omega}\dfrac{d}{dt}$
- $\varepsilon = \dfrac{e}{c}$
- $\overline{F_\varepsilon} = \dfrac{F_\varepsilon}{Mc\Omega^2}$
- $\overline{F_\phi} = \dfrac{F_\phi}{Mc\Omega^2}$

The above system of equations is composed of two 2nd order nonlinear equations. To solve this system, the two 2nd order equations are transformed into four 1st order equations.

Let

$$\boldsymbol{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} \varepsilon \\ \phi \\ \dot{\varepsilon} \\ \dot{\phi} \end{Bmatrix}$$

$$\overline{\Omega} = \Omega\sqrt{\frac{c}{g}}$$

Then the equations of motion become:

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_3 = x_1 x_4^2 + \overline{F_\varepsilon} + \dfrac{1}{\overline{\Omega}^2}\cos x_2 \\ \dot{x}_4 = \dfrac{-2x_3x_4}{x_1} + \overline{F_\phi} - \dfrac{1}{\overline{\Omega}^2}\sin x_2 \end{cases} \quad (4)$$

For an infinitely short journal bearing, the hydrodynamic force has a radial component F_ε, and a tangential component F_ϕ as shown in Fig. 1.

Assuming the Half-Sommerfeld boundary conditions, the two components of the dynamic oil-film reaction force of the journal are [7]:

$$\begin{cases} \overline{F_\varepsilon} = \dfrac{F_\varepsilon}{Mc\Omega^2} = -\dfrac{\mu RL^3}{2Mc^3\Omega}\left[\dfrac{2\left(1-2\dot{\phi}\right)\varepsilon^2}{\left(1-\varepsilon^2\right)^2} + \dfrac{\pi\dot{\varepsilon}\left(1+2\varepsilon^2\right)}{\left(1-\varepsilon^2\right)^{2.5}}\right] \\ \overline{F_\phi} = \dfrac{F_\phi}{Mc\Omega^2} = \dfrac{\mu RL^3}{2Mc^3\Omega\varepsilon}\left[\dfrac{\pi\left(1-2\dot{\phi}\right)\varepsilon^2}{2\left(1-\varepsilon^2\right)^{1.5}} + \dfrac{4\dot{\varepsilon}\varepsilon^2}{\left(1-\varepsilon^2\right)^2}\right] \end{cases} \quad (5)$$

Substituting the expression of the film reaction components into equation (4) and using the non dimensional bearing modulus $\Gamma = \dfrac{\mu RL^3}{2Mc^{2.5}g^{0.5}}$ which is independent of the rotor speed, we obtain:

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_3 = x_1 x_4^2 - \dfrac{\Gamma}{\overline{\Omega}}\left[\dfrac{(1-2x_4)x_1^2}{\left(1-x_1^2\right)^2} + \dfrac{\pi x_3\left(1+2x_1^2\right)}{\left(1-x_1^2\right)^{2.5}}\right] + \dfrac{1}{\overline{\Omega}^2}\cos x_2 \\ \dot{x}_4 = \dfrac{-2x_3x_4}{x_1} + \dfrac{\Gamma}{\overline{\Omega}x_1^2}\left[\dfrac{\pi(1-2x_4)x_1^2}{2\left(1-x_1^2\right)^{1.5}} + \dfrac{4x_3x_1^2}{\left(1-x_1^2\right)^2}\right] - \dfrac{1}{\overline{\Omega}^2}\sin x_2 \end{cases} \tag{6}$$

For an infinitely long journal bearing approximation, the non dimensional expressions for the hydrodynamic force components are [1]:

$$\begin{cases} \overline{F}_\varepsilon = \dfrac{F_\varepsilon}{Mc\Omega^2} = -\dfrac{6\mu LR^3}{Mc^3\Omega}\left\{\dfrac{2\varepsilon^2\left(1-2\dot{\phi}\right)}{\left(1-\varepsilon^2\right)\left(2+\varepsilon^2\right)} + \dfrac{\left[\pi^2\left(2+\varepsilon^2\right)-16\right]\dot{\varepsilon}}{\pi\left(2+\varepsilon^2\right)\left(1-\varepsilon^2\right)^{1.5}}\right\} \\ \overline{F}_\phi = \dfrac{F_\phi}{Mc\Omega^2} = \dfrac{6\mu LR^3}{Mc^3\Omega\varepsilon}\left\{\dfrac{\pi\varepsilon^2\left(1-2\dot{\phi}\right)}{\left(2+\varepsilon^2\right)\left(1-\varepsilon^2\right)^{0.5}} + \dfrac{4\dot{\varepsilon}\varepsilon^2}{\left(2+\varepsilon^2\right)\left(1-\varepsilon^2\right)}\right\} \end{cases} \tag{7}$$

The equations of motion of an infinitely long journal bearing is obtained by substituting equations (7) into equation (4) and using a non dimensional parameter $s = \dfrac{\mu LR^3}{M c^{2.5} g^{0.5}}$,

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_3 = x_1 x_4^2 - \dfrac{6s}{\overline{\Omega}}\left\{\dfrac{2x_1^2(1-2x_4)}{\left(1-x_1^2\right)\left(2+x_1^2\right)} + \dfrac{\left[\pi^2\left(2+x_1^2\right)-16\right]x_3}{\pi\left(2+x_1^2\right)\left(1-x_1^2\right)^{1.5}}\right\} + \dfrac{1}{\overline{\Omega}^2}\cos x_2 \\ \dot{x}_4 = \dfrac{-2x_3x_4}{x_1} + \dfrac{6s}{\overline{\Omega}x_1^2}\left\{\dfrac{\pi(1-2x_4)x_1^2}{\left(2+x_1^2\right)\left(1-x_1^2\right)^{0.5}} + \dfrac{4x_3x_1^2}{\left(2+x_1^2\right)\left(1-x_1^2\right)}\right\} - \dfrac{1}{\overline{\Omega}^2}\sin x_2 \end{cases} \tag{8}$$

2. LINEAR ANALYSIS

A linear analysis in the case of the Half-Sommerfeld boundary approximation can be used to determine the stability boundaries for both short and long bearings.

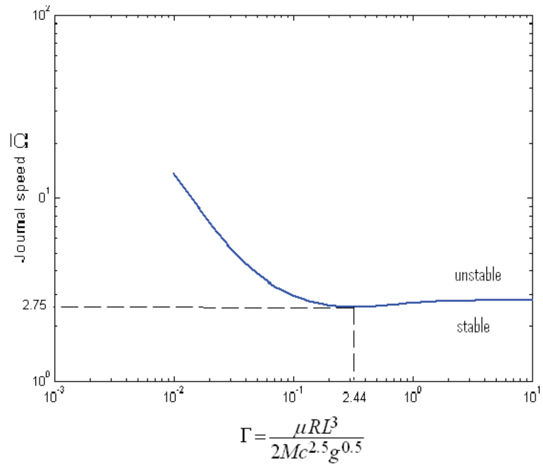

Fig. 2. Stability boundaries for an infinitely short journal bearing

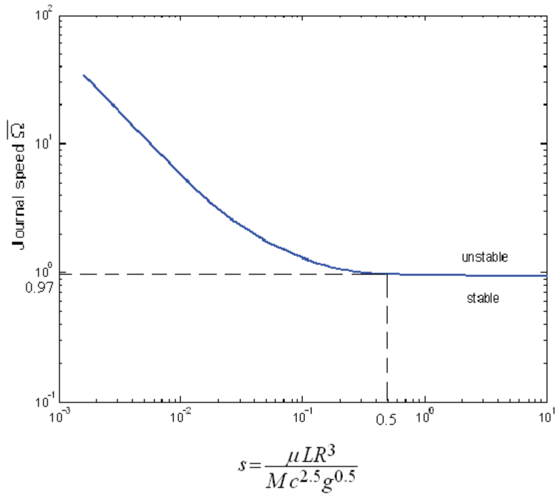

Fig. 3. Stability boundaries for an infinitely long journal bearing

Figures 2 and 3 can be used to determine the stability threshold speed of a bearing.
The non dimensional parameter Γ or s is calculated, then the non dimensional journal speed $\overline{\Omega}$ at the stability boundary is determined.

3. NONLINEAR ANALYSIS

To determine if stable or unstable limit cycles exist in the neighbourhood of the stability boundaries, a nonlinear analysis is applied using Hopf bifurcation theory.

A bifurcation is a qualitative change in the features of a system, such as the number and type of solutions, under the variation of the system parameters [1].

It has been shown that for a nonlinear system, a Hopf bifurcation must appear as the bifurcation from a fixed point to a limit cycles. The size, shape and stability of the limit cycles have to be determined.

Using Hopf bifurcation analysis, the regions of subcritical and supercritical stability can be predicted in teams of the non dimensional bearing parameter.

3.1. Application of Hopf Bifurcation Theory to a Rotor-Bearing System

The Hopf bifurcation theory (HBT) is concerned with the bifurcation of the periodic orbits from the equilibrium points of a system whose behaviour is described by the system of ordinary differential equation $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, \lambda)$.

Using Hopf bifurcation theory, one can investigate the existence of small amplitude periodic solutions of the equations (6, 8) which describe the motion of a rotor supported in fluid bearings. It is shown that the existence of stable limit cycles for rotor speed in excess of the threshold speed is confined to a specific region of parameter space. Outside this region, unstable limit cycles exist below the threshold speed [6].

Size and Stability of the Periodic Solutions of Journal Orbit

A Hopf bifurcation subroutine developed by Hassard et al. [2] to calculate the typical bifurcation parameters using HBT has been used. This theory is based on six parameters used to determine the shape, size and stability of the periodic solutions close to the bifurcation point.

These parameters are expressed as follow:

γ represents the parameter that gives the range of the existence of the periodic solutions of journal orbit. If $\gamma < 0$, periodic solutions exist for $\bar{\Omega} < \bar{\Omega}_s$; if $\gamma > 0$, periodic solutions exist for $\bar{\Omega} > \bar{\Omega}_s$.

τ is the coefficient in the expansion of the periods of periodic solutions $T(\bar{\Omega})$.

β is the leading coefficient in the expansion of the characteristic exponent $S_p(\bar{\Omega})$ which gives the stability of the periodic solution. If $\beta < 0$, the periodic solution is orbital asymptotically stable. If $\beta > 0$, the periodic solution is orbital-asymptotically unstable.

ω_0 is defined as $\beta(\bar{\Omega}_s)$.

The vector $\boldsymbol{V}$ contains the eigenvector of the Jacobian matrix at the stationary point when $\bar{\Omega} = \bar{\Omega}_s$. It corresponds to the eigenvalue $i\omega_0$.

The approximate periodic solutions can be expressed in the neighbourhood of $\bar{\Omega}_s$ using Hopf parameters [2].

$$\boldsymbol{x}(t,\bar{\Omega}) = \boldsymbol{x}_s(\bar{\Omega}_s) + \left(\frac{\bar{\Omega}-\bar{\Omega}_s}{\gamma}\right)^{0.5} \mathrm{Re}\left(e^{2\pi i t/T}\boldsymbol{V}\right) + \boldsymbol{O}\left(\bar{\Omega}-\bar{\Omega}_s\right) \tag{9}$$

where

$$\mathrm{T}(\bar{\Omega}) = \frac{2\pi}{\omega_0}\left[1 + \tau\left(\frac{\bar{\Omega}-\bar{\Omega}_s}{\gamma}\right) + \boldsymbol{O}\left(\bar{\Omega}-\bar{\Omega}_s\right)^2\right]$$

The characteristic exponent is expressed as:

$$S_p(\bar{\Omega}) = \beta\left(\frac{\bar{\Omega}-\bar{\Omega}_s}{\gamma}\right) + \boldsymbol{O}\left(\bar{\Omega}-\bar{\Omega}_s\right)^2$$

3.2. Infinitely short journal bearing

It is shown that for a journal speeds above a threshold speed; the system can exhibit supercritical bifurcations and so stable limit cycles. Unstable limit cycles however, exist for rotor speeds below the threshold speed.

The nonlinear parameters provided predict that for $\Gamma < 0.588$, there exist supercritical bifurcations for non dimensional speed $\bar{\Omega}$ greater than the critical value $\bar{\Omega}_s$. For $\Gamma \geq 0.588$, subcritical bifurcations exist for $\bar{\Omega}$ less than $\bar{\Omega}_s$ (Figure.4).

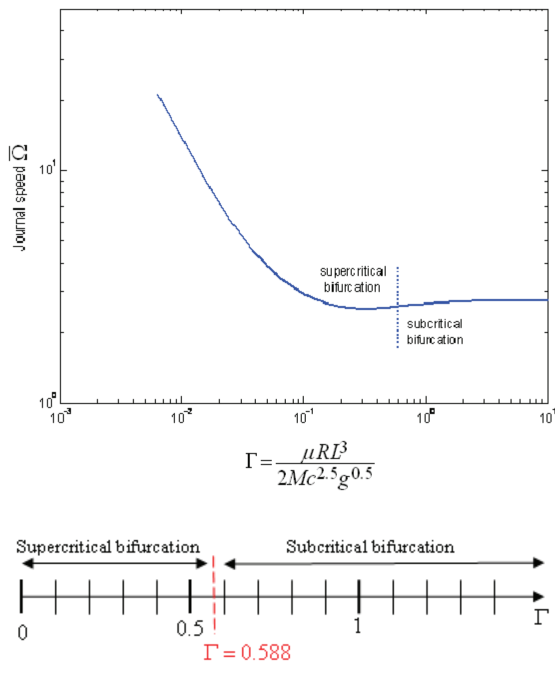

Fig. 4. Supercritical and subcritical bifurcations for short journal bearing

In order to describe the Hopf bifurcation behaviour of the journal bearing, Figure 5 illustrates the supercritical bifurcation profiles for several values of Γ. Stable solutions are shown by solid lines and unstable solutions by dashed lines.

For $\Gamma = 0.4$ with $\overline{\Omega} \le \overline{\Omega}_s$ $(=2.54)$ the rotor is stable at the position of steady-state eccentricity ratio ε_s. With $\overline{\Omega} > \overline{\Omega}_s$ the supercritical bifurcation appears close to the bifurcation point. Similar features are predicted for $\Gamma = 0.1$ and $\Gamma = 0.2$.

The amplitude of the periodic solution corresponding to a specific running speed $\overline{\Omega}$ is bounded by $\delta = \varepsilon_s \pm \sqrt{\dfrac{\left(\overline{\Omega} - \overline{\Omega}_s\right)}{\gamma}}$.

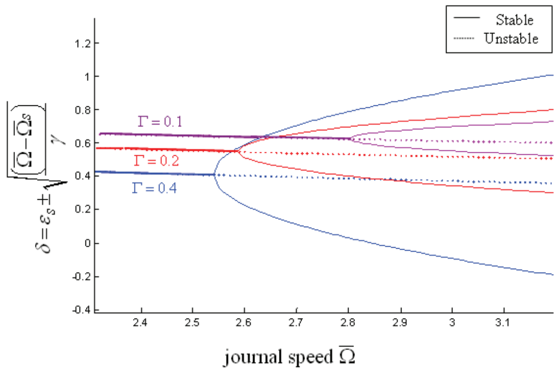

Fig. 5. Supercritical bifurcation profiles for several values of Γ

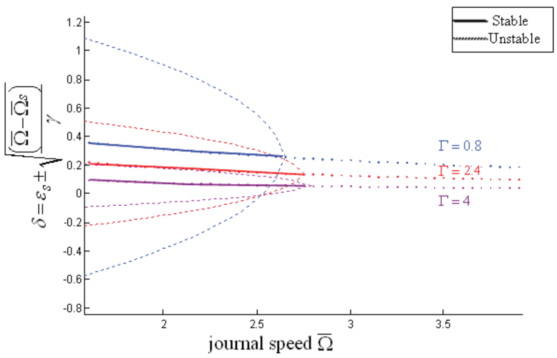

Fig. 6. Subcritical bifurcation profiles for several values of Γ

Figure 6 illustrates the subcritical bifurcation diagrams for several values of Γ. Stable solutions are shown by solid lines and unstable solutions by dashed lines. For $\Gamma = 0.8$ with $\overline{\Omega} > \overline{\Omega}_s$ $(=2.6)$, the equilibrium position of the rotor is unstable. There is unstable periodic solution for $\overline{\Omega} < \overline{\Omega}_s$; this is the subcritical behavior predicted by Hopf bifurcation theory. Similar features are observed for $\Gamma = 2.4$ and $\Gamma = 4$.

3.3. Infinitely long journal bearing

In this case, there are three separate regions of the parameter space s (Figure7). The Hopf bifurcation analysis, predicted that for $0 < s \le 0.05$ and $s > 0.79$ subcritical bifurcations appear for $\overline{\Omega} < \overline{\Omega}_s$ and therefore the bifurcated periodic orbit is unstable.

For $0.05 < s \le 0.79$, supercritical bifurcations occur for $\overline{\Omega} > \overline{\Omega}_s$. The bifurcated periodic orbits are stable.

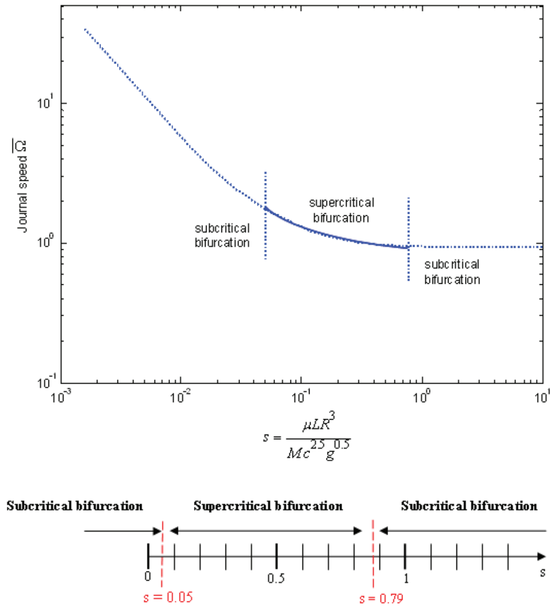

Fig. 7. Supercritical and subcritical bifurcations for long journal bearing

Figures (8-9) show the bifurcation profiles, which depict the amplitudes of the periodic solutions corresponding to running speeds close to the critical speed $\overline{\Omega}_s$ for several values of s.

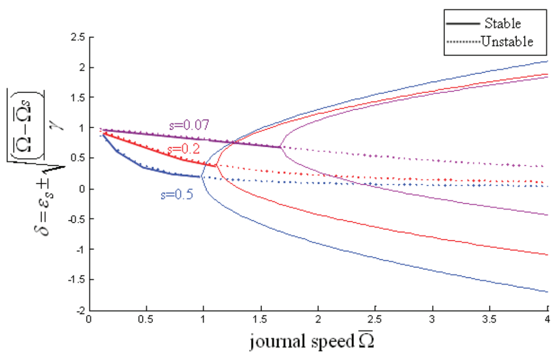

Fig. 8. Supercritical bifurcation profiles for several values of s

Figure. 8 shows the supercritical bifurcation diagrams.

Stable solutions are shown by solid lines and unstable solutions by dashed lines.

The supercritical bifurcation occurs close to the bifurcation point when $\bar{\Omega} > \bar{\Omega}_s$.

The predicted limit cycles grow as the value of s increases.

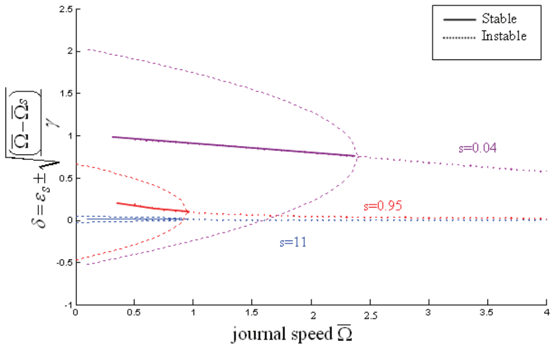

Fig. 9. Subcritical bifurcation profiles for several values of s

In order to describe the subcritical behavior, Figure 9 illustrates the subcritical profiles for several values of s. Below the threshold speed $\bar{\Omega}_s$, unstable limit cycles appear.

The predicted limit cycles grow as the value of s decreases.

4. NUMERICAL INTEGRATION

A numerical integration is carried out to verify the results obtained using HBT and to investigate how the whirl orbits develop at speeds well away from the threshold speed. The equations of motion are integrated using a variable-order Runge-Kutta method. The function ode45 is used to solve the system of ordinary differential equations (6) or (8) under Matlab.

Example 1: A short journal bearing

Consider the rotor-bearing system whose specifications are listed in Table 1 [5]. This rotor-bearing system consists of a rigid rotor symmetrically supported by two identical plain journal bearings. We apply Hopf bifurcation theory to determine its stability.

Table 1. Specification of the rotor-bearing system

$\mu(Pa.s)$	$c(m)$	$D(m)$	$L(m)$	$M(Kg)$
0.0212	0.125 10^{-3}	0.25	0.125	1936.8

This rotor-bearing has:

- A journal parameter: $\Gamma = 2.44$
- A threshold speed : 7360 *rpm*

The Hopf bifurcation parameters were obtained and the results are shown in Table 2.

Table 2. Hopf Bifurcation Parameters

$\bar{\Omega}_s$	γ	τ	β	$x_s(\bar{\Omega}_s)$	V
2.744	-19.95	0.14	2.27	0.093 1.452 0 0	1.0+0.0i 0.6-11i 0 +0.5i 5.52+0.31i

The value of the threshold speed calculated is similar to the value of $\bar{\Omega}_s$ determined by the linear analysis as shown in Fig.2.

In Table 2, $\gamma < 0$ and $\beta > 0$ unstable periodic solutions exist for $\bar{\Omega} < \bar{\Omega}_s = 2.744$.

To understand the behaviour of bearing instability, the journal is released from positions inside and outside the unstable periodic orbit for $\bar{\Omega} < \bar{\Omega}_s$.

Consider the non dimensional running speeds $\bar{\Omega} = 2$ which is less than $\bar{\Omega}_s$ $(=2.744)$.

The equations were integrated for two values of initial conditions. One is located inside the periodic limit orbit close to the equilibrium position. The other one is situated outside the periodic limit cycle (Figure 6).

Figures 10-11 represent the trajectory of the journal centre (the circle at $\varepsilon = 1$ is the clearance circle and represents the orbit of the journal centre when the journal surface is in contact with the bearing side).

According to the definition of the unstable periodic solution, for $\bar{\Omega} = 2 < \bar{\Omega}_s$ if the journal is released from a position inside the unstable periodic solution (Figure. 10), the system tends to asymptotically approach the steady state equilibrium position.
If the journal is released from a position outside the unstable periodic solution (Figure. 11), the orbit of the journal tends to become unstable.

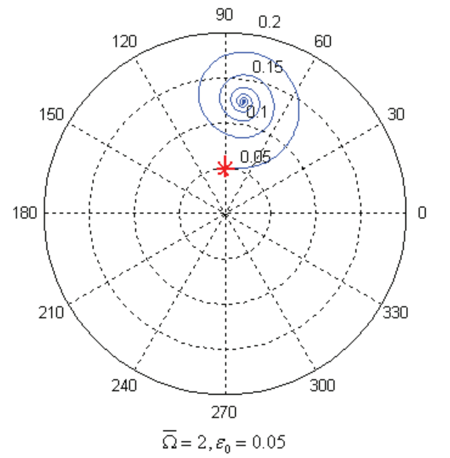

Fig. 10. The periodic solutions of the equations of motion for $\overline{\Omega} = 2, \varepsilon_0 = 0.05$

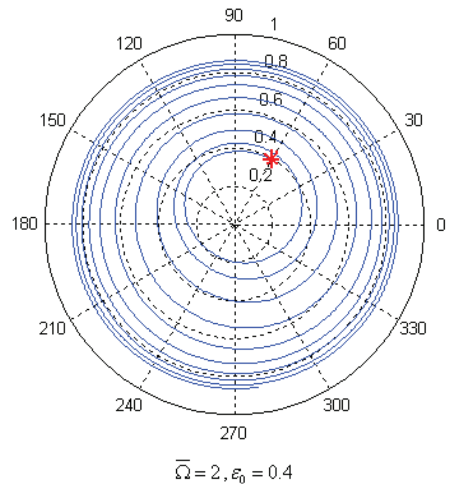

Fig. 11. The periodic solutions of the equations of motion for $\overline{\Omega} = 2, \varepsilon_0 = 0.4$

Example 2: A long journal bearing

Consider a bearing with a modulus $s = 0.5$.

In Table 3, $\gamma > 0$ *et* $\beta < 0$, then stable periodic solutions exist for $\overline{\Omega} > \overline{\Omega}_s = 0.98$.

Table 3. Hopf Bifurcation Parameters

$\overline{\Omega}_s$	γ	τ	β	$x_s(\overline{\Omega}_s)$	V
0.98	0.9	2.05	-1.12	0.21 1.43 0 0	1.0+0.0i 0.5-2.15i 0 +1.1i 2.38+5.9i

The value of the threshold speed calculated $\overline{\Omega}_s = 0.98$ is similar to the value determined by the linear analysis (Figure 3).

For rotor speeds immediately above $\overline{\Omega}_s$ a stable limit cycles appear, independent of the initial conditions (Figures 12-13).

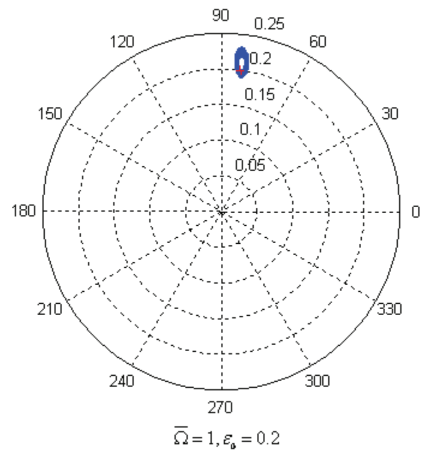

Fig. 12. The periodic solutions of the equations of motion for $\overline{\Omega} = 1, \varepsilon_0 = 0.2$

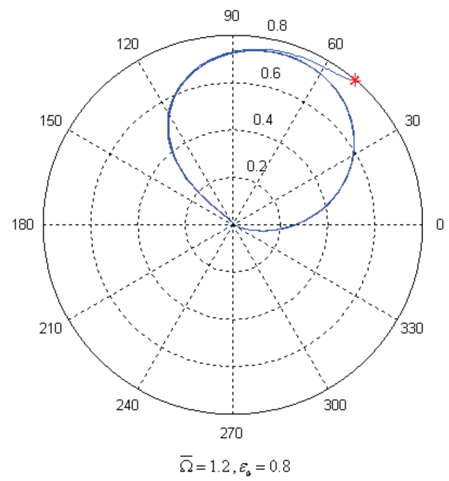

Fig. 13. The periodic solutions of the equations of motion for $\overline{\Omega} = 1.2, \varepsilon_0 = 0.8$

It is shown that the size of the limit cycles increase with rotor speed.

5. SUMMARY

The analysis of weakly nonlinear stability of both infinitely short and long journal bearings is presented.

Linear analysis may be used to determine the stability boundaries and the threshold speed for a particular bearing parameter.

Using the Hopf bifurcation theory, the nonlinear behaviours in the neighbourhood of the linear stability boundaries are predicted.

The onset of oil whirl for a rigid rotor bearing system supported by two identical hydrodynamic bearings is a bifurcation phenomenon. The existence of supercritical limit cycles or subcritical limit cycles can be established for journal speeds close to the threshold speed.

A numerical investigation supported the results of analytical analysis.

For the occurrence of supercritical bifurcation, a stable, small-amplitude whirl orbits appears when the rotor speed exceeds its threshold value and the size of the limit cycles increase with rotor speed.

When subcritical bifurcation occurs, unstable limit cycles with large amplitude are predicted at journal speed below its threshold speed.

For the practising engineer, it is interesting to choose specific operating parameters of a system in a manner that sustain subcritical bifurcation.

The effect of shaft flexibility on the stability boundaries and bifurcation regions is challenging future extension.

REFERENCES

[1] Dokhane A.: *BWR stability and bifurcation analysis using a novel reduced order model and the system code RAMONA*. Ph.D thesis, polytechnic Federal school of Lausanne, 2004.

[2] Hassard, B. D., Kazarinoff, N. D., Wan, Y. H.: *Theory and Applications of Hopf Bifurcation.* London Mathematical Society Lecture Notes 41, Cambridge University Press, New York, 1981.

[3] Hollis P., Taylor D. L.: *Hopf bifurcation to limit cycles in fluid film bearings*. ASME J Tribol, 1986, Vol. 108, pp. 184–9.

[4] Myers C. J.: *Bifurcation theory applied to oil whirl in plain cylindrical journal bearings.* ASME J Tribol, 1984, Vol. 5, pp. 244–9.

[5] Stolarski A.: *Tribology in Machine Design*. Butterworth-Heinemann, 2000.

[6] Wang J. K.: *On the stability of rotor bearing systems based on Hopf bifurcation.* Ph.D thesis, Faculty of the Louisiana State University, 2005.

[7] Wang J. K., Khonsari M .M.: *Application of Hopf bifurcation theory to rotor-bearing systems with consideration of turbulent effects.* ASME J Tribol, 2006, Vol. 39, pp. 701-14.