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PLANAR NONAUTONOMOUS POLYNOMIAL EQUATIONS V. THE ABEL EQUATION

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Abstract. We give a full description of the dynamics of the Abel equation $\dot{z} = z^3 + f(t)$ for some special complex valued f. We also prove the existence of at least three periodic solutions for equations of the form $\dot{z} = z^n + f(t)$ for odd $n \geq 5$.

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1. INTRODUCTION

The dynamics of the nonautonomous planar polynomial equation

$$\dot{z} = \sum_{j=0}^{n} a_j(t) z^j. \tag{1.1}$$

may be quite complicated (see e.g. [2,3,8]). The only exception is the Riccati equation ($n = 2$), where the Poincaré map is just a Möbius transformation [1]. The present paper is a continuation of [4–7] and is devoted to the full description of the dynamics of the Abel equation of the form

$$\dot{z} = v(t, z) = z^3 + f(t), \tag{1.2}$$

where $f \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ is T-periodic. We consider only the simplest case where the equation has three periodic solutions and every other solution is heteroclinic to the periodic ones or blows up.

The method we use is quite geometrical. We investigate the behaviour of the vector field on the boundary of some special sets. Then by the Denjoy–Wolff fixed point theorem we obtain the existence of a periodic solution which is asymptotically stable or asymptotically unstable. The shape of the sets has a great influence on the final result, i.e. the range of coefficient f which fullfils assumptions of obtained theorems. Thus it seems that presented results may be strengtened but probably with different methods.

We also consider the equation

$$\dot{z} = u(t,z) = z^n + f(t), \tag{1.3}$$

where $n > 3$ is odd. We adapt the method from the Abel case and obtain the existence of at least three periodic solutions. Unfortunately, the sectors we investigate do not cover the whole plain, so we are not able to describe the whole dynamics.

The paper is organised as follows. In Section 2 we give definitions and introduce notion. Later we investigate the Abel equation. Section 4 is devoted to the higher order polynomials.

2. DEFINITIONS

2.1. PROCESSES

Let X be a topological space and $\Omega \subset \mathbb{R} \times X \times \mathbb{R}$ be an open set.

By a *local process* on X we mean a continuous map $\varphi : \Omega \longrightarrow X$, such that three conditions are satisfied:

(i) $I_{(\sigma,x)} = \{t \in \mathbb{R} : \ (\sigma,x,t) \in \Omega\}$ is an open interval containing 0, for every $\sigma \in \mathbb{R}$ and $x \in X$,
(ii) $\varphi(\sigma,\cdot,0) = \mathrm{id}_X$, for every $\sigma \in \mathbb{R}$,
(iii) $\varphi(\sigma,x,s+t) = \varphi(\sigma+s,\varphi(\sigma,x,s),t)$, for every $x \in X$, $\sigma \in \mathbb{R}$ and $s,t \in \mathbb{R}$ such that $s \in I_{(\sigma,x)}$ and $t \in I_{(\sigma+s,\varphi(\sigma,x,s))}$.

For abbreviation, we write $\varphi_{(\sigma,t)}(x)$ instead of $\varphi(\sigma,x,t)$.

Let M be a smooth manifold and let $v : \mathbb{R} \times M \longrightarrow TM$ be a time-dependent vector field. We assume that v is so regular that for every $(t_0,x_0) \in \mathbb{R} \times M$ the Cauchy problem

$$\dot{x} = v(t,x), \tag{2.1}$$

$$x(t_0) = x_0 \tag{2.2}$$

has a unique solution. Then equation (2.1) generates a local process φ on X by $\varphi_{(t_0,t)}(x_0) = x(t_0,x_0,t+t_0)$, where $x(t_0,x_0,\cdot)$ is the solution of the Cauchy problem (2.1), (2.2).

Let T be a positive number. In the sequel T denotes the period. We assume that v is T-periodic in t. It follows that the local process φ is T-periodic, i.e.,

$$\varphi_{(\sigma+T,t)} = \varphi_{(\sigma,t)} \text{ for all } \sigma,t \in \mathbb{R},$$

hence there is a one-to-one correspondence between T-periodic solutions of (2.1) and fixed points of the Poincaré map $\varphi_{(0,T)}$.

2.2. BASIC NOTIONS

Let $g : M \longrightarrow M$ and $n \in \mathbb{N}$. We denote by g^n the n-th iterate of g, and by g^{-n} the n-th iterate of g^{-1} (if exists).

We say that the point z_0 is *attracting* (*repelling*) *for* g *in the set* $W \subset M$ if the equality $\lim_{n\to\infty} g^n(w) = z_0$ ($\lim_{n\to\infty} g^{-n}(w) = z_0$) holds for every $w \in W$ (i.e. every w is attracted (repelled) by z_0).

We call a T-periodic solution of (2.1) *attracting* (*repelling*) *in the set* $W \subset M$ if the corresponding fixed point of the Poincaré map $\varphi_{(0,T)}$ is attracting (repelling) in the set W.

Let $-\infty \le \alpha < \omega \le \infty$ and $s : (\alpha, \omega) \longrightarrow \mathbb{C}$ be a full solution of (1.1). We call s *forward blowing up* (shortly *f.b.*) or *backward blowing up* (*b.b.*) if $\omega < \infty$ or $\alpha > -\infty$, respectively. If $-\infty < \alpha < \omega < \infty$, then the solution is called *backward forward blowing up* (*b.f.b.*).

We define the sector

$$\mathcal{S}(\alpha, \beta) = \{z \in \mathbb{C} : \alpha < \mathrm{Arg}(z) < \beta\},$$

where $-\pi \le \alpha < \beta \le \pi$. Moreover, for $0 < \alpha \le \pi$ we define $\mathcal{S}(\alpha) = \mathcal{S}(-\alpha, \alpha)$ and $\widehat{\mathcal{S}}(\alpha)$ to be a set symmetric with respect to the origin to sector $\mathcal{S}(\alpha)$. Obviously, $0 \notin \mathcal{S}(\alpha, \beta)$.

Let us recall that the inner product of two vectors $a, b \in \mathbb{C}$ is given by the formula $\langle a, b\rangle = \mathfrak{Re}(a\bar{b}) = \mathfrak{Re}(\bar{a}b)$.

3. THE ABEL EQUATION

3.1. MAIN RESULT

In the present section we state the main theorem of the paper.

Theorem 3.1. *Let $T > 0$ and $f \in \mathcal{C}(\mathbb{R}, \mathbb{C})$, $f \not\equiv 0$ be T-periodic. We write*

$$R = \min\{-\mathfrak{Re}(f(t)) : t \in \mathbb{R}\},$$
$$M = \max\{|\mathfrak{Im}(f(t))| : t \in \mathbb{R}\}.$$

If

$$M \le \frac{1 + \sqrt{33}}{6\sqrt{6 + \sqrt{33}}} R, \tag{3.1}$$

then equation (1.2) *has three T-periodic solutions. One of them is asymptotically unstable and two other are asymptotically stable. Every other solution is heteroclinic between them or blows up. There are no b.f.b. solutions.*

Proof. By (3.1), one gets $f(\mathbb{R}) \subset \widehat{\mathcal{S}}\left(\frac{\pi}{2}\right)$. Thus, by [7, Theorem 4], the equation (1.2) has exactly one T-periodic solution ξ in $\mathcal{S}\left(\frac{\pi}{2}\right)$. It is asymptotically unstable and repelling in the whole $\mathcal{S}\left(\frac{\pi}{2}\right)$. Every other solution staying inside $\mathcal{S}\left(\frac{\pi}{2}\right)$ for some time is f.b. solution or leaves this set through the imaginary axis and enters the set $\widehat{\mathcal{S}}\left(\frac{\pi}{2}\right)$.

To finish the proof, it is enough to show that:

— there is exactly one T-periodic solution χ inside the set $\mathcal{S}\left(\frac{\pi}{2}, \pi\right)$, it is asymptotically stable,
— there is exactly one T-periodic solution ν inside the set $\mathcal{S}\left(-\pi, -\frac{\pi}{2}\right)$, it is asymptotically stable,
— every other solution staying inside $\widehat{\mathcal{S}}\left(\frac{\pi}{2}\right)$ for some time is attracted by χ or ν or is f.b.,
— every solution which is attracted by χ or ν is heteroclinic between ξ and χ or between ξ and ν, respectively, or is a b.b. solution.

We do that in two steps. First of all we define two subsets W and Z of $\widehat{\mathcal{S}}\left(\frac{\pi}{2}\right)$ which contains asymptotically stable T-periodic solutions. Then we investigate the behaviour of the vector field v outside W and Z.

I. We assume that $f(t) \in \mathbb{C} \setminus \mathbb{R}$ for some $t \in \mathbb{R}$, i.e.

$$M > 0 \tag{3.2}$$

holds. Write

$$\begin{aligned} k =& \sqrt{6+\sqrt{33}}, \\ p =& \frac{7+\sqrt{33}}{4}. \end{aligned}$$

Thus there exists $c > 0$ such that

$$M = c^3(3k^2 - 1) \tag{3.3}$$

holds.

Let $\gamma > 0$ and V be a triangle with vertex in points $-kc + ic$, ipc and the origin (see Figure 1). We define sets W and Z by

$$W_\gamma = \mathrm{cl}\left(\left\{z \in \mathbb{C} : -\mathfrak{Im}\left(\frac{1}{z}\right) \le \gamma, \mathfrak{Im}(z) \ge c, \mathfrak{Re}(z) \le 0\right\} \setminus V\right), \tag{3.4}$$

$$Z_\gamma = \{z \in \mathbb{C} : \overline{z} \in W_\gamma\}, \tag{3.5}$$

$$W = \bigcup_{\gamma>0} W_\gamma = \mathrm{cl}\left(\{z \in \mathbb{C} : \mathfrak{Re}[z] \le 0, \mathfrak{Im}[z] \ge c\} \setminus V\right), \tag{3.6}$$

$$Z = \{z \in \mathbb{C} : \overline{z} \in W\}. \tag{3.7}$$

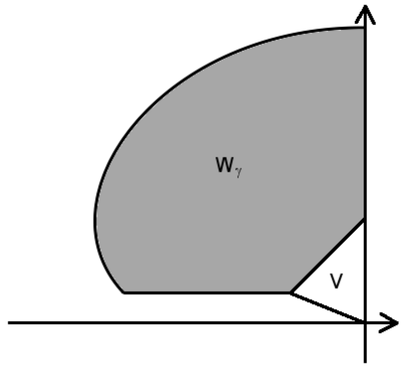

Fig. 1. The set W_γ is marked in grey

We show that the vector field v points inward W_γ or is tangent to its boundary at every point of ∂W_γ provided that γ is small enough. To do this, let us divide ∂W_γ into four parts

$$\begin{aligned}\Gamma_1 &= \partial W_\gamma \cap \left\{ z \in W : -\mathfrak{Im}\left(\frac{1}{z}\right) = \gamma \right\},\\ \Gamma_2 &= \partial W_\gamma \cap \{ z \in W : \mathfrak{Re}(z) = 0 \},\\ \Gamma_3 &= \partial W_\gamma \cap V,\\ \Gamma_4 &= \partial W_\gamma \cap \{ z \in W : \mathfrak{Im}(z) = c \}.\end{aligned}$$

For γ small enough, the term z^3 is the dominating term in v at every point of Γ_1, so by calculations similar to the ones from the proof of [7, Theorem 4], the vector field v points inwards W_γ in every point of Γ_1.

By (3.1), for every z such that $\mathfrak{Re}(z) = 0$ one gets

$$\mathfrak{Re}[v(t,z)] = \mathfrak{Re}[f(t)] < 0,$$

so the vector field v points inwards W_γ in every point of Γ_2.

Now we parameterize Γ_3 by

$$s_3 : [c, pc] \ni o \mapsto -k\frac{pc - o}{p-1} + io.$$

An outward orthogonal vector to W_γ is given by $n_3 = 1 - \frac{k}{p-1}i$. Since

$$\mathfrak{Re}\left[\left(-k\frac{pc-o}{p-1} + io\right)^3 \left(1 + \frac{k}{p-1}i\right)\right] \le \mathfrak{Re}\left[s_3^3(pc)\left(1 + \frac{k}{p-1}i\right)\right] = \frac{kp^3c^3}{p-1}$$

holds, thus, by (3.3) and (3.1), one gets

$$\begin{aligned}\langle v(t,s_3(o)),n_3(o)\rangle &= \mathfrak{Re}\left[\left(\left(-k\frac{pc-o}{p-1}+io\right)^3+f(t)\right)\left(1+\frac{k}{p-1}i\right)\right]\leq\\ &\leq \mathfrak{Re}\left[\left(-k\frac{pc-o}{p-1}+io\right)^3\left(1+\frac{k}{p-1}i\right)\right]+\\ &\quad+\mathfrak{Re}\left[f(t)\left(1+\frac{k}{p-1}i\right)\right]\leq\\ &\leq \frac{kp^3c^3}{p-1}-R+M\frac{k}{p-1}=\\ &=-R+\frac{Mk}{p-1}\left(\frac{p^3c^3}{c^3\left(3k^2-1\right)}+1\right)=\\ &=-R+Mk\frac{p^3+3k^2-1}{(p-1)\left(3k^2-1\right)}=0.\end{aligned}$$

It is easy to observe that the inequality may hold only in the point $ipc = s_3(pc)$. In every other point of Γ_3 the inequality is strict.

Let us now define

$$s_4 : [-\rho_{\gamma,c}, -kc] \ni o \mapsto o + ic,$$

where $\rho_{\gamma,c}$ is such that s_4 is a parameterization of Γ_4. An outward orthogonal vector to W_γ is given by $n_4 = -i$. Now, by (3.3), one gets

$$\begin{aligned}\langle v(t,s_4(o)),n_4(o)\rangle &= \mathfrak{Re}\left[i(o+ic)^3+if(t)\right]=\mathfrak{Re}\left[-3o^2c+c^3+if(t)\right]\leq\\ &\leq c^3\left(1-3k^2\right)+|\mathfrak{Im}[f(t)]|=c^3\left(1-3k^2\right)+M=0.\end{aligned}$$

It is easy to observe that the equality may hold only in the point $-kc+ic = s_4(-kc)$. In every other point of Γ_4 the inequality is strict.

Finally, by the Denjoy–Wolff fixed point theorem, there exists exactly one T-periodic solution inside the set W_γ. It is asymptotically stable and attracting in the whole W_γ. Since $W_\gamma \subset W_\rho$ for every $0 < \rho \leq \gamma$, there exists exactly one T-periodic solution χ inside the set W. It is asymptotically stable and attracting in the whole W.

By the symmetry of the vector field z^3, one may modify the above calculations and obtain the existence of exactly one T-periodic solution ν inside the set Z. The solution is asymptotically stable and attracting in the whole Z.

Now we show that every solution starting in the set $\widehat{\mathbb{S}}\left(\frac{\pi}{2}\right) \setminus (W \cup Z)$ is attracted by χ or ν or is f.b. Write

$$\hat{V} = \{z \in \mathbb{C} : \overline{z} \in V\}$$

and

$$K = \mathrm{cl}\left(\widehat{\mathbb{S}}\left(\frac{\pi}{2}\right) \setminus \left(W \cup Z \cup V \cup \hat{V}\right)\right).$$

Every solution starting in V enters K or W. Indeed, let

$$\hat{s}_a : [c, pc] \ni o \mapsto a\left(-k\frac{pc-o}{p-1}+io\right)$$

for $a \in [0,1]$, then

$$V = \bigcup_{a\in[0,1]} \hat{s}_a([c,pc]).$$

A solution starting in $\hat{s}_a((c,pc))$ for some $a \in [0,1]$ points towards $\hat{s}_b([c,pc])$ for $b > a$. To see this let us observe that for the orthogonal vector to $\hat{s}_a((c,pc))$ given by $n_a = 1 - \frac{k}{p-1}i$ one gets

$$\mathfrak{Re}\left[\hat{s}_a^3(o)\left(1+\frac{k}{p-1}i\right)\right] < \mathfrak{Re}\left[\hat{s}_a^3(pc)\left(1+\frac{k}{p-1}i\right)\right] = \frac{ka^3p^3c^3}{p-1},$$

thus, by calculations similar to the ones for s_3, one obtains

$$\langle v(t,\hat{s}_a(o)), n_a\rangle = \mathfrak{Re}\left[\left(a^3\left(-k\frac{pc-o}{p-1}+io\right)^3 + f(t)\right)\left(1+\frac{k}{p-1}i\right)\right] \leq$$
$$\leq -R + Mk\frac{a^3p^3+3k^2-1}{(p-1)(3k^2-1)} < 0.$$

Similarly, every solution starting in $\hat{V}$ enters K or Z.

By the definition of K and (3.1), for every solution η the inequality

$$\frac{d}{dt}\mathfrak{Re}[\eta(t)] < 0$$

holds, provided that $\eta(t) \in K$. To see this let us observe that

$$\frac{d}{dt}\mathfrak{Re}[\eta(t)] = \mathfrak{Re}[v(t,\eta(t))] = \mathfrak{Re}\left[\eta^3(t)\right] + \mathfrak{Re}[f(t)] < 0$$

is satisfied. Finally, every solution starting in K enters W, Z, V or $\hat{V}$ but its real part decreases as long as it stays in K. Moreover, when $\mathfrak{Re}[\eta(t)]$ is small enough, the solution behaves qualitatively the same as in the case of the unperturbed vector field $v(t,z) = z^3$. Thus, by the Ważewski method (see the proof of [7, Theorem 4]), η enters W or Z or is f.b., so is attracted by χ or ν or is f.b.

To finish the proof let us observe, that by (3.1), every solution passing through the imaginary axis is heteroclinic between ξ and χ or between ξ and ν or is being repelled by ξ and is f.b.

Analysing vector field v in the cone $\mathcal{S}\left(\frac{\pi}{2}-\epsilon, \frac{\pi}{2}+\epsilon\right)$ one gets the existence of b.b. solutions (by the time reversing symmetry, the dynamics is similar to the one in the set K). Since, for the unperturbed equation (i.e. $v(t,z) = z^3$) these solutions stays in imaginary axis, the perturbation f, pushes them inside the set W, so finally, they are attracted by χ.

II. Now $M = 0$ and the real axis is invariant. Let $\mathfrak{Re}[f(t)] < 0$ for every $t \in \mathbb{R}$. We fix $c > 0$ and follow the construction from point I. If c is small enough, then f is a dominating term on Γ_3, so vector fields point inside W at every point of its boundary. Moreover f is a dominating term in the set V, so the qualitative behaviour of solutions is the same as in the point I.

If $f(t) = 0$ for some $t \in \mathbb{R}$, then it is still possible, to fix $c > 0$ so small that $\varphi_{(0,T)}(\Gamma_3) \subset W$ and $\varphi_{(0,T)}(V) \subset K \cup V \cup W$, so, qualitatively, the situation is the same as above. □

3.2. FURTHER REMARKS

Let us see some applications of Theorem 3.1.

Example 3.2. The equation

$$\dot{z} = z^3 - 10 + e^{it}$$

has three 2π-periodic solutions. One of them is asymptotically unstable (it is contained in $\mathcal{S}\left(\frac{\pi}{2}\right)$) and two other are asymptotically stable (one is contained in $\mathcal{S}\left(\frac{\pi}{2}, \pi\right)$ and the other in $\mathcal{S}\left(-\pi, -\frac{\pi}{2}\right)$). Every other solution is heteroclinic between them or blows up.

Example 3.3. The equation

$$\dot{z} = z^3 + 10 + i\sin(t)$$

has three 2π-periodic solutions. One of them is asymptotically unstable (it is contained in $\widehat{\mathcal{S}}\left(\frac{\pi}{2}\right)$) and two other are asymptotically stable (one is contained in $\mathcal{S}\left(0, \frac{\pi}{2}\right)$ and the other in $\mathcal{S}\left(-\frac{\pi}{2}, 0\right)$). Every other solution is heteroclinic between them or blows up. To see this, one needs to use the change of variables

$$w = -z. \tag{3.8}$$

Using the change of variables (3.8) one can obtain the following corollary.

Corollary 3.4. *Let $T > 0$, $P \in \mathbb{R}$ and $f \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ be T-periodic. Then the equation*

$$\dot{z} = z^3 + f(t) + P$$

has three 2π-periodic solutions provided that $|P|$ is big enough. One of them is asymptotically unstable and two other are asymptotically stable. Every other solution is heteroclinic between them or blows up.

Example 3.5. The equation

$$\dot{z} = z^3 + 4i + e^{it}$$

has three 2π-periodic solutions. One of them is asymptotically stable (it is contained in $\mathcal{S}(0, \pi)$) and two other are asymptotically unstable (one is contained in $\mathcal{S}\left(-\frac{\pi}{2}, 0\right)$ and the other in $\mathcal{S}\left(-\pi, -\frac{\pi}{2}\right)$). Every other solution is heteroclinic between them or blows up. To see this, one needs to reverse the time, i.e. use the change of variables given by

$$x(t) = z(-t). \tag{3.9}$$

Combining changes of variables (3.8) and (3.9), one can prove the following corollary.

Corollary 3.6. *Let $T > 0$, $P \in \mathbb{R}$ and $f \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ be T-periodic. Then the equation*

$$\dot{z} = z^3 + f(t) + iP$$

has three 2π-periodic solutions provided that $|P|$ is big enough. One of them is asymptotically stable and two other are asymptotically unstable. Every other solution is heteroclinic between them or blows up.

By (3.1), Theorem 3.1 may be applied to some f satisfying the inclusion

$$f(\mathbb{R}) \subset \operatorname{cl} \widehat{\mathcal{S}} \left(\arctan \left(\frac{1+\sqrt{33}}{6\sqrt{6+\sqrt{33}}} \right) \right). \tag{3.10}$$

Since $\frac{1+\sqrt{33}}{6\sqrt{6+\sqrt{33}}} \approx 0,328$, the above sector seems to be quite narrow. On the other hand, the condition (3.1) depends heavily on the shape of the set W.

This leads to the following open problem.

Open Problem 3.7. What is the infimum over $\alpha \geq 0$ such that the equation (1.2) with $f \in \widehat{\mathcal{S}}(\alpha)$, has dynamics different from the one presented in Theorem 3.1?

The following example shows that the infimum may be at most $\frac{\pi}{4}$.

Example 3.8. The equation

$$\dot{z} = z^3 - e^{-i\frac{\pi}{4}}$$

has only two isolated periodic (constant) solutions $z \equiv e^{i\frac{7}{12}\pi}$ and $z \equiv e^{-i\frac{\pi}{12}}$. Moreover, it has a centre at $e^{-i\frac{3}{4}\pi}$.

4. HIGHER DEGREE POLYNOMIALS

The method presented in the previous section may be applied to equation (1.3). In this case, one may obtain the existence of three periodic solutions, but the full description of dynamics is impossible, because considered sectors do not cover the whole plain.

The following statement is the main theorem in the present section.

Theorem 4.1. *Let $n > 3$ be odd, $m \in \{0, 1, \ldots, n-1\}$, $P \in \mathbb{R}$, $T > 0$ and $f \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ be T-periodic. Then the equation*

$$\dot{z} = z^n + f(t) + Pe^{i\frac{\pi}{n-1}m} \tag{4.1}$$

has at least three T-periodic solutions, provided that $|P|$ is big enough. If m is even, then at least one periodic solution is asymptotically unstable and at least two of them are asymptotically stable. If m is odd, then at least one periodic solution is asymptotically stable and at least two of them are asymptotically unstable. There are heteroclinic solutions between unstable and stable ones.

Example 4.2. By Theorem 4.1, the equation

$$\dot{z} = z^9 + e^{it} + P$$

has at least three 2π-periodic solutions, provided that $|P|$ is big enough. One periodic solution is asymptotically unstable and the other two are asymptotically stable.

Example 4.3. By Theorem 4.1, the equation

$$\dot{z} = z^7 + e^{it} + Pe^{i\frac{\pi}{6}}$$

has at least three 2π-periodic solutions, provided that $|P|$ is big enough. One periodic solution is asymptotically stable and the other two are asymptotically unstable.

Theorem 4.1 is a straightforward consequence of the following technical lemma and changes of variables of the form $x = ze^{i\frac{\pi}{n-1}m}$ and (3.9).

Lemma 4.4. *Let $n > 3$ be odd, $T > 0$ and $f \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ be T-periodic. We write*

$$\begin{aligned} R &= \min\left\{-\mathfrak{Re}(f(t)) : t \in \mathbb{R}\right\}, \\ M &= \max\left\{|\mathfrak{Im}(f(t))| : t \in \mathbb{R}\right\}, \\ H(n) &= \min\left\{H(n,k,p) : p > 1, k \geq \cot\left(\frac{\pi}{n}\right)\right\}, \end{aligned}$$

where

$$H(n,k,p) = \frac{1}{p-1}\left[\left|p\cot\left(\frac{\pi}{n-1}\right) - k\right| + \frac{p^n\left[k\sin\left(\frac{\pi}{n-1}\right) - \cos\left(\frac{\pi}{n-1}\right)\right]}{\sin^n\left(\frac{\pi}{n-1}\right)\mathfrak{Im}\left[(i-k)^n\right]}\right].$$

If

$$MH(n) \leq R, \tag{4.2}$$

$$M < R\tan\left(\frac{\pi}{n-1}\right) \tag{4.3}$$

hold then equation (1.3) has:

— *exactly one T-periodic solution inside $\mathcal{S}\left(\frac{\pi}{n-1}\right)$ – it is asymptotically unstable and repelling in the whole $\mathcal{S}\left(\frac{\pi}{n-1}\right)$,*

— *infinitely many f.b. solutions inside $\mathcal{S}\left(\frac{\pi}{n-1}\right)$,*

— *exactly two T-periodic solutions inside $\widehat{\mathcal{S}}\left(\frac{2\pi}{n-1}\right)$ – they are asymptotically stable – one of them is contained in the sector $\mathcal{S}\left(\frac{n-3}{n-1}\pi, \pi\right)$ and the other in $\mathcal{S}\left(-\pi, -\frac{n-3}{n-1}\pi\right)$,*

— *infinitely many b.b. solutions inside $\mathcal{S}\left(\frac{n-3}{n-1}\pi, \pi\right)$,*

— *infinitely many b.b. solutions inside* $\mathcal{S}\left(-\pi, -\frac{n-3}{n-1}\pi\right)$,

— *infinitely many f.b. solutions inside* $\widehat{\mathcal{S}}\left(\frac{2\pi}{n-1}\right)$.

Moreover, every solution starting in $\widehat{\mathcal{S}}\left(\frac{2\pi}{n-1}\right)$ *is either a f.b. solution or is attracted by an asymptotically stable periodic one.*

Proof. The concept of the proof is similar to the one from the proof of Theorem 3.1. The differences comes from the fact that now the considered sectors are narrower then previously.

Let us fix n. By (4.3), one gets $f(\mathbb{R}) \subset \widehat{\mathcal{S}}\left(\frac{\pi}{n-1}\right)$. Thus, by [7, Theorem 4], equation (1.3) has exactly one T-periodic solution ξ in $\mathcal{S}\left(\frac{\pi}{n-1}\right)$. It is asymptotically unstable and repelling in the whole $\mathcal{S}(\frac{\pi}{n-1})$. Every other solution staying inside $\mathcal{S}(\frac{\pi}{n-1})$ for some time is a f.b. solution or leaves the set through its boundary.

Now we need to investigate the dynamics in $\widehat{\mathcal{S}}(\frac{2\pi}{n-1})$. By the symmetries of the term z^n we do it in $\mathcal{S}(\frac{n-3}{n-1}\pi, \pi)$. Qualitatively, the situation in $\mathcal{S}(-\pi, -\frac{n-3}{n-1}\pi)$ is the same.

Let $M > 0$ (the case $M = 0$ is similar to the one in the proof of Theorem 3.1). In the sequel we write k, p (where $p > 1$, $k \geq \cot\left(\frac{\pi}{n}\right)$) to denote numbers such that $H(n) = H(n, k, p)$ is satisfied. Let $c > 0$ be such that

$$M = c^n \mathfrak{Im}\left[(i-k)^n\right] \tag{4.4}$$

holds.

The crucial point is to define the set $W_{\gamma,\epsilon} \subset \mathcal{S}\left(\frac{n-3}{n-1}\pi, \pi\right)$ such that the vector field u points inwards at every point of its boundary. Write

$$\begin{aligned}
W_{\gamma,\epsilon} &= \mathrm{cl}\left(\left\{z \in \mathbb{C} : \mathrm{Arg}[z] \geq \frac{n-3}{n-1}\pi + \epsilon, -\mathfrak{Im}\left(z^{\frac{1-n}{2}}\right) \leq \gamma, \mathfrak{Im}(z) \geq c\right\} \setminus V\right), \\
Z_{\gamma,\epsilon} &= \left\{z \in \mathbb{C} : \overline{z} \in W_{\gamma,\epsilon}\right\}, \\
W &= \mathrm{cl} \bigcup_{\gamma>0,\epsilon>0} W_{\gamma,\epsilon} = \mathrm{cl}\left(\left\{z \in \mathbb{C} : \mathrm{Arg}[z] \geq \frac{n-3}{n-1}\pi, \mathfrak{Im}[z] \geq c\right\} \setminus V\right), \\
Z &= \left\{z \in \mathbb{C} : \overline{z} \in W\right\},
\end{aligned}$$

where V is the triangle with vertices at points $V_0 = 0$, $V_1 = -kc + ic$ and $V_2 = \left[i - \cot\left(\frac{2\pi}{n-1}\right)\right] \frac{pc\left[k - \cot\left(\frac{\pi}{n-1}\right)\right]}{(p-1)\cot\left(\frac{2\pi}{n-1}\right) + k - p\cot\left(\frac{\pi}{n-1}\right)}$ (see Figure 2).

We show that in every point of the boundary of $W_{\gamma,\epsilon}$ the vector field u points inwards $W_{\gamma,\epsilon}$ or is tangent to the boundary, provided that $\gamma > 0$ is small enough and $\epsilon \in \left(0, \frac{\pi}{n-1}\right)$. To do this, let us divide ∂W_γ into four parts

$$\Gamma_1 = \partial W_{\gamma,\epsilon} \cap \left\{ z \in W : -\mathfrak{Im}\left(z^{\frac{1-n}{2}}\right) = \gamma \right\},$$
$$\Gamma_2 = \partial W_{\gamma,\epsilon} \cap \left\{ z \in W : \mathrm{Arg}[z] = \frac{n-3}{n-1}\pi + \epsilon \right\},$$
$$\Gamma_3 = \partial W_{\gamma,\epsilon} \cap V,$$
$$\Gamma_4 = \partial W_{\gamma,\epsilon} \cap \{ z \in W : \mathfrak{Im}(z) = c \}.$$

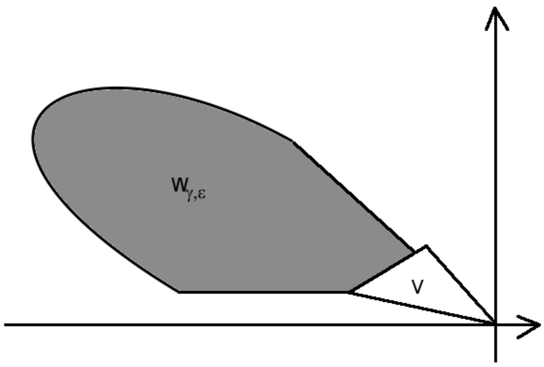

Fig. 2. The set $W_{\gamma,\epsilon}$ is marked in grey

For any $\epsilon \in \left(0, \frac{\pi}{n-1}\right)$ there exists γ small enough such that the term z^n is the dominating term in u at every point of Γ_1, so by calculations similar to the ones from the proof of [7, Theorem 4], the vector field v points inwards $W_{\gamma,\epsilon}$ in every point of Γ_1.

For every $z \in \Gamma_2$ an outward orthogonal vector to $W_{\gamma,\epsilon}$ at z is given by $n_2 = -ie^{i\frac{n-3}{n-1}\pi + i\epsilon}$. Since $z = |z|e^{i\frac{n-3}{n-1}\pi + i\epsilon}$, then, by (4.3), one gets

$$\begin{aligned}\mathfrak{Re}\left[\overline{n}_2 u(t,z)\right] &= \mathfrak{Re}\left[ie^{-i\frac{n-3}{n-1}\pi - i\epsilon}\left(|z|^n e^{in\frac{n-3}{n-1}\pi + in\epsilon} + f(t)\right)\right] = \\ &= |z|^n \mathfrak{Re}\left[ie^{i(n-1)\frac{n-3}{n-1}\pi + i(n-1)\epsilon}\right] + \mathfrak{Re}\left[ie^{-i\frac{n-3}{n-1}\pi - i\epsilon} f(t)\right] \leq \\ &\leq 0 + |f(t)|\mathfrak{Re}\left[ie^{i\frac{2}{n-1}\pi + i\,\mathrm{Arg}[-f(t)]}\right] < 0,\end{aligned}$$

so the vector field u points inwards $W_{\gamma,\epsilon}$ in every point of Γ_2.

Now we parameterize Γ_3 by

$$s_3 : [c, \rho_{\epsilon,c}] \ni o \mapsto io - \frac{(pc - o)k + (o - c)p\cot\left(\frac{\pi}{n-1}\right)}{p-1},$$

here $\rho_{\epsilon,c}$ is such that s_3 is a parameterization of a whole of Γ_3. An outward orthogonal vector to $W_{\gamma,\epsilon}$ is given by $n_3 = 1 + i\frac{p\cot(\frac{\pi}{n-1})-k}{p-1}$. Since, for any $o \in [c, \rho_{\epsilon,c}]$ the inequality

$$\mathfrak{Re}\left[(s_3(o))^n\,\overline{n}_3\right] < \mathfrak{Re}\left[s_3^n(pc)\overline{n}_3\right] = \frac{p^n c^n}{\sin^n\left(\frac{\pi}{n-1}\right)} \frac{k\sin\left(\frac{\pi}{n-1}\right) - \cos\left(\frac{\pi}{n-1}\right)}{p-1}$$

holds, thus, by (4.4) and (4.2), one gets

$$\begin{aligned}\langle u(t, s_3(o)), n_3\rangle =& \mathfrak{Re}\left[\left((s_3(o))^n + f(t)\right)\overline{n}_3\right] < \\ &< \frac{p^n c^n}{\sin^n\left(\frac{\pi}{n-1}\right)} \frac{k\sin\left(\frac{\pi}{n-1}\right) - \cos\left(\frac{\pi}{n-1}\right)}{p-1} + \\ &+ \mathfrak{Re}\left[f(t)\left(1 - i\frac{p\cot\left(\frac{\pi}{n-1}\right) - k}{p-1}\right)\right] \leq \\ &\leq \frac{p^n c^n}{\sin^n\left(\frac{\pi}{n-1}\right)} \frac{k\sin\left(\frac{\pi}{n-1}\right) - \cos\left(\frac{\pi}{n-1}\right)}{p-1} - \\ &- R + \frac{M}{p-1}\left|p\cot\left(\frac{\pi}{n-1}\right) - k\right| = \\ &= -R + MH(n) \leq 0.\end{aligned}$$

Let now define

$$s_4 : [-\rho_{\gamma,c}, -kc] \ni o \mapsto o + ic,$$

where $\rho_{\gamma,c}$ is such that s_4 is a parameterization of Γ_4. An outward orthogonal vector to W_γ is given by $n_4 = -i$. Now, by $k \geq \cot\left(\frac{\pi}{n}\right)$ and (4.4), one gets

$$\begin{aligned}\langle u(t, s_4(o)), n_4(o)\rangle &= \mathfrak{Re}\left[i(o + ic)^n + if(t)\right] \leq \\ &\leq \mathfrak{Re}\left[i(-kc + ic)^n + if(t)\right] = -c^n\mathfrak{Im}\left[(i-k)^n\right] + |\mathfrak{Im}[f(t)]| = 0.\end{aligned}$$

It is easy to observe that the equality may hold only in the point $-kc + ic = s_4(-kc)$. In every other point of Γ_4 the inequality is strict.

Finally, by the Denjoy–Wolff fixed point theorem, there exists exactly one T-periodic solution inside the set $W_{\gamma,\epsilon}$. It is asymptotically stable and attracting in the whole of $W_{\gamma,\epsilon}$. Since $W_{\gamma,\epsilon} \subset W_{\rho,\delta}$ for every $0 < \rho \leq \gamma$, $0 < \delta \leq \epsilon$ there exists exactly one T-periodic solution χ inside the set W. It is asymptotically stable and attracting in the whole W.

By the symmetry of the vector field z^n, one may modify the above calculations and obtain the existence of exactly one T-periodic solution ν inside the set Z. The solution is asymptotically stable and attracting in the whole of Z.

To show that every solution starting in the set $\widehat{\mathcal{S}}\left(\frac{2\pi}{n-1}\right) \setminus (W \cup Z)$ is attracted by χ or ν or is a f.b. it is enough to follow the method from the proof of Theorem 3.1.

Existence of a b.b. solution may be obtained by analysing the dynamics in the sectors $\mathcal{S}\left(\frac{n-2}{n-1}\pi - \delta, \frac{n-2}{n-1}\pi + \delta\right)$ and $\mathcal{S}\left(-\frac{n-2}{n-1}\pi + \delta, -\frac{n-2}{n-1}\pi - \delta\right)$ for some small positive δ. □

Remark 4.5. Since $n > 3$ is odd, the set $\widehat{\mathcal{S}}\left(\frac{2\pi}{n-1}\right)$ is contained in $\{z \in \mathbb{C} : \mathfrak{Re}[z] < 0\}$. It is not true for $n = 3$, so a different proof for Theorem 3.1 is needed.

Lemma 4.4 may be used directly to investigate the dynamics of the equations. The main difficulty is the calculation of $H(n)$. Let us observe that

$$H(5,k,p) = \frac{1}{p-1}\left[|p-k| + \frac{4p^5(k-1)}{5k^4 - 10k^2 + 1}\right]$$

holds. Thus the numerical calculations give

$$H(5) = H(5, k_0, p_0) < 1.66, \quad k_0 \approx 4.55, \quad p_0 \approx 2.92.$$

Example 4.6. By Lemma 4.4, the equation

$$\dot{z} = z^5 - 1 + \frac{3}{5} i \sin(t)$$

has at least three 2π-periodic solutions. One periodic solution is asymptotically unstable and the other two are asymptotically stable.

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