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GENERATING THE EXPONENTIALLY STABLE C_0-SEMIGROUP IN A NONHOMOGENEOUS STRING EQUATION WITH DAMPING AT THE END

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Abstract. Small vibrations of a nonhomogeneous string of length one with left end fixed and right one moving with damping are described by the one-dimensional wave equation

$$\begin{cases} v_{tt}(x,t) - \frac{1}{\rho(x)} v_{xx}(x,t) = 0, & x \in [0,1], \quad t \in [0,\infty), \\ v(0,t) = 0, & v_x(1,t) + h v_t(1,t) = 0, \\ v(x,0) = v_0(x), & v_t(x,0) = v_1(x), \end{cases}$$

where ρ is the density of the string and h is a complex parameter. This equation can be rewritten in an operator form as an abstract Cauchy problem for the closed, densely defined operator B acting on a certain energy space H. It is proven that the operator B generates the exponentially stable C_0-semigroup of contractions in the space H under assumptions that $\operatorname{Re} h > 0$ and the density function is of bounded variation satisfying $0 < m \leq \rho(x)$ for a.e. $x \in [0,1]$.

Keywords: nonhomogeneous string, one-dimensional wave equation, exponentially stable C_0-semigroup, Hilbert space.

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1. INTRODUCTION

Let us consider a finite nonhomogeneous string of length one with left end fixed and right one moving with damping. If we denote $v = v(x,t)$ as a vertical position of the string in time on the interval $[0,1]$, then small vibrations are described by the wave equation

$$v_{tt}(x,t) - \frac{1}{\rho(x)} v_{xx}(x,t) = 0, \quad x \in [0,1], \quad t \in [0,\infty), \tag{1.1}$$

with boundary conditions

$$v(0,t) = 0, \quad v_x(1,t) + hv_t(1,t) = 0, \tag{1.2}$$

and initial conditions

$$v(x,0) = v_0(x) \quad v_t(x,0) = v_1(x). \tag{1.3}$$

We assume that the density function ρ is of bounded variation and satisfies

$$0 < m \le \rho(x) \le M \quad \text{for a.e.} \quad x \in [0,1]. \tag{1.4}$$

Parameter $h \in \mathbb{C}$ is allowed to be complex, since this kind of boundary conditions can be used to describe an action of "smart materials", e.g. piezoelectric actuators (see [9] and references therein). The functions v_0 and v_1 are initial position and velocity, respectively.

In what follows we will use the notation $C = C[0,1]$ for the space of continuous functions with the supremum norm $\|\cdot\|_C$ and $W_2^1[0,1]$ for the Sobolev space with the first derivative in $L_2[0,1]$. Let $\widehat{W}_2^1[0,1] = \{y \in W_2^1[0,1] : \ y(0) = 0\}$, with the scalar product

$$\langle u_1, u_2 \rangle_1 = \int_0^1 u_1'(x)\overline{u}_2'(x)dx, \quad u_j \in \widehat{W}_2^1[0,1], \quad j = 1,2, \tag{1.5}$$

and let $\widehat{L}_2[0,1]$ be the space $L_2[0,1]$ equipped with the scalar product

$$\langle v_1, v_2 \rangle_2 = \int_0^1 \rho(x)v_1(x)\overline{v}_2(x)dx, \quad v_j \in \widehat{L}_2[0,1], \quad j = 1,2. \tag{1.6}$$

We can rewrite problem (1.1)–(1.3) as an abstract Cauchy problem in a certain energy space (see, for instance [11]). As the energy space we take the Hilbert space

$$H = \widehat{W}_2^1[0,1] \oplus \widehat{L}_2[0,1].$$

Let the linear operator $B : \mathcal{D}(B) \to H$ be defined as follows

$$B = \begin{bmatrix} 0 & I \\ 1/\rho(x)\, D^2 & 0 \end{bmatrix}, \quad D = \frac{d}{dx}, \tag{1.7}$$

on the domain

$$\mathcal{D}(B) = \big\{(u,v) \in W_2^2[0,1] \oplus \widehat{W}_2^1[0,1] : \ u(0) = 0, \ u'(1) + hv(1) = 0\big\}. \tag{1.8}$$

Here I denotes the identity operator on $\widehat{W}_2^1[0,1]$. If we choose $V(t) = \begin{bmatrix} v(x,t) \\ v_t(x,t) \end{bmatrix}$, then problem (1.1)–(1.3) has the form

$$\frac{d}{dt}V(t) = BV(t), \quad t > 0, \tag{1.9}$$

$$V(0) = \begin{bmatrix} v_0(x) \\ v_1(x) \end{bmatrix}. \tag{1.10}$$

The operator B is unbounded, closed, densely defined and has a compact inverse (see Section 2). We will prove that B is a generator of the exponentially stable C_0-semigroup $T(t) = e^{Bt}$, $t \geq 0$, which means that a solution of problem (1.9)–(1.10) converges exponentially to zero with respect to the energy norm. Consequently, physical energy of the string decays exponentially in time.

This fact is a generalization of results from [4], where some estimations of solutions of the equation

$$y''(x) + \mu^2 \rho(x) y(x) = 0, \quad x \in [0,1], \quad \mu \in \mathbb{C}, \tag{1.11}$$

were provided. As a consequence of those estimations authors showed that the operator B induced by problem (1.1)–(1.3) generates the exponentially stable C_0-semigroup in the case when $h = 1$ (see [4, Theorem 4.1]). The problem of a string free at the left end and damped at the right end with $h = 1$ was considered in [1] for $\rho \in W_1^1[0,1]$, where completely different methods were used (results for the damped homogeneous string, i.e. $\rho \equiv 1$ can be found in [2]). When $\operatorname{Re} h > 0$ we can deal with a broader class of physical phenomena connected with the string equation.

We will use the following estimations to prove our main result.

Proposition 1.1 ([4, Proposition 1.3])**.** *If $y \in W_1^2[0,1]$ is the solution of* (1.11)*, ρ satisfies* (1.4) *and $f \in \widehat{W}_2^1[0,1]$, then for any $\mu \in \mathbb{C} \setminus \{0\}$ the following estimation is valid*

$$\left| \int_0^1 y(t,\mu)\rho(t) f(t)]dt \right| \leq 3|\mu|^{-2} \|y'\|_C \|f'\|_{L_2}.$$

Theorem 1.2 ([4, Theorem 3.1])**.** *Let the density functions ρ be of bounded variation $\mathcal{V}(\rho)$ and satisfying* (1.4)*. Then for $\mu \neq 0$ and any solution $y \in W_1^2[0,1]$ of the equation* (1.11) *the following inequalities hold for every $x \in [0,1]$:*

$$\Big(|\mu|^2 |y(0)|^2 + |y'(0)|^2/M\Big) e^{-2\alpha_0(|\tau|)} \leq |\mu|^2 |y(x)|^2 + |y'(x)|^2/m, \tag{1.12}$$

$$\Big(|\mu|^2 |y(x)|^2 + |y'(x)|^2/M\Big) \leq \Big(|\mu|^2 |y(0)|^2 + |y'(0)|^2/m\Big) e^{2\alpha_0(|\tau|)}, \tag{1.13}$$

where $\tau = \operatorname{Im} \mu$ and $\alpha_0(|\tau|) = \frac{\mathcal{V}(\rho)}{2m} + |\tau| \|\rho^{1/2}\|_{L_1}$.

2. GENERATING THE CONTRACTION C_0-SEMIGROUP

We will first investigate some properties of the operator B. Simple calculations reveal that the inverse $B^{-1} \colon H \to H$ of B is given by

$$B^{-1} = \begin{bmatrix} B_1 & B_2 \\ I_1 & 0 \end{bmatrix}.$$

The operator B_1 is defined by

$$\big(B_1 f\big)(x) = -h f(1) x, \quad x \in [0,1].$$

According to [6, §1.4.5, Theorem 2] there is a continuous embedding $\widehat{W}_2^1[0,1] \hookrightarrow C[0,1]$, thus B_1 is a one-dimensional, compact operator acting on $\widehat{W}_2^1[0,1]$. The operator $B_2\colon \widehat{L}_2[0,1] \to W_2^2[0,1]$ acts as follows

$$\big(B_2 g\big)(x) = \int\limits_0^x (x-t)\rho(t)g(t)dt - x\int\limits_0^1 \rho(t)g(t)dt, \quad x \in [0,1],$$

and is bounded. Moreover, by [6, §1.4.5, Theorem 2], we have a compact embedding $W_2^2[0,1] \hookrightarrow W_2^1[0,1]$. Note that $\big(B_2 g\big)(0) = 0$, hence $B_2\colon \widehat{L}_2[0,1] \to \widehat{W}_2^1[0,1]$ is compact. Here I_1 denotes a compact embedding $I_1 : \widehat{W}_2^1[0,1] \hookrightarrow \widehat{L}_2[0,1]$ which exists again by [6, §1.4.5, Theorem 2]. This implies that B^{-1} is compact on H and in particular B is closed. One can show that $\overline{\mathcal{D}(B)} = \overline{\operatorname{ran}(B^{-1})} = H$ and B is densely defined. As a conclusion, the spectrum of B consists of at most a countable number of eigenvalues with the accumulation point at infinity.

Let us recall that an operator B is *dissipative* in the Hilbert space H, if for all $x \in \mathcal{D}(B)$

$$\operatorname{Re}\langle Bx, x\rangle \le 0. \tag{2.1}$$

For more information about dissipative operators, see [3, Chapter II], [7, Chapter I], [8]. It is well known that a densely defined, maximal dissipative operator generates a contraction C_0-semigroup (see [3, Chapter II, Theorem 3.15]). We will use this fact to prove our first result.

Theorem 2.1. *If the density function ρ satisfies* (1.4) *and* $\operatorname{Re} h \ge 0$. *Then the operator B generates the contraction C_0-semigroup in the space H.*

Proof. We showed that B is densely defined. It suffices to prove that B is maximal dissipative. Let $w = (u,v)$ be from the domain of B. Using integration by parts and (1.8), we obtain

$$\begin{aligned}\langle Bw, w\rangle_H &= \Big\langle (v, u''/\rho), (u,v)\Big\rangle_H = \langle v, u\rangle_1 + \langle u''/\rho, v\rangle_2 = \\ &= \int\limits_0^1 v'(x)\overline{u'(x)}dx + \int_0^1 u''(x)\overline{v(x)}dx = \\ &= \int\limits_0^1 v'(x)\overline{u'(x)}dx + u'(1)\overline{v(1)} - \int_0^1 u'(x)\overline{v'(x)}dx = \\ &= \langle v, u\rangle_1 - \langle u, v\rangle_1 - h|v(1)|^2 = 2i\operatorname{Im}\langle v,u\rangle_1 - h|v(1)|^2,\end{aligned}$$

thus

$$\operatorname{Re}\langle Bw, w\rangle_H = -\operatorname{Re} h|v(1)|^2.$$

Consequently, the operator B is dissipative whenever $\operatorname{Re} h \ge 0$. Since the inverse of B is bounded, zero belongs to the resolvent set $\rho(B)$. The resolvent set is open, therefore there exists $\lambda > 0$ in $\rho(B)$, which implies B is maximal dissipative. This ends the proof. □

Remark 2.2. In what follows we exclude the case when $\operatorname{Re} h = 0$, because in this case the operator B is skew-adjoint and there is no exponential stability of $T(t)$.

3. GENERATING THE EXPONENTIALLY STABLE C_0-SEMIGROUP

Recall the definition of a stable C_0-semigroup (see [3, Chapter V, Definition 1.1]).

Definition 3.1. The C_0-semigroup $T(t)$ is *exponentially stable* if

$$\omega_0 = \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\| < 0. \tag{3.1}$$

It follows that, if $\omega_0 < 0$ then for all $0 < \omega < |\omega_0|$, there exists $M_\omega > 0$ such that

$$\|T(t)\| \le M_\omega e^{-\omega t}, \quad t \ge 0.$$

The well known Gearhart theorem [3, Chapter V, Theorem 1.11] states that the C_0-semigroup of operators in the Hilbert space with a generator B is exponentially stable if and only if $\mathbb{C}_+ = \{\mu \in \mathbb{C};\ \operatorname{Re}\mu > 0\} \subset \rho(B)$ and

$$\sup_{\operatorname{Re}\mu>0} \|(B - \mu I)^{-1}\| < \infty.$$

We will use the following proposition, which is a consequence of the Gearhart theorem, to prove that B is the generator of the exponential stable C_0-semigroup.

Proposition 3.2. *Let B be the linear operator acting on the Hilbert space H which generates a uniformly bounded C_0-semigroup of operators $T(t)$, $t \ge 0$. Suppose that there exists the resolvent on the imaginary axis, which is uniformly bounded, i.e.*

$$\|(B - i\tau I)^{-1}\| \le r, \quad \tau \in \mathbb{R}, \quad r > 0. \tag{3.2}$$

Then the semigroup $T(t)$ is exponentially stable and moreover for any $0 < \delta < r^{-1}$ there exists a constant $M_\delta > 0$ such that

$$\|T(t)\| \le M_\delta e^{-\delta t}, \quad t \ge 0. \tag{3.3}$$

Proof. Since the operator B is a generator of the uniformly bounded semigroup, the Hille-Yosida theorem states that $\mathbb{C}_+$ is in the resolvent set and there exists $r_1 > 0$ such that

$$\|(B - \mu I)^{-1}\| \le \frac{r_1}{\operatorname{Re}\mu} \quad \text{for} \quad \operatorname{Re}\mu > 0. \tag{3.4}$$

Let $0 < \delta < r^{-1}$. Then thanks to the inequality (3.2) for $\mu = s + i\tau$ such that $|s| \le \delta < r^{-1}$, the resolvent exists and is given by

$$(B - \mu I)^{-1} = (B - i\tau I)^{-1}\big[I - s(B - i\tau I)^{-1}\big]^{-1}. \tag{3.5}$$

From (3.2) and (3.5) we obtain the estimation

$$\|(B - \mu I)^{-1}\| \le r(1 - \delta r)^{-1}, \quad |s| \le \delta. \tag{3.6}$$

Combining (3.4) and (3.6), we obtain for any λ such that $\operatorname{Re}\lambda \geq 0$ the following relation

$$\|(B+\delta-\lambda I)^{-1}\| = \|(B-(\lambda-\delta)I)^{-1}\| \leq c_\delta = \max\{r_1\delta^{-1}, r(1-\delta r)^{-1}\}.$$

Thus the Gearhart theorem yields the exponential stability of the semigroup $e^{(B+\delta)t}$ generated by the operator $B+\delta$. By Definition 3.1,

$$0 > \lim_{t\to\infty}\frac{1}{t}\log\|e^{(B+\delta)t}\| = \lim_{t\to\infty}\frac{1}{t}\log\|e^{Bt}e^{\delta t}\| = \lim_{t\to\infty}\frac{1}{t}\log\|e^{Bt}\| + \delta, \tag{3.7}$$

so $T(t)$ is exponentially stable and for all $0 < \delta < r^{-1}$ there exists $M_\delta > 0$ such that

$$\|T(t)\| \leq M_\delta e^{-\delta t}, \quad t \geq 0. \tag{3.8}$$

□

For the convenience of further calculations we introduce the operator A defined on $D(A) = D(B)$ by

$$A = i\begin{bmatrix} 0 & -I \\ -1/\rho(x)\,D^2 & 0 \end{bmatrix}, \tag{3.9}$$

so that $B = iA$. More information about the operator A can be found in [4, 10, 11]. By Theorem 2.1 and Proposition 3.2, it is sufficient now to prove that the resolvent of the operator A exists on the real axis and is bounded, but we will also provide the lower bound of the spectrum of the operator A.

Lemma 3.3. *Let the density function ρ be of bounded variation $\mathcal{V}(\rho)$ satisfying* (1.4) *and let $h = a + ib$ be such that $a = \operatorname{Re} h > 0$. Then all the eigenvalues $\mu = s + i\tau$ of the operator A are uniformly separated from the real axis and the following inequality holds*

$$\tau \geq c(1 + 4c\|\rho^{1/2}\|_{L_1})^{-1} > 0,$$

where $c = \frac{am^2}{M^2(m+|h|^2)}e^{-\frac{2\mathcal{V}(\rho)}{m}}$.

Proof. Since B is dissipative, then $\operatorname{Im}\langle Aw, w\rangle_H \geq 0$ and every eigenvalue μ of the operator A satisfies $\operatorname{Im}\mu \geq 0$. The operator B has a compact inverse, hence $\mu = 0$ is not an eigenvalue of A. We want to prove that the spectrum of A is separated from the real axis. If μ is an eigenvalue of A with an eigenfunction $w = (u, v)$, then by (3.9) we get $v = i\mu u$ and u satisfies

$$u''(x) + \mu^2\rho(x)u(x) = 0, \quad x \in [0, 1], \tag{3.10}$$

with boundary conditions

$$u(0) = 0, \quad U[u](\mu) := u'(1) + i\mu h u(1) = 0. \tag{3.11}$$

Multiplying by $\overline{u}$, dividing by μ and integrating by parts (3.10), thanks to (3.11) we obtain

$$ih|u(1)|^2 = \mu\int_0^1 \rho(x)|u(x)|^2dx - \frac{1}{\mu}\int_0^1 |u'(x)|^2dx. \tag{3.12}$$

By taking the imaginary part of the above equation we obtain the equality

$$a|u(1)|^2 = \tau \int\limits_0^1 \rho(x)|u(x)|^2 dx + \frac{\tau}{|\mu|^2} \int\limits_0^1 |u'(x)|^2 dx. \tag{3.13}$$

Thus (1.4) implies

$$a|u(1)|^2 \leq \tau \left(M\|u(x)\|_{L_2}^2 + \frac{\|u'(x)\|_{L_2}^2}{|\mu|^2} \right). \tag{3.14}$$

We want now to obtain a lower bound of $|u(1)|$. Using Theorem 1.2 and (3.11) we have

$$\begin{aligned} |\mu|^2|u(1)|^2\big(1+|h|^2m^{-1}\big) &= |\mu|^2|u(1)|^2 + |u'(1)|^2/m \geq \\ &\geq \Big(|\mu|^2|u(0)|^2 + |u'(0)|^2/M\Big)e^{-2\alpha_0(\tau)} \geq \\ &\geq \frac{m}{M^2}|\mu|^2\Big(M|u(x)|^2 + \frac{|u'(x)|^2}{|\mu|^2}\Big)e^{-4\alpha_0(\tau)}, \end{aligned}$$

thus the integration yields

$$|u(1)|^2 \geq \frac{m^2}{M^2(m+|h|^2)}\Big(M\|u(x)\|_{L_2}^2 + \frac{\|u'(x)\|_{L_2}^2}{|\mu|^2}\Big)e^{-4\alpha_0(\tau)}. \tag{3.15}$$

Combining (3.14) with (3.15) and using the inequality $e^{-x} \geq 1 - x$ for $x \geq 0$, we obtain

$$\begin{aligned} \tau &\geq \frac{am^2}{M^2(m+|h|^2)} e^{-\frac{2\mathcal{V}(\rho)}{m}} e^{-4\tau\|\rho^{1/2}\|_{L_1}} \geq \\ &\geq \frac{am^2}{M^2(m+|h|^2)} e^{-\frac{2\mathcal{V}(\rho)}{m}} (1 - 4\tau\|\rho^{1/2}\|_{L_1}), \end{aligned}$$

hence

$$\tau \geq c(1 + 4c\|\rho^{1/2}\|_{L_1})^{-1} > 0,$$

with a constant $c = \frac{am^2}{M^2(m+|h|^2)} e^{-\frac{2\mathcal{V}(\rho)}{m}}$. □

We can now state and prove our main theorem.

Theorem 3.4. *Let the density function ρ be of bounded variation $\mathcal{V}(\rho)$ satisfying* (1.4) *and* $\operatorname{Re} h > 0$. *Then the operator B generates the exponentially stable C_0-semigroup in the space H.*

Proof. Since $B = iA$ and the resolvent of the operator A exists on the real axis, by Proposition 3.2 it suffices to show that

$$\|(A - sI)^{-1}\|_H \leq r, \quad s \in \mathbb{R}, \quad r > 0. \tag{3.16}$$

The resolvent $(A - \mu I)^{-1}$ of the operator A is defined by an equation

$$\big(A - \mu I\big)(u, v) = (f, g), \tag{3.17}$$

where $(u, v) \in \mathcal{D}(A)$ and $(f, g) \in H$. Hence, our problem is reduced to finding a solution of the boundary value problem

$$u''(x) + \mu^2 \rho(x) u(x) = L(x, \mu), \tag{3.18}$$

$$u(0) = 0, \quad U[u](\mu) = u'(1) + i\mu h u(1) = -ihf(1), \tag{3.19}$$

where

$$L(x, \mu) = \rho(x)[ig(x) - \mu f(x)], \tag{3.20}$$

and v is expressed by the formula

$$v(x) = i\big(f(x) + \mu u(x)\big). \tag{3.21}$$

Let $y_1 = y_1(x, \mu)$, $y_2 = y_2(x, \mu)$ be a fundamental system of solutions of equation (3.10) such that

$$y_1(0, \mu) = 0, \quad y_1'(0, \mu) = 1, \quad y_2(0, \mu) = -1, \quad y_2'(0, \mu) = 0. \tag{3.22}$$

A particular solution $y_0 = y_0(x, \mu)$ of the nonhomogeneous equation (3.18) is given by the formula

$$y_0(x, \mu) = y_2(x, \mu) \int_0^x y_1(t, \mu) L(t, \mu) dt + y_1(x, \mu) \int_x^1 y_2(t, \mu) L(t, \mu) dt. \tag{3.23}$$

We want to find the solution of the problem (3.18)–(3.19) of the form $y(x, \mu) = Cy_1(x, \mu) + y_0(x, \mu)$. By using boundary conditions (3.19), we obtain

$$u(x, \mu) = -\frac{ihf(1) + U[y_0](s)}{U[y_1](\mu)} y_1(x, \mu) + y_0(x, \mu). \tag{3.24}$$

Thus the resolvent exists for all μ, which does not coincide with the roots of the analytic function $U[y_1](\mu)$. In particular $U[y_1](0) = 1$ and the inverse of the operator A exists and is bounded, which has been shown in Section 2.

For an arbitrary $w = (f, g) \in H$ and $s \in \mathbb{R}$, (3.21) implies the following estimation of the resolvent

$$\|(A - sI)^{-1} w\|_H^2 = \int_0^1 |u'(x, s)|^2 dx + \int_0^1 \rho(x) |v(x, s)|^2 dx \leq \tag{3.25}$$

$$\leq \|u'(x, s)\|_{L_2}^2 + 2Ms^2 \|u(x, s)\|_{L_2}^2 + 2M \|f(x)\|_{L_2}^2. \tag{3.26}$$

In order to establish the main result we need the following estimations

$$\|u'(x, s)\|_{L_2} \leq c\|w\|_H, \quad \|u(x, s)\|_{L_2} \leq cs^{-1} \|w\|_H.$$

According to (3.24), we need a lower bound of $|U[y_1](s)|$. In the case when $s \in \mathbb{R}$, the functions y_1 and y_2 are real-valued, and

$$|U[y_1](s)|^2 = \big(y_1'(1,s)\big)^2 + s^2a^2y_1^2(1,s) + s^2b^2y_1^2(1,s) - 2bsy_1(1,s)y_1'(1,s).$$

When $b \neq 0$, using the Cauchy inequality $2xy \leq \xi x^2 + \frac{y^2}{\xi}$ in the last term yields

$$|U[y_1](s)|^2 \geq \big(y_1'(1,s)\big)^2\big(1-\xi^{-1}\big) + y_1^2(1,s)s^2(a^2+b^2-b^2\xi).$$

However we can always find ξ such that $1 < \xi < 1 + \frac{a^2}{b^2}$. Writing

$$k = \min\Big\{m\big(1-\xi^{-1}\big), a^2+b^2-b^2\xi\Big\} > 0,$$

we find that $k = \frac{1}{2}\Big(|h|^2 + m - \sqrt{(m-|h|^2)^2+4mb^2}\Big) \geq k_0 = \frac{a^2m}{|h|^2+m} > 0$. Again by Theorem 1.2 and (3.22), we obtain

$$\begin{aligned}|U[y_1](s)|^2 &\geq k_0\left(s^2y_1^2(1,s) + \frac{\big(y_1'(1,s)\big)^2}{m}\right) \geq \\ &\geq k_0C_1^{-1}\left(s^2y_1^2(0,s) + \frac{\big(y_1'(0,s)\big)^2}{M}\right) = k_0C_1^{-1}M^{-1} > 0,\end{aligned}$$

where $C_1 = e^{\mathcal{V}(\rho)/m}$. Consequently, we have

$$|U[y_1](s)|^{-1} \leq c_0 = (k_0^{-1}C_1M)^{\frac{1}{2}}. \tag{3.27}$$

If $b = 0$, take $k_0 = \min\{m, a^2\}$.

In what follows we will need an even real function $\mathbb{R} \ni s \mapsto z(s)$ and its estimation. Define

$$z^2(s) = s^2\|y_1(x,s)\|_C^2 + \|y_1'(x,s)\|_C^2 + \|y_2(x,s)\|_C^2 + s^{-2}\|y_2'(x,s)\|_C^2.$$

In the same way as in [4, (4.10)] we obtain

$$z(s) \leq c_1 = 2(M_1(m^{-1}+1)C_1)^{\frac{1}{2}}, \tag{3.28}$$

where $M_1 = \max\{1, M\}$. From the definition of z we obtain

$$\begin{aligned}&\|y_1(x,s)\|_{L_2} \leq \|y_1(x,s)\|_C \leq z(s)|s|^{-1} \leq c_1|s|^{-1},\\ &\|y_1'(x,s)\|_{L_2} \leq \|y_1(x,s)\|_C \leq z(s) \leq c_1,\\ &\|y_2(x,s)\|_{L_2} \leq \|y_2(x,s)\|_C \leq z(s) \leq c_1,\\ &\|y_2'(x,s)\|_{L_2} \leq \|y_2'(x,s)\|_C \leq z(s)|s| \leq c_1|s|.\end{aligned} \tag{3.29}$$

We can now estimate $|U[y_0](s)|$. Using Proposition 1.1 and (1.4), (3.29), we get

$$\begin{aligned}
|U[y_0](s)| &= |y_2'(1,s) + ishy_2(1,s)| \left| \int_0^1 y_1(t,s)L(t,s)dt \right| \leq \\
&\leq c_1|s|(1+|h|) \left| \int_0^1 y_1(t,s)\rho(t)[ig(t) - sf(t)]dt \right| \leq \\
&\leq c_1|s|(1+|h|) \left(c_1\sqrt{M}|s|^{-1}\|g\|_{\widehat{L}_2} + |s| \left| \int_0^1 y_1(t,s)\rho(t)f(t)]dt \right| \right) \leq \\
&\leq c_1^2(1+|h|)\big(\sqrt{M}\|g\|_{\widehat{L}_2} + 3\|f'\|_{L_2}\big). \qquad (3.30)
\end{aligned}$$

The estimation of $\|y_0\|_{L_2}$ is exactly the same as in the case when $h = 1$. It has been proved in [4] in an analogous way as in (3.30) that the following inequality is true

$$\|y_0\|_{L_2} \leq c_1^2|s|^{-1}\Big(2\sqrt{M}\|g\|_{\widehat{L}_2} + 6\|f'\|_{L_2}\Big). \qquad (3.31)$$

Combining estimations (3.27), (3.29), (3.30), (3.31), we obtain

$$\begin{aligned}
\|u(x,s)\|_{L_2} &\leq \frac{|h||f(1)| + |U[y_0](s)|}{|U[y_1](s)|}\|y_1(x,\mu)\|_{L_2} + \|y_0(x,\mu)\|_{L_2} \leq \\
&\leq c_0c_1|s|^{-1}\Big(|h|\|f'\|_{L_2} + c_1^2(1+|h|)\big(\sqrt{M}\|g\|_{\widehat{L}_2} + 3\|f'\|_{L_2}\big)\Big) + \\
&+ c_1^2|s|^{-1}\Big(2\sqrt{M}\|g\|_{\widehat{L}_2} + 6\|f'\|_{L_2}\Big). \qquad (3.32)
\end{aligned}$$

Finally let us estimate $\|u'(x,s)\|_{L_2}$. Thanks to (3.24) we have

$$\begin{aligned}
u'(x,\mu) &= -\frac{ihf(1) + U[y_0](s)}{U[y_1](s)}y_1'(x,\mu) + y_2'(x,s)\int_0^x y_1(t,s)L(t,s)dt + \\
&+ y_1'(x,s)\int_x^1 y_2(t,s)L(t,s)dt, \qquad (3.33)
\end{aligned}$$

and analogously as in (3.32) we obtain

$$\begin{aligned}
\|u'(x,s)\|_{L_2} &\leq c_0c_1\Big(|h|\|f'\|_{L_2} + c_1^2(1+|h|)\big(\sqrt{M}\|g\|_{\widehat{L}_2} + 3\|f'\|_{L_2}\big)\Big) + \\
&+ c_1^2\Big(2\sqrt{M}\|g\|_{\widehat{L}_2} + 6\|f'\|_{L_2}\Big). \qquad (3.34)
\end{aligned}$$

The estimations (3.32) and (3.34) give us

$$\|(A - sI)^{-1}w\|_H \leq r\|w\|_H,$$

hence the theorem is proved. □

Remark 3.5. In general, the constant r has a rather complicated form

$$r^2 = 2\max\Big\{\big(k_1^2(1+2M)+M\big), k_2^2(1+2M)\Big\},$$

where

$$k_1 = c_0c_1|h| + 3c_0c_1^3(1+|h|) + 6c_1^2, \quad k_2 = c_0c_1^3(1+|h|)\sqrt{M} + 2c_1^2\sqrt{M}.$$

If $M \le 1$ then we can take $r^2 = 6(k_1^2 + 1/3)$.

Remark 3.6. When the density function ρ satisfies (1.4) and belongs to the space $W_1^1[0,1]$, in order to prove Theorem 3.4 one can use asymptotic expressions for fundamental solutions y_1 and y_2 of the equation (3.10) from [5]. In this way one can obtain estimations of the same growth as in (3.27), (3.29), (3.30), and this allows us to complete the proof in the same way as in the proof of Theorem 3.4.

As a consequence of the previous theorem we obtain some information about the solutions of the problem (1.9)–(1.10) (see [3, Chapter II, Proposition 6.2]).

Corollary 3.7. *Under the assumptions of Theorem 3.4 there exists positive constant r such that for any mild solution of the problem* (1.9)–(1.10) *with initial data* $\big(v_0(x), v_1(x)\big) \in H$ *the following estimations are true*

$$\|V(t)\|_H \le \|V(0)\|_H, \quad \|V(t)\|_H \le M_\delta \|V(0)\|_H e^{-\delta t}, \quad \forall \delta < r^{-1}, \quad t \ge 0.$$

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