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BOUNDARY VALUE PROBLEMS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS AND INCLUSIONS WITH NONLOCAL AND FRACTIONAL INTEGRAL BOUNDARY CONDITIONS

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Abstract. This paper studies the boundary value problem of nonlinear fractional differential equations and inclusions of order $q \in (1,2]$ with nonlocal and integral boundary conditions. Some new existence and uniqueness results are obtained by using fixed point theorems.

Keywords: fractional differential equations, fractional differential inclusions, nonlocal conditions, fractional integral boundary conditions, existence, contraction principle, nonlinear contraction.

Mathematics Subject Classification: 34A08, 26A33, 34A60.

1. INTRODUCTION

Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. These characteristics of the fractional derivatives make the fractional-order models more realistic and practical than the classical integer-order models. In recent years, boundary value problems for nonlinear fractional differential equations have been addressed by several researchers. As a matter of fact, fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. [26, 32–34]. For some recent development on the topic, see [1–13, 15, 27, 29, 35, 36] and the references therein.

As a first problem in this paper we discuss the existence and uniqueness of solutions for a boundary value problem of nonlinear fractional differential equations of order $q \in (1, 2]$ with nonlocal and integral boundary conditions given by:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), & 0 < t < 1, \quad 1 < q \le 2, \\ x(0) = x_0 + g(x), & x(1) = \alpha I^p x(\eta), \quad 0 < \eta < 1, \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $g : C([0, 1], \mathbb{R}) \to \mathbb{R}$, $\alpha \in \mathbb{R}$ is such that $\alpha \neq \Gamma(p+2)/\eta^{p+1}$, Γ is the Euler gamma function and I^p is the Riemann-Liouville fractional integral of order p. The fractional integral boundary conditions were introduced recently in [24].

Nonlocal conditions were initiated by Bitsadze [16]. As remarked by Byszewski [18–20], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. For example, $g(x)$ may be given by $g(x) = \sum_{i=1}^{p} c_i x(t_i)$ where $c_i, i = 1, \ldots, p$, are given constants and $0 < t_1 < \ldots < t_p \le T$. For recent papers on nonlocal fractional boundary value problems the interested reader is referred to [10, 14, 15, 37] and the references cited therein.

In Section 3 we give some sufficient conditions for the uniqueness of solutions and for the existence of at least one solution of problem (1.1). The first result is based on Banach's contraction principle and the second on a fixed point theorem due to D. O'Regan. A concrete example is also provided to illustrate the possible application of the established analytical results.

In Section 4, we extend the results to cover the multi-valued case, considering the following boundary value problem for fractional order differential inclusions with nonlocal and fractional integral boundary conditions

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), & 0 < t < 1, \quad 1 < q \le 2, \\ x(0) = x_0 + g(x), & x(1) = \alpha I^p x(\eta), \quad 0 < \eta < 1, \end{cases} \tag{1.2}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$.

Existence results for the problem (1.2), are presented when the right hand side is convex as well as nonconvex valued. The first result relies on the Nonlinear Alternative for contractive maps. In the second result, we shall combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel, in Section 3 we prove our main results for the single-valued case and in Section 4 we prove our main results for the multi-valued case.

2. PRELIMINARIES

Let us recall some basic definitions of fractional calculus [26, 32, 34].

Definition 2.1. For an at least n-times differentiable function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2. The Riemann-Liouville fractional integral of order q is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

Definition 2.3. The Riemann-Liouville fractional derivative of order $q > 0$ for a continuous function $g : (0, \infty) \to \mathbb{R}$ is defined by

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}} ds, \quad n = [q]+1,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

Lemma 2.4. *For $q > 0$, the general solution of the fractional differential equation $^cD^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}, \ i = 0, 1, \ldots, n-1$ $(n = [q]+1)$.

In view of Lemma 2.4, it follows that

$$I^q \ {}^cD^q x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1} \tag{2.1}$$

for some $c_i \in \mathbb{R}, \ i = 0, 1, \ldots, n-1$ $(n = [q]+1)$.

To define the solution for the problem (1.1), we find the solution for its associated linear problem.

Lemma 2.5. *Assume that $\alpha \neq \dfrac{\Gamma(p+2)}{\eta^{p+1}}$. For a given $y \in C([0,1], \mathbb{R})$ the unique solution of the boundary value problem*

$$\begin{cases} {}^cD^q x(t) = y(t), \quad 0 < t < 1, \quad 1 < q \leq 2, \\ x(0) = x_0 + g(x), \quad x(1) = \alpha I^p x(\eta), \quad 0 < \eta < 1, \end{cases} \tag{2.2}$$

is given by

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s) ds -$$
$$- \frac{\Gamma(p+2)t}{\Gamma(q)[\Gamma(p+2) - \alpha\eta^{p+1}]} \int_0^1 (1-s)^{q-1} y(s) ds + \tag{2.3}$$
$$+ \frac{\alpha p(p+1)t}{\Gamma(q)[\Gamma(p+2) - \alpha\eta^{p+1}]} \int_0^\eta \int_0^s (\eta - s)^{p-1} (s-r)^{q-1} y(r) dr ds +$$
$$+ (1-t)[x_0 + g(x)].$$

Proof. For some constants $c_0, c_1 \in \mathbb{R}$, we have [26]

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s) ds - c_0 - c_1 t. \tag{2.4}$$

From $x(0) = x_0 + g(x)$ we have $c_0 = -(x_0 + g(x))$. Using the Riemann-Liouville integral of order p for (2.4) we have

$$I^p x(t) = \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} \left[\int_0^s \frac{(s-r)^{q-1}}{\Gamma(q)} y(r) dr - c_0 - c_1 s \right] ds =$$
$$= \frac{1}{\Gamma(p)} \frac{1}{\Gamma(q)} \int_0^t \int_0^s (t-s)^{p-1} (s-r)^{q-1} y(r) dr ds - c_0 \frac{t^p}{\Gamma(p+1)} - c_1 \frac{t^{p+1}}{\Gamma(p+2)}.$$

Applying the second boundary condition of (2.2) we get

$$c_1 = \frac{\Gamma(p+2)}{[\Gamma(p+2) - \alpha\eta^{p+1}]} \left[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} y(s) ds - \right.$$
$$\left. - \frac{\alpha}{\Gamma(p)\Gamma(q)} \int_0^\eta \int_0^s (\eta - s)^{p-1} (s-r)^{q-1} g(r) dr ds + \frac{\Gamma(p+1) - \alpha\eta^p}{\Gamma(p+1)} [x_0 + g(x)] \right] =$$
$$= \frac{\Gamma(p+2)}{[\Gamma(p+2) - \alpha\eta^{p+1}]} \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} y(s) ds -$$
$$- \frac{\alpha p(p+1)}{\Gamma(q)[\Gamma(p+2) - \alpha\eta^{p+1}]} \int_0^\eta \int_0^s (\eta - s)^{p-1} (s-r)^{q-1} g(r) dr ds + [x_0 + g(x)].$$

Substituting in (2.4) the values of c_0 and c_1, we obtain (2.3). □

We denote by $\mathcal{C} = C([0,1],\mathbb{R})$ the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$.

In view of Lemma 2.5, we define an operator $F : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}(Fx)(t) = {} & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s))ds - \\ & - \frac{\Gamma(p+2)t}{\Gamma(q)[\Gamma(p+2)-\alpha\eta^{p+1}]} \int_0^1 (1-s)^{q-1} f(s,x(s))ds + \\ & + \frac{\alpha p(p+1)t}{\Gamma(q)[\Gamma(p+2)-\alpha\eta^{p+1}]} \int_0^\eta \int_0^s (\eta-s)^{p-1}(s-r)^{q-1} f(r,x(r))drds + \\ & + (1-t)[x_0 + g(x)], \quad t \in [0,1].\end{aligned}$$

Define two operators from $\mathcal{C} \to \mathcal{C}$, respectively, by

$$\begin{aligned}(F_1x)(t) = {} & \\ = {} & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s))ds - \\ & - \frac{\Gamma(p+2)t}{\Gamma(q)[\Gamma(p+2)-\alpha\eta^{p+1}]} \int_0^1 (1-s)^{q-1} f(s,x(s))ds + \\ & + \frac{\alpha p(p+1)t}{\Gamma(q)[\Gamma(p+2)-\alpha\eta^{p+1}]} \int_0^\eta \int_0^s (\eta-s)^{p-1}(s-r)^{q-1} f(r,x(r))drds, \quad t \in [0,1].\end{aligned} \tag{2.5}$$

and

$$(F_2x)(t) = (1-t)[x_0 + g(x)], \quad t \in [0,1]. \tag{2.6}$$

Clearly

$$(Fx)(t) = (F_1x)(t) + (F_2x)(t), \quad t \in [0,1]. \tag{2.7}$$

3. EXISTENCE RESULTS – THE SINGLE-VALUED CASE

Theorem 3.1. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that:*

(A_1) $|f(t,x) - f(t,y)| \le L|x-y|$, *for all $t \in [0,1]$, $L > 0$, $x, y \in \mathbb{R}$;*

(A_2) *there exist a positive constant $\ell < 1/2$ and a continuous function $\phi : [0,\infty) \to (0,\infty)$ such that $\phi(z) \le \ell z$ and $|g(u)-g(v)| \le \phi(\|u-v\|)$ for all $u, v \in C([0,1])$;*

(A_3)

$$\gamma = \left[\frac{L}{\Gamma(q+1)}\left\{1+\frac{\Gamma(p+2)}{|\Gamma(p+2)-\alpha\eta^{p+1}|}\right\}\frac{L\alpha\eta^{p+q}\Gamma(p+2)}{\Gamma(p+q+1)|\Gamma(p+2)-\alpha\eta^{p+1}|}+2\ell\right] < 1.$$

Then the boundary value problem (1.1) *has a unique solution.*

Proof. For $x, y \in \mathcal{C}$ and for each $t \in [0,1]$, from the definition of F and assumptions (A_1) and (A_2), we obtain

$$\begin{aligned}
&|(Fx)(t)-(Fy)(t)| \leq \\
&\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,x(s))-f(s,y(s))|ds+ \\
&\quad + \frac{\Gamma(p+2)t}{\Gamma(q)|\Gamma(p+2)-\alpha\eta^{p+1}|}\int_0^1 (1-s)^{q-1}|f(s,x(s))-f(s,y(s))|ds+ \\
&\quad + \frac{\alpha p(p+1)t}{\Gamma(q)|\Gamma(p+2)-\alpha\eta^{p+1}|}\int_0^\eta\int_0^s (\eta-s)^{p-1}(s-m)^{q-1}|f(m,x(m))-f(m,y(m))|dmds+ \\
&\quad + |1-t||g(x)-g(y)| \leq \\
&\leq L\|x-y\|\left[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds + \frac{\Gamma(p+2)}{\Gamma(q)|\Gamma(p+2)-\alpha\eta^{p+1}|}\int_0^1 (1-s)^{q-1}ds + \right. \\
&\left. + \frac{\alpha p(p+1)}{\Gamma(q)|\Gamma(p+2)-\alpha\eta^{p+1}|}\int_0^\eta\int_0^s (\eta-s)^{p-1}(s-m)^{q-1}dmds\right] + \\
&\quad + 2\ell\|x-y\| = \\
&= L\|x-y\|\left\{\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+1)}\frac{\Gamma(p+2)}{|\Gamma(p+2)-\alpha\eta^{p+1}|} + \right. \\
&\quad \left. + \frac{1}{\Gamma(q+1)}\frac{\alpha p(p+1)\eta^{p+q}B(q+1,p)}{|\Gamma(p+2)-\alpha\eta^{p+1}|}\right\} + 2\ell\|x-y\| = \\
&= L\|x-y\|\left\{\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+1)}\frac{\Gamma(p+2)}{|\Gamma(p+2)-\alpha\eta^{p+1}|} + \right. \\
&\quad \left. + \frac{\alpha\eta^{p+q}\Gamma(p+2)}{\Gamma(p+q+1)|\Gamma(p+2)-\alpha\eta^{p+1}|}\right\} + 2\ell\|x-y\| \leq \\
&\leq \left[\frac{L}{\Gamma(q+1)}\left\{1+\frac{\Gamma(p+2)}{|\Gamma(p+2)-\alpha\eta^{p+1}|}\right\} + \right. \\
&\quad \left. + \frac{L\alpha\eta^{p+q}\Gamma(p+2)}{\Gamma(p+q+1)|\Gamma(p+2)-\alpha\eta^{p+1}|} + 2\ell\right]\|x-y\|,
\end{aligned}$$

where we used the computation

$$\int_0^\eta \int_0^s (\eta - s)^{p-1}(s-r)^{q-1} dr ds = \frac{1}{q}\eta^{p+q} B(q+1, p),$$

where B is the beta function and the property of beta function $B(q+1,p) = \dfrac{\Gamma(q+1)\Gamma(p)}{\Gamma(p+q+1)}$. Hence

$$\|Fx - Fy\| \leq \gamma \|x - y\|.$$

As $\gamma < 1$, by (A_3), F is a contraction map from the Banach space $\mathcal{C}$ into itself. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Example 3.2. Consider the following fractional boundary value problem

$$\begin{cases} {}^cD^{3/2}x(t) = \dfrac{1}{(t+2)^2}\dfrac{|x|}{1+|x|} + 1 + \sin^2 t, \quad t \in [0,1], \\ x(0) = \dfrac{1}{2} + \dfrac{1}{16}x(\xi), \quad x(1) = \sqrt{3} I^{5/2} x\left(\dfrac{1}{3}\right). \end{cases} \tag{3.1}$$

Here, $q = 3/2$, $\alpha = \sqrt{3}$, $p = 5/2$, $\eta = 1/3$ and $f(t,x) = \dfrac{1}{(t+2)^2}\dfrac{|x|}{1+|x|} + 1 + \sin^2 t$. As $\alpha = \sqrt{3} \neq \Gamma(p+2)/\eta^{p+1} = \Gamma(9/2)/(1/3)^{7/2}$ and $|f(t,x) - f(t,y)| \leq \dfrac{1}{4}|x-y|$, therefore, (A_1) is satisfied with $L = \dfrac{1}{4}$. Since

$$\begin{aligned} \gamma = \Bigg[&\frac{L}{\Gamma(q+1)} \left\{ 1 + \frac{\Gamma(p+2)}{|\Gamma(p+2) - \alpha\eta^{p+1}|} \right\} + \\ &+ \frac{L\alpha\eta^{p+q}\Gamma(p+2)}{\Gamma(p+q+1)|\Gamma(p+2) - \alpha\eta^{p+1}|} + 2\ell \Bigg] \approx 0.5017842 < 1, \end{aligned}$$

by the conclusion of Theorem 3.1, the boundary value problem (3.1) has a unique solution on $[0,1]$.

Next, we introduce the fixed point theorem which was established by O'Regan in [30]. This theorem will be adopted to prove the next main result.

Lemma 3.3. *Denote by U an open set in a closed, convex set C of a Banach space E. Assume $0 \in U$. Also assume that $F(\bar{U})$ is bounded and that $F : \bar{U} \to C$ is given by $F = F_1 + F_2$, in which $F_1 : \bar{U} \to E$ is continuous and completely continuous and $F_2 : \bar{U} \to E$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|F_2(x) - F_2(y)\| \leq \phi(\|x-y\|)$ for all $x, y \in \bar{U}$). Then, either:*

(C1) *F has a fixed point $u \in \bar{U}$; or*
(C2) *there exist a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(u)$, where $\bar{U}$ and ∂U, respectively, represent the closure and boundary of U.*

Let

$$\Omega_r = \{x \in C([0,1], \mathbb{R}) : \|x\| < r\},$$

and denote the maximum number by

$$M_r = \max\{|f(t,x)| : (t,x) \in [0,1] \times [-r,r]\}.$$

Theorem 3.4. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that (A_1) and (A_2) hold. In addition we assume that:*

(A_4) $g(0) = 0$;
(A_5) *there exists a nonnegative function $p \in C([0,1], \mathbb{R})$ and a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that*

$$|f(t,u)| \le p(t)\psi(|u|) \quad \text{for any} \quad (t,u) \in [0,1] \times \mathbb{R};$$

(A_6) $\sup\limits_{r \in (0,\infty)} \dfrac{r}{2|x_0| + p_0\psi(r)} > \dfrac{1}{1-2\ell}$, *where*

$$p_0 = \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} p(s)ds + \frac{\Gamma(p+2)}{|\Gamma(p+2) - \alpha\eta^{p+1}|} \int_0^1 (1-s)^{q-1} p(s)ds +$$
$$+ \frac{\alpha p(p+1)}{|\Gamma(p+2) - \alpha\eta^{p+1}|} \int_0^\eta \int_0^s (\eta - s)^{p-1}(s-r)^{q-1} p(r)drds.$$

Then the boundary value problem (1.1) *has at least one solution on* $[0,1]$.

Proof. Consider the operator $F : \mathcal{C} \to \mathcal{C}$ as that defined in (2.7), that is,

$$(Fx)(t) = (F_1x)(t) + (F_2x)(t), \quad t \in [0,1],$$

where the operators F_1 and F_2 are defined respectively in (2.5) and (2.6).

From (A_6) there exists a number $r_0 > 0$ such that

$$\frac{r_0}{2|x_0| + p_0\psi(r_0)} > \frac{1}{1-2\ell}. \tag{3.2}$$

We shall prove that the operators F_1 and F_2 satisfy all the conditions in Lemma 3.3.
Step 1. The operator F_1 is continuous and completely continuous. We first show that $F_1(\bar{\Omega}_{r_0})$ is bounded. For any $x \in \bar{\Omega}_{r_0}$ we have

$$\|F_1 x\| \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s,x(s))| ds +$$

$$+ \frac{\Gamma(p+2)t}{\Gamma(q)|\Gamma(p+2) - \alpha\eta^{p+1}|} \int_0^1 (1-s)^{q-1} |f(s,x(s))| ds +$$

$$+ \frac{\alpha p(p+1)t}{\Gamma(q)|\Gamma(p+2) - \alpha\eta^{p+1}|} \int_0^\eta \int_0^s (\eta - s)^{p-1} (s-r)^{q-1} |f(r,x(r))| dr ds \leq$$

$$\leq M_r \left[\frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} ds + \frac{\Gamma(p+2)}{|\Gamma(p+2) - \alpha\eta^{p+1}|} \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} ds + \right.$$

$$\left. + \frac{\alpha p(p+1)}{\Gamma(q)|\Gamma(p+2) - \alpha\eta^{p+1}|} \int_0^\eta \int_0^s (\eta - s)^{p-1} (s-r)^{q-1} dr ds \right] \leq$$

$$\leq M_r \left\{ \frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+1)} \frac{\Gamma(p+2)}{|\Gamma(p+2) - \alpha\eta^{p+1}|} + \right.$$

$$\left. + \frac{\alpha\eta^{p+q}\Gamma(p+2)}{\Gamma(p+q+1)|\Gamma(p+2) - \alpha\eta^{p+1}|} \right\}.$$

This proves that $F_1(\bar{\Omega}_{r_0})$ is uniformly bounded.

In addition for any $t_1, t_2 \in [0,1], t_1 < t_2$, we have:

$$|(F_1 x)(t_2) - (F_1 x)(t_1)| \leq$$

$$\leq \frac{1}{\Gamma(q)} \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] |f(s,x(s))| ds +$$

$$+ \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} |f(s,x(s))| ds +$$

$$+ \frac{\Gamma(p+2)|t_2 - t_1|}{\Gamma(q)|\Gamma(p+2) - \alpha\eta^{p+1}|} \int_0^1 (1-s)^{q-1} |f(s,x(s))| ds +$$

$$+\frac{\alpha p(p+1)|t_2-t_1|}{\Gamma(q)|\Gamma(p+2)-\alpha\eta^{p+1}|}\int_0^\eta\int_0^s(\eta-s)^{p-1}(s-r)^{q-1}|f(r,x(r))|drds\le$$

$$\le\frac{M_r}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]ds+\frac{M_r}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}ds+$$

$$+\frac{M_r\Gamma(p+2)|t_2-t_1|}{\Gamma(q)|\Gamma(p+2)-\alpha\eta^{p+1}|}\int_0^1(1-s)^{q-1}ds+$$

$$+\frac{M_r\alpha p(p+1)|t_2-t_1|}{\Gamma(q)|\Gamma(p+2)-\alpha\eta^{p+1}|}\int_0^\eta\int_0^s(\eta-s)^{p-1}(s-r)^{q-1}drds,$$

which is independent of x and tends to zero as $t_2-t_1\to 0$. Thus, F_1 is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $F_1(\bar{\Omega}_{r_0})$ is a relatively compact set. Now, let $x_n\subset\bar{\Omega}_{r_0}$ with $\|x_n-x\|\to 0$. Then the limit $\|x_n(t)-x(t)\|\to 0$ uniformly valid on $[0,1]$. From the uniform continuity of $f(t,x)$ on the compact set $[0,1]\times[-r_0,r_0]$ it follows that $\|f(t,x_n(t))-f(t,x(t))\|\to 0$ is uniformly valid on $[0,1]$. Hence $\|F_1x_n-F_1x\|\to 0$ as $n\to\infty$ which proves the continuity of F_1. Hence Step 1 is completely proved.

Step 2. The operator $F_2:\bar{\Omega}_{r_0}\to C([0,1],\mathbb{R})$ is contractive. This is a consequence of (A_2).

Step 3. The set $F(\bar{\Omega}_{r_0})$ is bounded. By (A_2) and (A_4) imply that

$$\|F_2(x)\|\le 2(|x_0|+\ell r_0),$$

for any $x\in\bar{\Omega}_{r_0}$. This, with the boundedness of the set $F_1(\bar{\Omega}_{r_0})$ implies that the set $F(\bar{\Omega}_{r_0})$ is bounded.

Step 4. Finally, it is to show that the case (C2) *in Lemma 3.3 does not occur.* To this end, we suppose that (C2) holds. Then, we have that there exists $\lambda\in(0,1)$ and $x\in\partial\Omega_{r_0}$ such that $x=\lambda Fx$. So, we have $\|x\|=r_0$ and

$$x(t)=\lambda\Bigg[\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,x(s))ds-$$

$$-\frac{\Gamma(p+2)t}{\Gamma(q)[\Gamma(p+2)-\alpha\eta^{p+1}]}\int_0^1(1-s)^{q-1}f(s,x(s))ds+$$

$$+\frac{\alpha p(p+1)t}{\Gamma(q)[\Gamma(p+2)-\alpha\eta^{p+1}]}\int_0^\eta\int_0^s(\eta-s)^{p-1}(s-r)^{q-1}f(r,x(r))drds+$$

$$+(t+1)[x_0+g(x)]\Bigg],\quad t\in[0,1].$$

With hypotheses $(A_4)-(A_6)$, we have

$$r_0 \le \frac{\psi(r_0)}{\Gamma(q)} \left[\int_0^t (t-s)^{q-1} p(s)ds + \frac{\Gamma(p+2)}{|\Gamma(p+2)-\alpha\eta^{p+1}|} \int_0^1 (1-s)^{q-1} p(s)ds + \right.$$
$$\left. + \frac{\alpha p(p+1)}{|\Gamma(p+2)-\alpha\eta^{p+1}|} \int_0^\eta \int_0^s (\eta-s)^{p-1}(s-r)^{q-1} p(r)drds \right] + 2(|x_0| + \ell r_0),$$

which implies

$$r_0 \le 2\ell r_0 + 2|x_0| + p_0\psi(r_0).$$

Thus,

$$\frac{r_0}{2|x_0| + p_0\psi(r_0)} \le \frac{1}{1-2\ell},$$

which contradicts (3.2). Consequently, we have proved that the operators F_1 and F_2 satisfy all the conditions in Lemma 3.3. Hence, the operator F has at least one fixed point $x \in \bar{\Omega}_{r_0}$, which is the solution of the boundary value problem (1.1). The proof is complete. □

4. EXISTENCE RESULTS – THE MULTI-VALUED CASE

Let us recall some basic definitions on multi-valued maps [21], [25].

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0;1] \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y-z| : z \in G(t)\}$$

is measurable.

Let $L^1([0,1],\mathbb{R})$ be the Banach space of measurable functions $x : [0,1] \to \mathbb{R}$, which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)|dt$.

Definition 4.1. A function $x \in AC^1([0,1],\mathbb{R})$ is a solution of the problem (1.1) if $x(0) = x_0 + g(x),\ x(1) = \alpha I^p x(\eta)$, and there exists a function $f \in L^1([0,1],\mathbb{R})$ such that $f(t) \in F(t,x(t))$ a.e. on $[0,1]$ and

$$\begin{aligned} x(t) = &\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s)ds - \\ &-\frac{\Gamma(p+2)t}{\Gamma(q)|\Gamma(p+2)-\alpha\eta^{p+1}|}\int_0^1 (1-s)^{q-1} f(s)ds + \\ &+\frac{\alpha p(p+1)t}{\Gamma(q)|\Gamma(p+2)-\alpha\eta^{p+1}|}\int_0^\eta\int_0^s (\eta-s)^{p-1}(s-r)^{q-1} f(r)drds + \\ &+(1-t)[x_0+g(x)]. \end{aligned} \tag{4.1}$$

4.1. THE CARATHÉODORY CASE

Definition 4.2. A multivalued map $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;
(ii) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,1]$.

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0,1],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le \varphi_\alpha(t)$$

for all $\|x\|_\infty \le \alpha$ and for a.e. $t \in [0,1]$.

For each $y \in C([0,1],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,1],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,1]\}.$$

The following lemma will be used in the sequel.

Lemma 4.3 ([28]). *Let X be a Banach space. Let $F : [0,T]\times\mathbb{R} \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1],X)$ to $C([0,1],X)$. Then the operator*

$$\Theta\circ S_F : C([0,1],X) \to \mathcal{P}_{cp,c}(C([0,1],X)),\quad x \mapsto (\Theta\circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,1],X)\times C([0,1],X)$.

To prove our main result in this section we will use the following form of the Nonlinear Alternative for contractive maps [31, Corollary 3.8].

Theorem 4.4. *Let X be a Banach space, and D a bounded neighborhood of $0 \in X$. Let $Z_1 : X \to \mathcal{P}_{cp,c}(X)$ (here $\mathcal{P}_{cp,c}(X)$ denotes the family of all nonempty, compact and convex subsets of X) and $Z_2 : \bar{D} \to \mathcal{P}_{cp,c}(X)$ two multi-valued operators satisfying*

(a) *Z_1 is contraction, and*
(b) *Z_2 is u.s.c and compact.*

Then, if $G = Z_1 + Z_2$, either

(i) *G has a fixed point in $\bar{D}$ or*
(ii) *there is a point $u \in \partial D$ and $\lambda \in (0,1)$ with $u \in \lambda G(u)$.*

Theorem 4.5. *Assume that:*

(H_1) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory multivalued map;*
(H_2) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,1],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|) \quad \textit{for each } (t,x) \in [0,1] \times \mathbb{R};$$

(H_3) *there exists a constant $L_g < 1/2$ such that*

$$|g(x) - g(y)| \le L_g|x-y| \quad \textit{for all} \quad x, y \in \mathbb{R};$$

(H_4) *there exists a number $M > 0$ such that*

$$\frac{(1-2L_g)M}{\Lambda\psi(M) + 2|x_0|} > 1, \tag{4.2}$$

where

$$\Lambda = \frac{1}{\Gamma(q)}\left[\int_0^1 (1-s)^{q-1}p(s)ds + \frac{\Gamma(p+2)}{|\Gamma(p+2)-\alpha\eta^{p+1}|}\int_0^1 (1-s)^{q-1}p(s)ds + \right.$$
$$\left. + \frac{\alpha p(p+1)}{|\Gamma(p+2)-\alpha\eta^{p+1}|}\int_0^\eta\int_0^s (\eta-s)^{p-1}(s-r)^{q-1}p(r)drds\right].$$

Then the boundary value problem (1.2) has at least one solution on $[0,1]$.

Proof. Transform the problem (1.2) into a fixed point problem. Consider the operator $\mathcal{N} : C([0,1],\mathbb{R}) \longrightarrow \mathcal{P}(C([0,1],\mathbb{R}))$ defined by

$$\mathcal{N}(x) = \Bigg\{ h \in C([0,1],\mathbb{R}) : h(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds -$$
$$- \frac{\Gamma(p+2)t}{\Gamma(q)[\Gamma(p+2) - \alpha\eta^{p+1}]} \int_0^1 (1-s)^{q-1} f(s)ds +$$
$$+ \frac{\alpha p(p+1)t}{\Gamma(q)[\Gamma(p+2) - \alpha\eta^{p+1}]} \int_0^\eta \int_0^s (\eta-s)^{p-1}(s-r)^{q-1} f(r)drds +$$
$$+ (1-t)[x_0 + g(x)] \Bigg\}$$

for $f \in S_{F,x}$.

Now, we define two operators as follows: $\mathcal{A} : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ by

$$\mathcal{A}x(t) = (1-t)(x_0 + g(x)), \tag{4.3}$$

and the multi-valued operator $\mathcal{B} : C([0,1],\mathbb{R}) \longrightarrow \mathcal{P}(C([0,1],\mathbb{R}))$ by

$$\mathcal{B}(x) = \Bigg\{ h \in C([0,1],\mathbb{R}) : \tag{4.4}$$
$$h(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds -$$
$$- \frac{\Gamma(p+2)t}{\Gamma(q)[\Gamma(p+2) - \alpha\eta^{p+1}]} \int_0^1 (1-s)^{q-1} f(s)ds +$$
$$+ \frac{\alpha p(p+1)t}{\Gamma(q)[\Gamma(p+2) - \alpha\eta^{p+1}]} \int_0^\eta \int_0^s (\eta-s)^{p-1}(s-r)^{q-1} f(r)drds \Bigg\}.$$

Then $\mathcal{N} = \mathcal{A} + \mathcal{B}$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 4.4 on $[0,1]$. For better readability, we break the proof into a sequence of steps and claims.

Step 1. We show that $\mathcal{A}$ *is a contraction on* $C([0,1],\mathbb{R})$. Let $x, y \in C([0,1],\mathbb{R})$. Then

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| = |1-t||g(x) - g(y)| \leq 2|g(x) - g(y)| \leq 2L_g|x-y|.$$

Taking supremum over t,

$$\|\mathcal{A}x - \mathcal{A}y\| \leq L_0\|x-y\|, \quad L_0 = 2L_g < 1.$$

This shows that $\mathcal{A}$ is a contraction, since $L_0 < 1$.

Step 2. We shall show that the operator $\mathcal{B}$ *is compact and convex valued and it is completely continuous.* This will be given in several claims.

Claim I. $\mathcal{B}$ *maps bounded sets into bounded sets in* $C([0,1],\mathbb{R})$. To see this, let $B_\rho = \{x \in C([0,1],\mathbb{R}) : \|x\| \le \rho\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then, for each $h \in \mathcal{B}(x), x \in B_\rho$, there exists $f \in S_{F,x}$ such that

$$x(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s)ds - \frac{\Gamma(p+2)t}{\Gamma(q)[\Gamma(p+2)-\alpha\eta^{p+1}]}\int_0^1 (1-s)^{q-1} f(s)ds + \\ + \frac{\alpha p(p+1)t}{\Gamma(q)[\Gamma(p+2)-\alpha\eta^{p+1}]}\int_0^\eta\int_0^s (\eta-s)^{p-1}(s-r)^{q-1} f(r)drds.$$

Then for $t \in [0,1]$ we have

$$\begin{aligned} |h(t)| \le{}& \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s)|ds + \\ &+ \frac{\Gamma(p+2)}{\Gamma(q)|\Gamma(p+2)-\alpha\eta^{p+1}|}\int_0^1 (1-s)^{q-1}|f(s)|ds + \\ &+ \frac{\alpha p(p+1)}{\Gamma(q)|\Gamma(p+2)-\alpha\eta^{p+1}|}\int_0^\eta\int_0^s (\eta-s)^{p-1}(s-r)^{q-1}|f(r)|drds \le \\ \le{}& \psi(\|x\|)\Bigg[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}p(s)ds + \frac{\Gamma(p+2)}{\Gamma(q)|\Gamma(p+2)-\alpha\eta^{p+1}|}\int_0^1 (1-s)^{q-1}p(s)ds + \\ &+ \frac{\alpha p(p+1)}{\Gamma(q)|\Gamma(p+2)-\alpha\eta^{p+1}|}\int_0^\eta\int_0^s (\eta-s)^{p-1}(s-r)^{q-1}p(r)drds\Bigg]. \end{aligned}$$

Thus,

$$\|h\| \le \frac{\psi(\rho)}{\Gamma(q)}\Bigg[\int_0^1 (t-s)^{q-1}p(s)ds + \frac{\Gamma(p+2)}{|\Gamma(p+2)-\alpha\eta^{p+1}|}\int_0^1 (1-s)^{q-1}p(s)ds + \\ + \frac{\alpha p(p+1)}{|\Gamma(p+2)-\alpha\eta^{p+1}|}\int_0^\eta\int_0^s (\eta-s)^{p-1}(s-r)^{q-1}p(r)drds\Bigg].$$

Claim II. Next we show that $\mathcal{B}$ *maps bounded sets into equi-continuous sets.* Let $t', t'' \in [0,1]$ with $t' < t''$ and $x \in B_\rho$. For each $h \in \mathcal{B}(x)$, we obtain

$$|h(t'') - h(t')| \leq$$

$$\leq \left| \psi(\|x\|) \int_0^{t'} \left[\frac{(t''-s)^{q-1} - (t'-s)^{q-1}}{\Gamma(q)} \right] p(s)ds + \psi(\|x\|) \int_{t'}^{t''} \frac{(t''-s)^{q-1}}{\Gamma(q)} p(s)ds \right| +$$

$$+ \psi(\|x\|) \frac{\Gamma(p+2)|t''-t'|}{|\Gamma(p+2) - \alpha\eta^{p+1}|} \int_0^1 (1-s)^{q-1} p(s)ds +$$

$$+ \psi(\|x\|) \frac{\alpha p(p+1)|t''-t'|}{|\Gamma(p+2) - \alpha\eta^{p+1}|} \int_0^{\eta} \int_0^{s} (\eta-s)^{p-1}(s-r)^{q-1} p(r)drds.$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $t'' - t' \to 0$. As $\mathcal{B}$ satisfies the above three assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{B} : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ is completely continuous.

Claim III. Next we prove that $\mathcal{B}$ *has a closed graph.* Let $x_n \to x_*, h_n \in \mathcal{B}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{B}(x_*)$. Associated with $h_n \in \mathcal{B}(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$h_n(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_n(s)ds - \frac{\Gamma(p+2)t}{\Gamma(q)[\Gamma(p+2) - \alpha\eta^{p+1}]} \int_0^1 (1-s)^{q-1} f_n(s)ds +$$

$$+ \frac{\alpha p(p+1)t}{\Gamma(q)[\Gamma(p+2) - \alpha\eta^{p+1}]} \int_0^{\eta} \int_0^{s} (\eta-s)^{p-1}(s-r)^{q-1} f_n(r)drds.$$

Thus it suffices to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$h_*(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_*(s)ds - \frac{\Gamma(p+2)t}{\Gamma(q)[\Gamma(p+2) - \alpha\eta^{p+1}]} \int_0^1 (1-s)^{q-1} f_*(s)ds +$$

$$+ \frac{\alpha p(p+1)t}{\Gamma(q)[\Gamma(p+2) - \alpha\eta^{p+1}]} \int_0^{\eta} \int_0^{s} (\eta-s)^{p-1}(s-r)^{q-1} f_*(r)drds.$$

Let us consider the linear operator $\Theta : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$f \mapsto \Theta(f)(t) = \\ = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds - \frac{\Gamma(p+2)t}{\Gamma(q)[\Gamma(p+2)-\alpha\eta^{p+1}]} \int_0^1 (1-s)^{q-1} f(s)ds + \\ + \frac{\alpha p(p+1)t}{\Gamma(q)[\Gamma(p+2)-\alpha\eta^{p+1}]} \int_0^\eta \int_0^s (\eta-s)^{p-1}(s-r)^{q-1} f(r)drds.$$

Observe that

$$\|h_n(t) - h_*(t)\| = \\ = \Bigg\| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}(f_n(s) - f_*(s))ds - \\ - \frac{\Gamma(p+2)t}{\Gamma(q)[\Gamma(p+2)-\alpha\eta^{p+1}]} \int_0^1 (1-s)^{q-1}(f_n(s) - f_*(s))ds + \\ + \frac{\alpha p(p+1)t}{\Gamma(q)[\Gamma(p+2)-\alpha\eta^{p+1}]} \int_0^\eta \int_0^s (\eta-s)^{p-1}(s-r)^{q-1}(f_n(r) - f_*(r))drds \Bigg\| \to 0,$$

as $n \to \infty$.

Thus, it follows by Lemma 4.3 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_*(s)ds - \frac{\Gamma(p+2)t}{\Gamma(q)[\Gamma(p+2)-\alpha\eta^{p+1}]} \int_0^1 (1-s)^{q-1} f_*(s)ds + \\ + \frac{\alpha p(p+1)t}{\Gamma(q)[\Gamma(p+2)-\alpha\eta^{p+1}]} \int_0^\eta \int_0^s (\eta-s)^{p-1}(s-r)^{q-1} f_*(r)drds$$

for some $f_* \in S_{F,x_*}$. Hence $\mathcal{B}$ has a closed graph (and therefore has closed values). As a result $\mathcal{B}$ is compact valued.

Therefore, the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 4.4 and hence an application of it yields that either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $x \in \lambda\mathcal{A}(x) + \lambda\mathcal{B}(x)$ for $\lambda \in (0,1)$, then there exists $f \in S_{F,x}$ such that

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds - \frac{\Gamma(p+2)t}{\Gamma(q)[\Gamma(p+2) - \alpha\eta^{p+1}]} \int_0^1 (1-s)^{q-1} f(s)ds +$$
$$+ \frac{\alpha p(p+1)t}{\Gamma(q)[\Gamma(p+2) - \alpha\eta^{p+1}]} \int_0^\eta \int_0^s (\eta - s)^{p-1} (s-r)^{q-1} f(r) dr ds +$$
$$+ (1-t)[x_0 + g(x)], \quad t \in [0,1].$$

Consequently, we have

$$|x(t)| \leq \frac{\psi(\|x\|)}{\Gamma(q)} \left[\int_0^1 (1-s)^{q-1} p(s)ds + \frac{\Gamma(p+2)}{|\Gamma(p+2) - \alpha\eta^{p+1}|} \int_0^1 (1-s)^{q-1} p(s)ds + \right.$$
$$\left. + \frac{\alpha p(p+1)}{|\Gamma(p+2) - \alpha\eta^{p+1}|} \int_0^\eta \int_0^s (\eta - s)^{p-1}(s-r)^{q-1} p(r) dr ds \right] + 2[|x_0| + L_g\|x\|].$$

If condition (ii) of Theorem 4.4 holds, then there exists $\lambda \in (0,1)$ and $x \in \partial B_r$ with $x = \lambda \mathcal{N}(x)$. Then, x is a solution of (2.7) with $\|x\| = M$. Now, the previous inequality implies

$$\frac{(1-2L_g)M}{\Lambda\psi(M) + 2|x_0|} \leq 1,$$

which contradicts (4.2). Hence, $\mathcal{N}$ has a fixed point in $[0,1]$ by Theorem 4.4, and consequently the boundary value problem (1.2) has a solution. This completes the proof. □

4.2. THE LOWER SEMI-CONTINUOUS CASE

As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with this problems is based on the nonlinear alternative of Leray-Schauder type together with the selection theorem of Bressan and Colombo [17] for lower semi-continuous maps with decomposable values.

Let us mention some auxiliary facts. Let X be a nonempty closed subset of a Banach space E and $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,1] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,1],\mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,1] = J$, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 4.6. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0,1] \times \mathbb{R}) \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ associated with F as

$$\mathcal{F}(x) = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,1]\},$$

which is called the Nemytskii operator associated with F.

Definition 4.7. Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Lemma 4.8 ([22])**.** *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,1],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Theorem 4.9. *Assume that $(H_2), (H_3), (H_4)$ and the following conditions hold:*

(H_5) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that:*
- (a) *$(t,x) \longmapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,*
- (b) *$x \longmapsto F(t,x)$ is lower semicontinuous for each $t \in [0,1]$.*

Then the boundary value problem (1.1) *has at least one solution on* $[0,1]$.

Proof. It follows from (H_2) and (H_5) that F is of l.s.c. type. Then from Lemma 4.8, there exists a continuous function $f : C([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,1],\mathbb{R})$.

Consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x(t)), & 0 < t < 1, \quad 1 < q \le 2, \\ x(0) = x_0 + g(x), & x(1) = \alpha I^p x(\eta), \quad 0 < \eta < 1. \end{cases} \tag{4.5}$$

Observe that if $x \in AC^1([0,1])$ is a solution of (4.5), then x is a solution to the problem (1.1). Now, we define two operators as follows: $\mathcal{A}' : C([0,1],E) \longrightarrow C([0,1],\mathbb{R})$ by

$$\mathcal{A}'x(t) = (1-t)(x_0 + g(x)), \tag{4.6}$$

and the multi-valued operator $\mathcal{B}' : C([0,1],E) \longrightarrow \mathcal{P}(C([0,1],\mathbb{R}))$ by

$$\begin{aligned} \mathcal{B}'x(t) = {} & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(x(s))ds - \\ & - \frac{\Gamma(p+2)t}{\Gamma(q)[\Gamma(p+2) - \alpha\eta^{p+1}]} \int_0^1 (1-s)^{q-1} f(x(s))ds + \\ & + \frac{\alpha p(p+1)t}{\Gamma(q)[\Gamma(p+2) - \alpha\eta^{p+1}]} \int_0^\eta \int_0^s (\eta-s)^{p-1}(s-r)^{q-1} f(x(r))drds. \end{aligned} \tag{4.7}$$

Now $\mathcal{A}', \mathcal{B}' : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ are continuous. Also the argument in Theorem 4.5 guarantees that $\mathcal{A}'$ and $\mathcal{B}'$ satisfy all the conditions of the Nonlinear Alternative for contractive maps in the single valued setting [23] and hence the problem (4.5) has a solution. $\square$

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