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NOTES ON TOPOLOGICAL INDICES OF GRAPH AND ITS COMPLEMENT

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Abstract. In this note, we derive the lower bound on the sum for Wiener index of bipartite graph and its bipartite complement, as well as the lower and upper bounds on this sum for the Randić index and Zagreb indices. We also discuss the quality of these bounds.

Keywords: Wiener index, Zagreb index, Randić index, bipartite graph, bipartite complement.

Mathematics Subject Classification: 05C12.

1. INTRODUCTION

Throughout this note, we consider simple graphs (not necessarily connected). Given a graph $G = (V, E)$ and its vertices $u, v \in V$, the distance $d(u, v)$ is the length of the shortest path between u and v; if u and v belong to different components of G, then we set $d(u, v) = +\infty$. The *Wiener index* $W(G)$ of G is the sum of distances of all unordered pairs of vertices of G. This graph invariant has found numerous applications in mathematical chemistry in connection with modeling the physical properties of compounds using the structural description of their molecules (see, for example, the survey papers [1] and [2]). Besides the chemical connections, the mathematical properties of the Wiener index are also studied as well. In the paper [3], it was proved that, for an n-vertex graph G and its complement $\overline{G}$, $W(G)+W(\overline{G}) \geq \frac{3}{2}n(n-1)$ this bound is sharp for $n \geq 5$. This result suggests a prove, for the Wiener index, a theorem of Nordhaus-Gaddum type (that is, the best lower and upper bounds for the sum or the product of values of a particular graph invariant for a graph and its complement); indeed, the upper bound $W(G) + W(\overline{G}) \leq \frac{n^3+3n^2+2n-6}{6}$ was proved in [7]. The results of Nordhaus-Gaddum type were proved for many graph invariants (among them vertex and edge chromatic number, domination and independence number) including chemical indices (see papers [6, 7]).

Our aim is to derive analogical results for the sum of selected chemical indices for bipartite graph and its bipartite complement. Recall that the *bipartite complement* of the bipartite graph $G = (X, Y; E)$ with respect to bipartitions X, Y is the bipartite graph $\widetilde{G} = (X, Y; \widetilde{E})$ with the same bipartitions such that, for each $x \in X$ and $y \in Y$, $xy \in \widetilde{E}$ if and only if $xy \notin E$. Besides Wiener index, in this paper, we will consider also *Zagreb indices* $M_1(G) = \sum_{v \in V(G)} \deg^2(v)$ and $M_2(G) = \sum_{uv \in E(G)} \deg(u) \cdot \deg(v)$, and the *Randić index* $R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{\deg(u)\deg(v)}}$. The next section contains Nordhaus-Gaddum type results for sums based on these indices with respect to bipartite complementation and a discussion on the quality of obtained bounds.

2. RESULTS

Theorem 2.1. *Let $G = (X, Y; E)$ be bipartite graph on n vertices. Then $W(G) + W(\widetilde{G}) \geq 2n(n-1)$.*

Proof. Let $|X| = p, |Y| = q$ and let $\widetilde{d}(u, v)$ be the distance of u and v in $\widetilde{G}$. Then

$$W(G) = \sum_{x \in X, y \in Y} d(u, v) + \sum_{x, x' \in X} d(x, x') + \sum_{y, y' \in Y} d(y, y'),$$

$$W(\widetilde{G}) = \sum_{x \in X, y \in Y} \widetilde{d}(u, v) + \sum_{x, x' \in X} \widetilde{d}(x, x') + \sum_{y, y' \in Y} \widetilde{d}(y, y').$$

Let $x \in X, y \in Y$ be two vertices from different bipartitions of G. If $xy \in E$, then $d(x, y) = 1$ and $xy \notin \widetilde{E}$, which implies (using the fact that $\widetilde{G}$ is also bipartite) $\widetilde{d}(x, y) \geq 3$. Similarly, if $xy \notin E$, then $d(x, y) \geq 3$ and $\widetilde{d}(x, y) = 1$. We conclude that each pair of vertices from different bipartitions of G contributes at least 4 to the sum $W(G) + W(\widetilde{G})$, thus, the total contribution of such pairs is at least $4pq$.

Now, let $x, x' \in X$ (the case when both vertices are from Y is symmetrical). Then $d(x, x') \geq 2, \widetilde{d}(x, x') \geq 2$; thus, any two vertices from the same bipartition of G contribute at least 4 to $W(G) + W(\widetilde{G})$, which yields a total contribution of at least $4\binom{p}{2}$ (and $4\binom{q}{2}$ regarding the bipartition Y).

Hence,

$$W(G) + W(\widetilde{G}) \geq 4pq + 4\binom{p}{2} + 4\binom{q}{2} = 2(p+q)(p+q-1) = 2n(n-1). \qquad \square$$

To discuss the sharpness of this bound, consider a finite projective plane F of order k, and let G_k be its incidence graph. It follows that G_k is a balanced bipartite, has $2(k^2 + k + 1)$ vertices, is $(k+1)$-regular and has diameter 3. Let x, y be from the same bipartition of G_k (there are $\binom{k^2+k+1}{2}$ such pairs). Then $d(x, y) = 2$ (because, in F, every two distinct points are incident with a line, and vice versa). Suppose that x, y are from different bipartitions of G_k. Then, for fixed x, there are $k+1$ vertices y

with $d(x,y) = 1$, and k^2 vertices y' with $d(x,y') = 3$. Calculating the Wiener index of G_k, we obtain

$$W(G_k) = 2 \cdot 2 \cdot \binom{k^2+k+1}{2} + 1 \cdot (k^2+k+1)(k+1) + 3 \cdot (k^2+k+1)k^2 =$$

$$= (k^2+k+1)(5k^2+3k+1).$$

Now, the graph $\widetilde{G}_k$ is again a balanced bipartite with $2(k^2+k+1)$ vertices, is k^2-regular and has diameter 3. Again, $\widetilde{d}(x,y) = 2$ for x, y from the same bipartition of G_k, $\widetilde{d}(x,y) = 3$ for x, y from different bipartitions (for fixed x, there are $k+1$ such pairs) and otherwise $\widetilde{d}(x,y) = 1$ (with k^2 pairs for fixed x). Hence,

$$W(\widetilde{G}_k) = 2 \cdot 2 \cdot \binom{k^2+k+1}{2} + 1 \cdot (k^2+k+1)k^2 + 3 \cdot (k^2+k+1)(k+1) =$$

$$= (k^2+k+1)(3k^2+5k+3).$$

Thus $W(G_k)+W(\widetilde{G}_k) = 4(k^2+k+1)(2k^2+2k+1) = 2n(n-1)$ for $n = k^2+k+1$. This implies that the lower bound in Theorem 2.1 is attained for all $n = 2(k^2+k+1)$, where k is a prime power.

On the other hand, there are values of n for which this lower bound is not attained. For example, there are 17 connected bipartite graphs on 6 vertices, and 44 connected bipartite graphs on 7 vertices (see [4]), but no 6-vertex connected bipartite graph has a connected bipartite complement, and there are only two 7-vertex connected bipartite graphs with connected components: the path P_7 and the 3-star T^* with branches of length 1, 2 and 3 (the graph B34 in [4, p. 192]). For these graphs, $\widetilde{P}_7 \cong P_7, \widetilde{T^*} \cong T^*$ and $W(P_7)+W(\widetilde{P}_7) = 2 \cdot 56 = 112 > 2 \cdot 7 \cdot (7-1)$, $W(T^*)+W(\widetilde{T^*}) = 2 \cdot 50 = 100 > 2 \cdot 7 \cdot (7-1)$.

Next, of 182 connected bipartite graphs on 8 vertices, 28 have connected bipartite complement; using the database of bipartite graphs (see [5]) and the Maple 14 software, we found out that the minimum value of $W(G) + W(\widetilde{G})$ equals 120 (being greater that $2 \cdot 8 \cdot 7 = 112$) for G being the graph obtained from the graph of a 3-cube by deleting edges of any 4-cycle. Similarly, when considering 9-vertex bipartite graphs, the minimum value 152 (being greater that $2 \cdot 9 \cdot 8 = 144$) results from only one graph which is obtained from the graph $K_{2,3}$ by attaching four vertices of degree 1 to vertices of any 4-cycle. For 10-vertex bipartite graphs, the minimum value is 188 (which is greater that $2 \cdot 10 \cdot 9 = 180$) and is obtained for five graphs, see Figure 1.

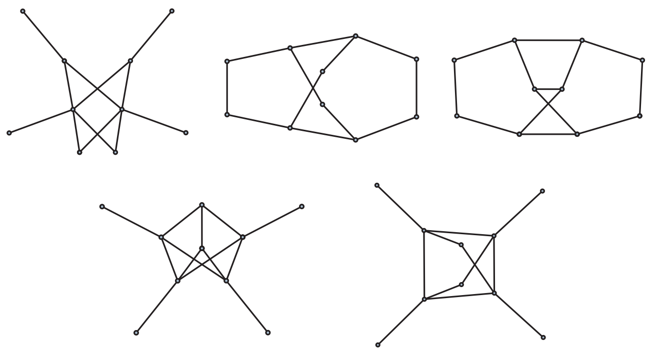

Fig. 1. 10-vertex connected bipartite graphs with minimal $W(G) + W(\widetilde{G})$

For 11-vertex bipartite graphs, the minimum value is 226 (which is greater that $2 \cdot 11 \cdot 10 = 220$) and is obtained for two complementary graphs, see Figure 2.

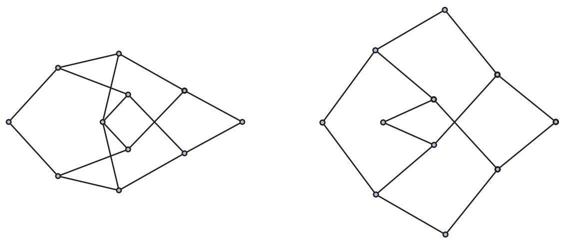

Fig. 2. 11-vertex connected bipartite graphs with minimal $W(G) + W(\widetilde{G})$

Herbert Vojčík (private communication) has performed the computer search on 12- and 13-vertex bipartite graphs and found many graphs attaining the equality in Theorem 2.1. It is an open question whether the equality in Theorem 2.1 can be attained for any $n \geq 12$.

In general, one might consider the notion of a non-standard complement also for non-bipartite graphs, in the following way: given a graph $G = (V_1, \ldots, V_k; E)$ with distinguished partition $\mathcal{X} = (V_1, \ldots, V_k)$ of its vertex set into k classes, its complement with respect to $\mathcal{X}$ is the graph $\widetilde{G}_{\mathcal{X}}$ with the same vertex set under the same partition such that, for each $x \in V_i$ and $y \in V_j$ with $i \neq j$, $xy \in E(\widetilde{G}_{\mathcal{X}})$ if and only if $xy \notin E$. However, in order to have such a complement independent of vertex set partitions, the partition $\mathcal{X}$ should be uniquely induced in a "generic way" by certain natural and reasonable graph characteristics. The standard graph complement of an n-vertex graph corresponds to the "most granular" partition $\mathcal{X} = (\{v_1\}, \{v_2\}, \ldots, \{v_n\})$. Another way how to obtain a uniquely defined complement is to consider the family of uniquely k-colourable graphs (that is, the graphs such that each of their k-colouring induces the same vertex partition); this generalizes the bipartite complement (as the family of uniquely colourable graphs also includes connected bipartite graphs).

For these graphs, we can prove the following analogue of Theorem 2.1.

Theorem 2.2. *Let G be a uniquely k-colourable graph on n vertices with $\mathcal{X}$ being the partition of $V(G)$ into k colour classes, and let p, q be nonnegative integers such that $n = pk + q, 0 \le p \le k-1$. Then $W(G) + W(\widetilde{G}_{\mathcal{X}}) \ge 3\binom{n}{2} + (k-q)\binom{p}{2} + q\binom{p+1}{2}$.*

Proof. Let $n_i, n = 1, \ldots, k$ be the number of vertices in the i-th colour class of G. Consider a pair x, y of vertices of G. If x, y are from the same colour class, then $d_G(x, y) \ge 2, d_{\widetilde{G}_{\mathcal{X}}}(x, y) \ge 2$; thus, the total contribution of such pairs to $W(G) + W(\widetilde{G}_{\mathcal{X}})$ is at least $4\sum\limits_{i=1}^{k} \binom{n_i}{2}$. If x, y are from different colour classes, then their contribution is at least 3 (since x, y are nonadjacent in G or $\widetilde{G}_{\mathcal{X}}$). Thus

$$W(G) + W(\widetilde{G}_{\mathcal{X}}) \ge 4\sum_{i=1}^{k}\binom{n_i}{2} + 3\left(\binom{n}{2} - \sum_{i=1}^{k}\binom{n_i}{2}\right) = 3\binom{n}{2} + \sum_{i=1}^{k}\binom{n_i}{2}.$$

With $\sum_{i=1}^{k} n_i = n$, it can be easily checked that the minimum of $\sum_{i=1}^{k}\binom{n_i}{2}$ is attained when any two numbers of $n_1, \ldots, n_k$ differ by at most one; this means that, of k variables $n_1, \ldots, n_k$, q are equal to $p+1$ and $k-q$ are equal to p. Thus $\sum_{i=1}^{k}\binom{n_i}{2} \ge (k-q)\binom{p}{2} + q\binom{p+1}{2}$. □

In the next, we turn our attention to Zagreb indices for bipartite graph and its bipartite complement.

Theorem 2.3. *Let $G = (P, Q; E)$ be a bipartite graph with $|P| = p$, $|Q| = q$ and let $\widetilde{G} = (P, Q; \widetilde{E})$ be its bipartite complement. Then*

$$\frac{pq(p+q)}{2} \le M_1(G) + M_1(\widetilde{G}) \le pq(p+q).$$

Proof. For the sum of the first Zagreb indices of G and $\widetilde{G}$, we have

$$M_1(G) + M_1(\widetilde{G}) = \sum_{u \in P\cup Q} \deg_G^2(u) + \sum_{u \in P\cup Q} \deg_{\widetilde{G}}^2(u) = \sum_{u\in P} \deg_G^2(u) + \sum_{u \in Q} \deg_G^2(u) +$$

$$+ \sum_{u\in P} \deg_{\widetilde{G}}^2(u) + \sum_{u\in Q} \deg_{\widetilde{G}}^2(u) = \sum_{u\in P} (\deg_G^2(u) + \deg_{\widetilde{G}}^2(u)) + \sum_{u\in Q} (\deg_G^2(u) + \deg_{\widetilde{G}}^2(u)).$$

By Jensen's inequality (applied on the convex function $f(x) = x^2$) we have $\deg_G^2(u) + \deg_{\widetilde{G}}^2(u) \ge \frac{(\deg_G(u) + \deg_{\widetilde{G}}(u))^2}{2}$; thus, we obtain

$$M_1(G) + M_1(\widetilde{G}) \ge \sum_{u\in P} \frac{(\deg_G(u) + \deg_{\widetilde{G}}(u))^2}{2} + \sum_{u\in Q} \frac{(\deg_G(u) + \deg_{\widetilde{G}}(u))^2}{2} =$$

$$= \sum_{u\in P}\frac{q^2}{2} + \sum_{u\in Q}\frac{p^2}{2} = \frac{pq^2}{2} + \frac{qp^2}{2} = \frac{pq(p+q)}{2}.$$

To prove the upper bound, we use the fact that $\deg^2_G(u)+\deg^2_{\widetilde{G}}(u)\le(\deg_G(u)+\deg_{\widetilde{G}}(u))^2$. Therefore,

$$M_1(G)+M_1(\widetilde{G})\le\sum_{u\in P}(\deg_G(u)+\deg_{\widetilde{G}}(u))^2+\sum_{u\in Q}(\deg_G(u)+\deg_{\widetilde{G}}(u))^2=$$
$$=\sum_{u\in P}q^2+\sum_{u\in Q}p^2=pq^2+qp^2=pq(p+q).\qquad\square$$

Note that, in the above theorem, both bounds are best possible. The lower bound is attained for a bipartite graph $G=(P,Q;E)$ such that, for any vertex $u\in P$, $\deg(u)=\frac{q}{2}$ and for $u\in Q$, $\deg(u)=\frac{p}{2}$. The equality in the upper bound holds for a complete bipartite graph $K_{p,q}$.

Theorem 2.4. *Let $G=(P,Q;E)$ be a bipartite graph with $|P|=p$, $|Q|=q$ and let $\widetilde{G}=(P,Q;\widetilde{E})$ be its bipartite complement. Then*

$$\left(\frac{pq}{2}\right)^2\le M_2(G)+M_2(\widetilde{G})\le(pq)^2.$$

Proof. Recall that

$$M_2(G)+M_2(\widetilde{G})=\sum_{uv\in E(G)}\deg_G(u)\cdot\deg_G(v)+\sum_{uv\in E(\widetilde{G})}\deg_{\widetilde{G}}(u)\cdot\deg_{\widetilde{G}}(v)$$

and that G and $\widetilde{G}$ together have pq edges. For an upper bound, we have

$$M_2(G)+M_2(\widetilde{G})\le\sum_{uv\in E(G)}qp+\sum_{uv\in E(\widetilde{G})}qp=pq(|E(G)|+|E(\widetilde{G})|)=(pq)^2.$$

To show the lower bound, we will use the approach and Lemma 2.1 from [7]: for the function $f:\ [0,a]\to\mathbb{R}$ defined by $f(x)=x^x(a-x)^{(a-x)}$ for $x\in(0,a)$ with $f(0)=f(a)=a^a$, it holds $f(x)\ge\left(\frac{a}{2}\right)^a$ for each $x\in[0,a]$.

By the arithmetic-geometric mean inequality, we have

$$\frac{M_2(G)+M_2(\widetilde{G})}{pq}\ge\sqrt[pq]{\left(\prod_{uv\in E(G)}\deg_G(u)\cdot\deg_G(v)\right)\left(\prod_{uv\in E(\widetilde{G})}\deg_{\widetilde{G}}(u)\cdot\deg_{\widetilde{G}}(v)\right)}$$

thus,

$$
\begin{aligned}
&M_2(G) + M_2(\widetilde{G}) \geq \\
&\geq pq \sqrt[pq]{\left(\prod_{uv \in E(G)} \deg_G(u) \cdot \deg_G(v)\right)\left(\prod_{uv \in E(\widetilde{G})} \deg_{\widetilde{G}}(u) \cdot \deg_{\widetilde{G}}(v)\right)} = \\
&= pq \sqrt[pq]{\left(\prod_{x \in V(G)} \deg_G(x)^{\deg_G(x)}\right)\left(\prod_{x \in V(\widetilde{G})} \deg_{\widetilde{G}}(x)^{\deg_{\widetilde{G}}(x)}\right)} = \\
&= pq \sqrt[pq]{\prod_{x \in P} \deg_G(x)^{\deg_G(x)} \prod_{x \in Q} \deg_G(x)^{\deg_G(x)} \prod_{x \in P} \deg_{\widetilde{G}}(x)^{\deg_{\widetilde{G}}(x)} \prod_{x \in Q} \deg_{\widetilde{G}}(x)^{\deg_{\widetilde{G}}(x)}} = \\
&= pq \sqrt[pq]{\prod_{x \in P} \deg_G(x)^{\deg_G(x)} (q - \deg_G(x))^{q - \deg_G(x)} \prod_{x \in Q} \deg_G(x)^{\deg_G(x)} (p - \deg_G(x))^{p - \deg_G(x)}} = \\
&= pq \sqrt[pq]{\prod_{x \in P} \deg_G(x)^{\deg_G(x)} (q - \deg_G(x))^{q - \deg_G(x)}} \times \\
&\quad \times \sqrt[pq]{\prod_{x \in Q} \deg_G(x)^{\deg_G(x)} (p - \deg_G(x))^{p - \deg_G(x)}} \geq \\
&\geq pq \sqrt[pq]{\prod_{x \in P} \left(\frac{q}{2}\right)^q} \sqrt[pq]{\prod_{x \in Q} \left(\frac{p}{2}\right)^p} = pq \cdot \frac{q}{2} \cdot \frac{p}{2} = \left(\frac{pq}{2}\right)^2.
\end{aligned}
$$

□

Note that both bounds are best possible and they are attained for the same graphs as in Theorem 2.4.

Theorem 2.5. *Let $G = (P, Q; E)$ be a bipartite graph with $|P| = p$, $|Q| = q$, $|E| = m$ and let $\widetilde{G} = (P, Q; \widetilde{E})$ be its bipartite complement. Then*

$$\sqrt{pq} \leq R(G) + R(\widetilde{G}) \leq (\sqrt{p} + \sqrt{q}) \left(\frac{\sqrt{m} + \sqrt{pq - m}}{2}\right).$$

Proof. For any edge $uv \in E(G)$, $\deg_G(u) \cdot \deg_G(v) \leq pq$, thus $\frac{1}{\sqrt{\deg_G(u) \cdot \deg_G(v)}} \geq \frac{1}{\sqrt{pq}}$; the same lower bound holds also for any edge $uv \in E(\widetilde{G})$. Hence,

$$R(G) + R(\widetilde{G}) \geq \sum_{uv \in E(G)} \frac{1}{\sqrt{pq}} + \sum_{uv \in E(\widetilde{G})} \frac{1}{\sqrt{pq}} = pq \cdot \frac{1}{\sqrt{pq}} = \sqrt{pq}.$$

On the other hand, to prove the upper bound we use the relation of arithmetic mean and root mean square. Denote by $N(u)$ and $\widetilde{N}(u)$ the set of the neighbours of a vertex

u in G and in $\widetilde{G}$, respectively. We have

$$R(G)+R(\widetilde{G})=\sum_{uv\in E(G)}\frac{1}{\sqrt{\deg_G(u)\deg_G(v)}}+\sum_{uv\in E(\widetilde{G})}\frac{1}{\sqrt{\deg_{\widetilde{G}}(u)\deg_{\widetilde{G}}(v)}}=$$

$$=\frac{1}{2}\sum_{u\in V(G)}\left(\sum_{v\in N(u)}\frac{1}{\sqrt{\deg_G(u)\deg_G(v)}}\right)+\frac{1}{2}\sum_{u\in V(G)}\left(\sum_{v\in\tilde{N}(u)}\frac{1}{\sqrt{\deg_{\widetilde{G}}(u)\deg_{\widetilde{G}}(v)}}\right)=$$

$$=\frac{1}{2}\sum_{u\in V(G)}\left(\frac{1}{\sqrt{\deg_G(u)}}\sum_{v\in N(u)}\frac{1}{\sqrt{\deg_G(v)}}\right)+\frac{1}{2}\sum_{u\in V(G)}\left(\frac{1}{\sqrt{\deg_{\widetilde{G}}(u)}}\sum_{v\in\tilde{N}(u)}\frac{1}{\sqrt{\deg_{\widetilde{G}}(v)}}\right)=$$

$$=\frac{1}{2}\sum_{u\in P}\left(\frac{1}{\sqrt{\deg_G(u)}}\sum_{v\in N(u)}\frac{1}{\sqrt{\deg_G(v)}}\right)+\frac{1}{2}\sum_{u\in Q}\left(\frac{1}{\sqrt{\deg_G(u)}}\sum_{v\in N(u)}\frac{1}{\sqrt{\deg_G(v)}}\right)+$$

$$+\frac{1}{2}\sum_{u\in P}\left(\frac{1}{\sqrt{\deg_{\widetilde{G}}(u)}}\sum_{v\in\tilde{N}(u)}\frac{1}{\sqrt{\deg_{\widetilde{G}}(v)}}\right)+\frac{1}{2}\sum_{u\in Q}\left(\frac{1}{\sqrt{\deg_{\widetilde{G}}(u)}}\sum_{v\in\tilde{N}(u)}\frac{1}{\sqrt{\deg_{\widetilde{G}}(v)}}\right)\leq$$

$$\leq\frac{1}{2}\sum_{u\in P}\frac{1}{\sqrt{\deg_G(u)}}\deg_G(u)+\frac{1}{2}\sum_{u\in Q}\frac{1}{\sqrt{\deg_G(u)}}\deg_G(u)+$$

$$+\frac{1}{2}\sum_{u\in P}\frac{1}{\sqrt{q-\deg_G(u)}}(q-\deg_G(u))+\frac{1}{2}\sum_{u\in Q}\frac{1}{\sqrt{p-\deg_G(u)}}(p-\deg_G(u))=$$

$$=\frac{1}{2}\sum_{u\in P}\sqrt{\deg_G(u)}+\frac{1}{2}\sum_{u\in Q}\sqrt{\deg_G(u)}+\frac{1}{2}\sum_{u\in P}\sqrt{q-\deg_G(u)}+\frac{1}{2}\sum_{u\in Q}\sqrt{p-\deg_G(u)}\leq$$

$$\leq\frac{p}{2}\sqrt{\frac{\sum\limits_{u\in P}\deg_G(u)}{p}}+\frac{q}{2}\sqrt{\frac{\sum\limits_{u\in Q}\deg_G(u)}{q}}+$$

$$+\frac{p}{2}\sqrt{\frac{\sum\limits_{u\in P}(q-\deg_G(u))}{p}}+\frac{q}{2}\sqrt{\frac{\sum\limits_{u\in Q}(p-\deg_G(u))}{q}}=$$

$$=\frac{\sqrt{p}}{2}\sqrt{m}+\frac{\sqrt{q}}{2}\sqrt{m}+\frac{\sqrt{p}}{2}\sqrt{pq-m}+\frac{\sqrt{q}}{2}\sqrt{pq-m}=(\sqrt{p}+\sqrt{q})\left(\frac{\sqrt{m}+\sqrt{pq-m}}{2}\right).$$

□

Note that the lower bound is attained for complete bipartite graphs. However, we do not know an example of a bipartite graph attaining the upper bound, and we conjecture that the best upper bound is $2\sqrt{pq}$.

In general, one might consider also the Nordhaus-Gaddum type inequality (in the case of a bipartite graph/bipartite complement) for a generalized Randić index $R_\alpha(G)=\sum_{uv\in E(G)}(\deg_G(u)\deg_G(v))^\alpha$ and to obtain results analogical to the ones in [7]; however, the upper bound in Theorem 2.3 of [7] seems to be incorrect – for example, putting $\alpha=-1$ and $G\cong K_{2,2}$, one has $R(G)=4\cdot\frac{1}{\sqrt{2\cdot 2}}=1$ and $R(\overline{G})=R(2K_2)=2\cdot\frac{1}{\sqrt{1\cdot 1}}=2$, thus $R(G)+R(\overline{G})=3$ whereas the upper bound for this sum is $\binom{4}{2}\left(\frac{4-1}{2}\right)^{2\cdot(-1)}=6\cdot\left(\frac{3}{2}\right)^{-2}=\frac{8}{3}<3$. Thus, for general α, the problem of best bounds remains open both for classical and bipartite complement version.

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REFERENCES

[1] A.A. Dobrynin, R. Entringer, I. Gutman, *Wiener index of trees: theory and applications*, Acta Appl. Math. **66** (2001), 211–249.

[2] A.A. Dobrynin, I. Gutman, S. Klavžar, P. Žigert, *Wiener index of hexagonal systems*, Acta Appl. Math. **72** (2002), 247–294.

[3] R.C. Entringer, D.E. Jackson, D.A. Snyder, *Distance in graphs*, Czech. Math. J. **26** (1976), 283–296.

[4] R.C. Read, R.J. Wilson, *An Atlas of Graphs*, Clarendon Press, Oxford, 1998.

[5] http://mapleta.maths.uwa.edu.au/∼gordon/remote/graphs/#bips

[6] Y. Yang, H. Zhang, D.J. Klein, *New Nordhaus-Gaddum-type results for the Kirchhoff index*, J. Math. Chem. **49** (2011) 8, 1587–1598.

[7] L. Zhang, B. Wu, *The Nordhaus-Gaddum-type inequalities for some chemical indices*, MATCH Commun. Math. Comput. Chem. **54** (2005), 189–194.

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