EIGENVALUE-EIGENFUNCTION PROBLEM FOR STEKLOV'S SMOOTHING OPERATOR AND DIFFERENTIAL-DIFFERENCE EQUATIONS OF MIXED TYPE

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Abstract. It is shown that any $\mu \in \mathbb{C}$ is an infinite multiplicity eigenvalue of the Steklov smoothing operator S_h acting on the space $L^1_{loc}(\mathbb{R})$. For $\mu \neq 0$ the eigenvalue-eigenfunction problem leads to studying a differential-difference equation of mixed type. An existence and uniqueness theorem is proved for this equation. Further a transformation group is defined on a countably normed space of initial functions and the spectrum of the generator of this group is studied. Some possible generalizations are pointed out.

Keywords: Steklov's smoothing operator, spectrum, eigenvalues, eigenfunctions, mixed-type differential-difference equations, initial function, method of steps, countably normed space, transformation group, generator.

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1. INTRODUCTION

In this paper it is proposed to study the spectrum of the Steklov smoothing operator S_h ($h > 0$) [1], acting on the space $L^1_{loc}(\mathbb{R})$ of locally summable functions:

$$(S_h u)(t) := \frac{1}{2h} \int\limits_{t-h}^{t+h} u(s)ds, \quad u \in L^1_{loc}(\mathbb{R}). \tag{1.1}$$

In Section 2 it is shown that for any $\mu \in \mathbb{C}$ the eigenvalue-eigenfunction problem $\mu u = S_h u$, $u \in L^1_{loc}(\mathbb{R})$, has nontrivial solutions. So any value $\mu \in \mathbb{C}$ appears to be an infinite multiplicity eigenvalue of the operator S_h. For $\mu = 0$ the eigenspace of S_h consists of all $2h$-periodic functions v from $L^1_{loc}(\mathbb{R})$ such that $\int_{-h}^{h} v(s)ds = 0$. For $\mu \neq 0$ the eigenfunctions of the Steklov operator S_h are smooth, i.e., infinitely differentiable.

In this case the eigenvalue-eigenfunction problem for S_h leads to studying solutions of the following differential-difference equation of mixed type:

$$\mu u'(t) = \frac{u(t+h) - u(t-h)}{2h}, \quad t \in \mathbb{R}. \tag{1.2}$$

In Section 3 by using the method of steps [2–4] we prove an existence and uniqueness theorem for equation (1.2). For any $\mu \in \mathbb{C} \setminus \{0\}$ this equation has a countably infinite solution set of the exponential form: $u_\lambda(t) = e^{\lambda t}$, where $\lambda \in \Lambda_\mu$. Here Λ_μ is a certain countably infinite unbounded subset of the complex plane $\mathbb{C}$ consisting of isolated points. These exponential functions u_λ are also eigenfunctions of the operator S_h corresponding to the eigenvalue $\mu \in \mathbb{C} \setminus \{0\}$. However these exponential functions and their finite linear combinations do not exhaust the corresponding eigenspace.

In Section 4 a transformation group $T(t)$, $t \in \mathbb{R}$, for equation (1.2) is constructed defined on a countably normed space $\mathfrak{I}_\mu$ of initial functions. A generator A of this transformation group is shown to have a purely point spectrum coinciding with the set Λ_μ. The corresponding eigenspace for any $\lambda \in \sigma(A) = \Lambda_\mu$ is one-dimensional and generated by a function $u_\lambda(t) = e^{\lambda t}$.

In the last Section 5 it is pointed out that a somewhat more general differential-difference equation of mixed type than equation (1.2) and a corresponding operator like Steklov's operator (1.1) can be studied in a similar manner.

The space $L^1_{loc}(\mathbb{R})$ is the natural maximal domain of definition of the Steklov smoothing operator when we consider the eigenvalue–eigenfunction problem for this operator. It seems that the Steklov operator has never been studied from this point of view before. Similarly, a relation between this operator and differential–difference equations of mixed type is established for the first time in this paper. This relation enables us to prove in Section 3 a general existence and uniqueness theorem for equation (1.2) in a rather simple way and then to show that any eigenspace of S_h is not exhausted by the exponential functions and their finite linear combinations. Historically the Steklov operator S_h was introduced and always considered like some type of smoothing operator acting in various function spaces (see, for example, [1]). In Section 2 we also use a little this property of the operator S_h to improve smoothness of functions.

2. STEKLOV'S SMOOTHING OPERATOR
AND A DIFFERENTIAL-DIFFERENCE EQUATION OF MIXED TYPE

If a function $v \in L^1_{loc}(\mathbb{R})$ and $h > 0$, then the Steklov smoothing operator S_h is defined by

$$(S_h v)(t) := \frac{1}{2h} \int_{t-h}^{t+h} v(s)ds. \tag{2.1}$$

The space $L^1_{loc}(\mathbb{R})$ is the maximal domain of definition for the operator S_h. It is clear that the function $S_h v$ is absolutely continuous (on every bounded interval

of the real line) [5], i.e., $S_h v \in AC(\mathbb{R})$, and if $v \in C^n(\mathbb{R})$, $n = 0, 1, \ldots, \infty$, then $S_h v \in C^{n+1}(\mathbb{R})$. Hence the operator S_h improves the smoothness.

Let us consider the eigenvalue-eigenfunction problem for the operator S_h:

$$S_h v = \mu v, \tag{2.2}$$

where $\mu \in \mathbb{C}$ and a function $v \in L^1_{loc}(\mathbb{R})$. We consider separately two cases: a) $\mu = 0$ and b) $\mu \neq 0$.

Case a) $\mu = 0$. This case is rather simple and a complete treatment of it is given. In this case equation (2.2) has the form

$$\int_{t-h}^{t+h} v(s)ds = 0. \tag{2.3}$$

As $S_h v \in AC(\mathbb{R})$, we can differentiate equation (2.3) with respect to t almost everywhere and it follows by the Lebesgue theorem that $v(t + h) - v(t - h) = 0$ for almost every $t \in \mathbb{R}$. Setting $t - h =: \tau$, we find that $v(\tau + 2h) = v(\tau)$ for a.e. $\tau \in \mathbb{R}$. Consequently, an eigenfunction v is periodic of period $T = 2h$.

Let $\varphi := v \mid_{[-h,h]}$ be "an initial function". Then v is obtained from φ by a $2h$-periodic extension to the entire real line $\mathbb{R}$. From (2.3) for $t = 0$ it also follows that

$$\int_{-h}^{h} \varphi(s)ds = 0. \tag{2.4}$$

The converse is also true: if a locally summable $2h$-periodic function v satisfies the condition

$$\int_{-h}^{h} v(s)ds = 0, \tag{2.5}$$

then v is an eigenfunction of the operator S_h, that is, $S_h v = 0$. Indeed, according to the $2h$-periodicity we have

$$(S_h v)(t) = \frac{1}{2h} \int_{t-h}^{t+h} v(s)ds = \frac{1}{2h} \int_{-h}^{h} v(s)ds = 0.$$

Thus, we proved the following simple theorem.

Theorem 2.1. *A function $v \in L^1_{loc}(\mathbb{R})$ is an eigenfunction of the Steklov operator S_h corresponding to the eigenvalue $\mu = 0$ if and only if v is a $2h$-periodic function and satisfies condition* (2.5), *that is, the integral of the function v over any interval of length $T = 2h$ is equal to zero.*

Case b) $\mu \neq 0$. Let us consider the following nonhomogeneous equation

$$\mu u(t) = (S_h u)(t) + F(t) \tag{2.6}$$

in the unknown function $u \in L^1_{loc}(\mathbb{R})$, where $F \in C^\infty(\mathbb{R})$ is a given function. If $F = 0$, a solution $u = u(t)$ to (2.6) is an eigenfunction of the Steklov operator S_h.

Proposition 2.2. *Let $\mu \neq 0$. If $F \in C^\infty(\mathbb{R})$, then any locally summable solution of equation* (2.6) *is infinitely differentiable, i.e., if a function $u \in L^1_{loc}(\mathbb{R})$ satisfies equation* (2.6)*, then $u \in C^\infty(\mathbb{R})$.*

Proof. If $u \in L^1_{loc}(\mathbb{R})$, then $S_h u \in C(\mathbb{R})$, and from (2.6) it follows that $u \in C(\mathbb{R})$. We now proceed by induction. Namely, if $u \in C^n(\mathbb{R})$ for some arbitrary $n \in \mathbb{N}$, then $S_h u \in C^{n+1}(\mathbb{R})$. Hence, according to (2.6) we conclude that $u = \mu^{-1}(S_h u + F) \in C^{n+1}(\mathbb{R})$. $\qquad\square$

Differentiating equation (2.6) with respect to t yields

$$\mu u'(t) = \frac{1}{2h} u(t+h) - \frac{1}{2h} u(t-h) + F'(t). \qquad (2.7)$$

Thus a solution of equation (2.6) satisfies the mixed-type differential-difference equation (2.7).

Now suppose that a function $u \in AC(\mathbb{R})$ and almost everywhere in $\mathbb{R}$ satisfies the equation

$$\mu u'(t) = \frac{1}{2h}(u(t+h) - u(t-h)) + f(t), \qquad (2.8)$$

where $f \in C^\infty(\mathbb{R})$. Then, if F is one of the primitives of f, equality (2.8) is written in the form

$$(\mu u(t) - (S_h u)(t) - F(t))' = 0. \qquad (2.9)$$

Integrating (2.9) from 0 to t, we obtain

$$\mu u(t) = (S_h u)(t) + F(t) + C_u \qquad (2.10)$$

with the constant

$$C_u = \mu u(0) - \frac{1}{2h} \int_{-h}^{h} u(s) ds - F(0)$$

not depending on t.

(Note that if any function $u(t)$ satisfies (2.6), then $C_u = 0$. In order to see that, it is sufficient to put $t = 0$ in (2.6).)

Consequently, according to Proposition 2.2 the function $u \in C^\infty(\mathbb{R})$ and by continuity satisfies (2.7) for all $t \in \mathbb{R}$. Therefore we established the validity of the following assertion.

Proposition 2.3. *Let $\mu \neq 0$. If $f \in C^\infty(\mathbb{R})$, then any absolutely continuous solution of equation* (2.8) *is infinitely differentiable, i.e., if a function $u \in AC(\mathbb{R})$ satisfies equation* (2.8) *almost everywhere in $\mathbb{R}$, then $u \in C^\infty(\mathbb{R})$ and* (2.8) *is fulfilled at every point $t \in \mathbb{R}$.*

We also proved the following theorem.

Theorem 2.4. *Let $\mu \neq 0$, a function $F \in C^\infty(\mathbb{R})$ and $f = F'$. If a function $u \in C^\infty(\mathbb{R})$ is a solution of equation (2.6), then $u(t)$ is a solution of equation (2.8). Conversely, if a function $u \in C^\infty(\mathbb{R})$ is a solution equation (2.8), then $u(t)$ satisfies equation (2.10). In particular, a solution $u = u(t)$ of (2.8) will be a solution of (2.6) if and only if the "initial" condition $\mu u(0) = S_h u(0) + F(0)$ is fulfilled, which means the correctness of (2.6) for $t = 0$.*

As any solution $u = u(t)$ of equation (2.6) satisfies the initial condition $C_u = 0$, the following problems

$$\mu u(t) = (S_h u)(t) + F(t), \quad u \in L^1_{loc}(\mathbb{R}), \tag{2.11}$$

and

$$\begin{cases} \mu u'(t) = (D_h u)(t) + F'(t), \quad u \in AC(\mathbb{R}), \\ \mu u(0) = (S_h u)(0) + F(0) \end{cases} \tag{2.12}$$

where the operator $(D_h u)(t) := (u(t+h) - u(t-h))/(2h)$, are equivalent, i.e., they have the same set of solutions. All these solutions are from $C^\infty(\mathbb{R})$.

Now let $F = 0$, and respectively $f = 0$ too. As a consequence of the previous result we obtain the following theorem.

Theorem 2.5. *Let $\mu \neq 0$. Then:*

a) *Every locally summable eigenfunction of the Steklov smoothing operator S_h corresponding to the eigenvalue μ is infinitely differentiable, i.e., if $v \in L^1_{loc}(\mathbb{R})$ and*

$$\mu v(t) = (S_h v)(t), \tag{2.13}$$

then $v \in C^\infty(\mathbb{R})$.

b) *Every absolutely continuous solution $v = v(t)$ to the equation*

$$\mu v'(t) = (D_h v)(t) \tag{2.14}$$

is infinitely differentiable, i.e., if $v \in AC(\mathbb{R})$ and satisfies (2.14) almost everywhere in $\mathbb{R}$, then $v \in C^\infty(\mathbb{R})$ and by continuity satisfies (2.14) for all $t \in \mathbb{R}$.

c) *If a function $v \in C^\infty(\mathbb{R})$ satisfies equation (2.13), then v also satisfies (2.14).*

d) *If a function $v \in C^\infty(\mathbb{R})$ satisfies equation (2.14), then v also satisfies the equation*

$$\mu v(t) = (S_h v)(t) + \widetilde{C_v}, \tag{2.15}$$

with the constant $\widetilde{C_v} = \mu v(0) - (S_h v)(0)$.

In particular, a solution of (2.14) is a solution of (2.13) if and only if $\widetilde{C_v} = 0$, i.e., if and only if the function $v = v(t)$ satisfies the "initial condition"

$$\mu v(0) = (S_h v)(0). \tag{2.16}$$

e) *As every solution $v = v(t)$ of equation (2.13) satisfies the initial condition (2.16), the following problems*

$$\mu v(t) = (S_h v)(t), \quad v \in L^1_{loc}(\mathbb{R}), \tag{2.17}$$

and

$$\begin{cases} \mu v'(t) = (D_h v)(t), \quad v \in AC(\mathbb{R}), \\ \mu v(0) = (S_h v)(0) \end{cases} \tag{2.18}$$

are equivalent, i.e., they have the same set of solutions. All these solutions belong to $C^\infty(\mathbb{R})$.

Theorem 2.6. *Let $\mu \neq 0$ be arbitrary and fixed and let $\lambda \in \mathbb{C}$. A function $u_\lambda(t) := e^{\lambda t}$ for $\lambda \neq 0$ is an eigenfunction of the Steklov operator S_h corresponding to the eigenvalue μ if and only if λ is a solution of the following equation*

$$2\mu(\lambda h) = e^{\lambda h} - e^{-\lambda h}, \quad \lambda \in \mathbb{C} \setminus \{0\}. \tag{2.19}$$

The solution set of equation (2.19) *is countably infinite. Therefore any number $\mu \in \mathbb{C} \setminus \{0\}$ is an infinite multiplicity eigenvalue of the Steklov operator S_h.*

Proof. For $\lambda \neq 0$

$$(S_h u_\lambda)(t) = \frac{1}{2h} \int\limits_{t-h}^{t+h} e^{\lambda t} dt = \frac{1}{2\lambda h} e^{\lambda t}(e^{\lambda h} - e^{-\lambda h}).$$

Consequently, the condition $(S_h u_\lambda)(t) = \mu u_\lambda(t)$ is written in the form (2.19) or as follows

$$P_\mu(\lambda) := e^{2\lambda h} - (2h\mu)\lambda e^{\lambda h} - 1 = 0, \qquad \lambda \neq 0.$$

As is known [4], quasipolynomials like $P_\mu(\lambda)$ have in $\mathbb{C}$ a denumerable set of zeros. $\quad\square$

Remark 2.7. A function $u_0(t) = 1$ corresponding to $\lambda = 0$ is an eigenfunction of S_h only for $\mu = 1$.

To sum up, any $\mu \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of the Steklov operator of at least denumerable multiplicity. In this case the functions $u_\lambda = u_\lambda(t)$ for $\lambda \in \Lambda_\mu \setminus \{0\}$, where the set

$$\Lambda_\mu := \{\lambda \in \mathbb{C} \colon 2\mu(\lambda h) = e^{\lambda h} - e^{-\lambda h}\}, \tag{2.20}$$

are linearly independent eigenfunctions of S_h corresponding to the eigenvalue $\mu \in \mathbb{C} \setminus \{0\}$. This means according to Theorem 2.3 that the functions $u_\lambda(t)$ for $\lambda \in \Lambda_\mu \setminus \{0\}$ are also solutions of equation (2.14) and hence the differential–difference equation of mixed type

$$\mu u'(t) = \frac{u(t+h) - u(t-h)}{2h} \tag{2.21}$$

for any $\mu \neq 0$ has at least a denumerable set of solutions $u_\lambda = u_\lambda(t)$, $\lambda \in \Lambda_\mu$. It is easily seen that the last assertion (also for $\lambda = 0$) is verified by a direct substitution of the function $u_\lambda(t) = e^{\lambda t}$ into equation (2.21).

In Section 3 it will be shown that for any $\mu \neq 0$ the corresponding eigenspace of the operator S_h is not exhausted by the exponential functions $u_\lambda(t) = e^{\lambda t}$ and their finite linear combinations.

3. CONSTRUCTION OF SOLUTIONS OF A MIXED-TYPE DIFFERENTIAL-DIFFERENCE EQUATION

Throughout this section $\mu \in \mathbb{C} \setminus \{0\}$. In this section we construct solutions of the following differential-difference equation of mixed type

$$\mu u'(t) = \frac{u(t+h) - u(t-h)}{2h} + f(t), \quad t \in \mathbb{R}, \tag{3.1}$$

where the function $f \in C^\infty(\mathbb{R})$. As shown in Section 2, if a function $u \in AC(\mathbb{R})$ satisfies equation (3.1) for almost all $t \in \mathbb{R}$, then $u \in C^\infty(\mathbb{R})$ and equation (3.1) is fulfilled for all $t \in \mathbb{R}$.

Let $u|_{[-h,h]} =: \varphi$ be an "initial function", which is smooth, i.e., $\varphi \in C^\infty[-h, h]$. Let us show that if we know the initial function $\varphi \in C^\infty[-h, h]$, then by the method of steps [2–4] we can construct the function $u = u(t)$ on the whole real line $\mathbb{R}$. For this purpose we write (3.1) in the form

$$u(t+h) = 2h\mu u'(t) + u(t-h) - 2hf(t). \tag{3.2}$$

Setting $t + h =: \tau$, we find from (3.2) that

$$u(\tau) = 2h\mu u'(\tau - h) + u(\tau - 2h) - 2hf(\tau - h). \tag{3.3}$$

If $\tau \in [h, 2h]$, then $(\tau - h) \in [0, h]$ and $(\tau - 2h) \in [-h, 0]$. On the interval $[-h, h]$ the function $u = u(t)$ is known. Therefore the right-hand side of (3.3) is uniquely determined for $\tau \in [h, 2h]$. Hence the values of $u = u(\tau)$ are found for $\tau \in [h, 2h]$. We continue by induction. If the function $u = u(\tau)$ is already known on the interval $[-h, mh]$ (for arbitrary and fixed $m \in \mathbb{N}$), then the right-hand side of (3.3) is determined for $\tau \in [mh, (m+1)h]$. It follows that the function is found for $\tau \in [mh, (m+1)h]$, i.e., the values of $u = u(\tau)$ are already determined on the interval $[-h, (m+1)h]$. Thus the method of steps allows us to find the solution of (3.1) on the interval $[h, nh]$ for any $n \in \mathbb{N}$, i.e., the function $u = u(t)$ is determined for all $\tau \geq h$ starting from the initial function $\varphi = u|_{[-h,h]}$.

Similarly, rewriting (3.1) in the form

$$u(t-h) = -2h\mu u'(t) + u(t+h) + 2hf(t) \tag{3.4}$$

and making the change of variables $s := t - h$, we obtain

$$u(s) = -2h\mu u'(s+h) + u(s+2h) + 2hf(s+h). \tag{3.5}$$

Now by applying the method of steps to (3.5), we also find the function $u = u(s)$ for all $s \leq -h$ starting from the initial function $\varphi = u|_{[-h,h]}$.

Suppose now that we have an arbitrary function $\varphi \in C^\infty[-h, h]$. Let us try to consider φ as an "initial function". By applying the method of steps to this function, we construct a function $u \in C^\infty[mh, (m+1)h]$, $m \in \mathbb{Z}$, satisfying equation (3.1) for $t \in (mh, (m+1)h)$, $m \in \mathbb{Z}$. But at the points $x_k = kh$, $k \in \mathbb{Z} \setminus \{0\}$, the function $u = u(t)$ and its derivatives can have discontinuities of the first kind. The following theorem is valid.

Theorem 3.1. *Let $\varphi \in C^\infty[-h, h]$ and $f \in C^\infty(\mathbb{R})$. A solution $u = u(t)$ of equation (3.1) satisfying the initial condition $u|_{[-h,h]} = \varphi$ and constructed by the method of steps belongs to $C^\infty(\mathbb{R})$ if and only if the following conditions are fulfilled*

$$\mu\varphi^{(n+1)}(0) = \frac{\varphi^{(n)}(h) - \varphi^{(n)}(-h)}{2h} + f^{(n)}(0), \quad n = 0, 1, 2, \dots \qquad (3.6)$$

Proof. Necessity. Let $u \in C^\infty(\mathbb{R})$ be a solution to (3.1) and $u(t) = \varphi(t)$ for $t \in [-h, h]$. By taking the n-th derivative of (3.1) and setting $t = 0$, we obtain

$$\mu u^{(n+1)}(0) = \frac{u^{(n)}(h) - u^{(n)}(-h)}{2h} + f^{(n)}(0). \qquad (3.7)$$

As $u|_{[-h,h]} = \varphi$, equation (3.7) is exactly (3.6).

Sufficiency. Let us assume that (3.6) is true. From (3.3) it follows that

$$u^{(n)}(\tau) = 2h\mu u^{(n+1)}(\tau - h) + u^{(n)}(\tau - 2h) - 2hf^{(n)}(\tau - h). \qquad (3.8)$$

Consequently for $u^{(n)}(h^+) := \lim_{\tau \longrightarrow h^+} u^{(n)}(\tau)$ we get

$$u^{(n)}(h^+) = 2h\mu u^{(n+1)}(0^+) + u^{(n)}(-h^+) - 2hf^{(n)}(0), \qquad (3.9)$$

i.e.,

$$u^{(n)}(h^+) = 2h\mu\varphi^{(n+1)}(0) + \varphi^{(n)}(-h) - 2hf^{(n)}(0). \qquad (3.10)$$

Therefore taking into account (3.10), the continuity condition $u^{(n)}(h^+) = u^{(n)}(h^-) \equiv \varphi^{(n)}(h)$ is written as

$$2h\mu\varphi^{(n+1)}(0) + \varphi^{(n)}(-h) - 2hf^{(n)}(0) = \varphi^{(n)}(h) \qquad (3.11)$$

or

$$\mu\varphi^{(n+1)}(0) = \frac{\varphi^{(n)}(h) - \varphi^{(n)}(-h)}{2h} + f^{(n)}(0), \qquad (3.12)$$

that is true according to our supposition.

Analogously with the help of (3.5) the continuity of $u^{(n)} = u^{(n)}(t)$ at the point $t = -h$, i.e., the validity of the equality $u^{(n)}(-h^-) = u^{(n)}(-h^+) \equiv \varphi^{(n)}(-h)$ is established.

We finish the proof by induction on k. To be exact, let us assume that for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ the equality $u^{(n)}(kh^+) = u^{(n)}(kh^-)$ is true for $k = \pm 1, \pm 2, \dots, \pm m$. Then according to (3.8) for arbitrary $n \in \mathbb{N}_0$ we have

$$u^{(n)}((m+1)h^+) = 2h\mu u^{(n+1)}(mh^+) + u^{(n)}((m-1)h^+) - 2hf^{(n)}(mh), \qquad (3.13)$$

and also

$$u^{(n)}((m+1)h^-) = 2h\mu u^{(n+1)}(mh^-) + u^{(n)}((m-1)h^-) - 2hf^{(n)}(mh). \qquad (3.14)$$

By the induction hypothesis $u^{(n+1)}(mh^+) = u^{(n+1)}(mh^-)$ and $u^{(n)}((m-1)h^+) = u^{(n)}((m-1)h^-)$. Therefore we see that the right-hand sides of (3.13) and (3.14) are

equal. It follows that $u^{(n)}((m+1)h^+) = u^{(n)}((m+1)h^-)$ for any $n \in \mathbb{N}_0$ as well. In the same manner by using the relation (3.5) we establish the validity of the equality $u^{(n)}(-(m+1)h^-) = u^{(n)}(-(m+1)h^+)$ for any $n \in \mathbb{N}_0$.

Thus it is shown that for any $n \in \mathbb{N}_0$ the equality $u^{(n)}(kh^+) = u^{(n)}(kh^-)$ is also true for $k = \pm(m+1)$. Consequently, by induction we infer that the equality $u^{(n)}(kh^+) = u^{(n)}(kh^-)$ is true for all $k \in \mathbb{Z}$. So the function $u = u(t)$ and all its derivatives are continuous at the points $t = kh$, $k \in \mathbb{Z}$. Hence $u \in C^\infty(\mathbb{R})$. The theorem is proved. $\square$

Actually Theorem 3.1 together with the preceding application of the method of steps represents the existence and uniqueness theorem for equation (3.1).

Remark 3.2. Let an initial function $\varphi \in C^\infty[-h, h]$ generate by the method of steps a function $u = u(t)$, $t \in \mathbb{R}$. We see from the proof of Theorem 3.1 that if the function $u = u(t)$ is infinitely differentiable at the points $t = \pm h$, then $u = u(t)$ is also infinitely differentiable at every point $t = \pm mh$, $m \in \mathbb{N}$, and hence $u \in C^\infty(\mathbb{R})$.

By combining Theorem 3.1 and Theorem 2.4 we obtain the following result.

Theorem 3.3. *Let $\mu \in \mathbb{C} \setminus \{0\}$ and $F \in C^\infty(\mathbb{R})$. If a function $\varphi \in C^\infty[-h, h]$ satisfies the condition*

$$\mu\varphi^{(n+1)}(0) = \frac{\varphi^{(n)}(h) - \varphi^{(n)}(-h)}{2h} + F^{(n+1)}(0), \quad n \in \mathbb{N}_0, \tag{3.15}$$

then a solution $u = u(t)$, $t \in \mathbb{R}$, to the problem

$$\begin{cases} \mu u'(t) = \dfrac{u(t+h) - u(t-h)}{2h} + F'(t), \\[2mm] u|_{[-h,h]} = \varphi \end{cases} \tag{3.16}$$

is also a solution to the equation

$$\mu u(t) = (S_h u)(t) + F(t), \tag{3.17}$$

if and only if $\mu\varphi(0) = (S_h\varphi)(0) + F(0)$, i.e., if and only if the condition

$$\mu\varphi(0) = \frac{1}{2h} \int_{-h}^{h} \varphi(t)dt + F(0) \tag{3.18}$$

is fulfilled.

Proof. By Theorem 3.1 with $f = F'$, a solution to the problem (3.16) with the function φ satisfying (3.15) belongs to $C^\infty(\mathbb{R})$. Consequently, according to Theorem 2.4 a solution to the problem (3.16) is also a solution of equation (3.17) if and only if $\mu u(0) = S_h u(0) + F(0)$, i.e., if and only if

$$\mu u(0) = \frac{1}{2h} \int_{-h}^{h} u(t)dt + F(0). \tag{3.19}$$

But $u(t) = \varphi(t)$, $t \in [-h, h]$, so (3.19) is identically rewritten as (3.18). $\qquad\square$

Corollary 3.4. *Taking in Theorem 3.3 the function $F = 0$, we establish the following assertion. Let $\mu \in \mathbb{C}\backslash\{0\}$. If a function $\varphi \in C^\infty[-h, h]$ satisfies the following condition*

$$\mu\varphi^{(n+1)}(0) = \frac{\varphi^{(n)}(h) - \varphi^{(n)}(-h)}{2h}, \quad n \in \mathbb{N}_0, \tag{3.20}$$

then a solution $u = u(t)$, $t \in \mathbb{R}$, to the problem

$$\begin{cases} \mu u'(t) = \dfrac{u(t + h) - u(t - h)}{2h}, \\ u|_{[-h,h]} = \varphi \end{cases} \tag{3.21}$$

is an eigenfunction of the Steklov operator, i.e., $(S_h u)(t) = \mu u(t)$, if and only if

$$\mu\varphi(0) = \frac{1}{2h} \int\limits_{-h}^{h} \varphi(t)dt. \tag{3.22}$$

Remark 3.5. It is easy to check by a direct substitution that for the function $u_\lambda(t) = e^{\lambda t}$, $\lambda \neq 0$, the conditions (3.20) and (3.22) are equivalent to the unique condition $2\mu(\lambda h) = e^{\lambda h} - e^{-\lambda h}$ determining the set Λ_μ in (2.20).

Now we are able to show that an eigenspace of the operator S_h contains a large collection of eigenfunctions other than the exponential functions $u_\lambda(t) = e^{\lambda t}$ and their finite linear combinations. Indeed, let $\psi = \psi(t)$, $t \in \mathbb{R}$, be a smooth function compactly supported in the interval $(0, h)$. Define the initial function φ as follows

$$\varphi(t) = \begin{cases} \psi(t), & \text{if } t \in [0, h], \\ -\psi(-t), & \text{if } t \in [-h, 0]. \end{cases}$$

Clearly, $\varphi \in C^\infty[-h, h]$. Being identically equal to zero in some neighborhoods of the points $t = \pm h$ and $t = 0$, the function φ satisfies condition (3.20). Since φ is odd, we have $\int_{-h}^{h} \varphi(t)dt = 0$, and hence condition (3.22) is fulfilled as well. Therefore, by Corollary 3.4, the solution $u = u(t)$ to the problem (3.21) is an eigenfunction of the operator S_h. As $u|_{[-h,h]} = \varphi$, the function $u = u(t)$ is identically equal to zero on some intervals of the real line. Consequently, $u = u(t)$ is not analytic, and thus it can not be a finite linear combination of the exponential eigenfunctions $u_\lambda(t) = e^{\lambda t}$.

4. TRANSLATION GROUP

Throughout this section $\mu \neq 0$. In a linear space $\Phi = C^\infty[-h, h]$ we define a denumerable system of seminorms $\| \cdot \|_m$, $m \in \mathbb{N}_0$, setting for $\varphi \in \Phi$

$$\|\varphi\|_0 := \max\{|\varphi(x)| : x \in [-h, h]\}, \tag{4.1}$$

$$\|\varphi\|_m := \max\{|\varphi^{(m)}(x)| : x \in [-h, h]\}, \quad m > 1. \tag{4.2}$$

Then Φ will be a complete countably normed space [6], in which the convergence $\varphi_n \xrightarrow{\Phi} \varphi$ for the elements $\varphi, \varphi_n \in \Phi$ means a uniform convergence of the functions and their derivatives of any order on the interval $[-h, h]$, i.e.,

$$\varphi_n^{(m)}(x) \xrightarrow[n \to \infty]{} \varphi^{(m)}(x)$$

uniformly on $[-h, h]$ for every $m \in \mathbb{N}_0$.

For $\mu \in \mathbb{C} \setminus \{0\}$ we define in Φ a subset

$$\mathfrak{I}_\mu := \{\varphi \in \Phi : 2h\mu\varphi^{(m+1)}(0) = \varphi^{(m)}(h) - \varphi^{(m)}(-h), \quad m \in \mathbb{N}_0\}. \tag{4.3}$$

It is easily seen that $\mathfrak{I}_\mu$ is a linear closed subspace of Φ. Here the closure of $\mathfrak{I}_\mu$ follows from the fact that if $\varphi_n \in \mathfrak{I}_\mu$ and $\varphi_n \xrightarrow{\Phi} \varphi$ as $n \longrightarrow \infty$, then for every $m \in \mathbb{N}_0$ the relation

$$2h\mu\varphi_n^{(m+1)}(0) = \varphi_n^{(m)}(h) - \varphi_n^{(m)}(-h)$$

also holds for the limit function φ. We say that $\mathfrak{I}_\mu$ is an initial space.

Note that the initial space $\mathfrak{I}_\mu$ is not empty. Besides the exponential functions $\varphi_\lambda(\theta) = e^{\lambda\theta}, \lambda \in \Lambda_\mu$, the space $\mathfrak{I}_\mu$ contains all functions from $C^\infty[-h, h]$ identically equal to zero in some neighborhoods of the points $t = \pm h$ and $t = 0$.

By Theorem 3.1 if $\varphi \in \mathfrak{I}_\mu$, then there exists a unique solution $u = u(t)$, $t \in \mathbb{R}$, to the equation $\mu u'(t) = (D_h u)(t)$ with the initial function φ, i.e., a solution satisfying the initial condition $u|_{[-h,h]} = \varphi$.

Let $\mu u'(t) = (D_h u)(t)$ and $u|_{[-h,h]} = \varphi$, where $\varphi \in \mathfrak{I}_\mu$. For $t \in \mathbb{R}$ we define a function $\varphi_t = \varphi_t(\theta) := u(\theta + t)$ of a variable $\theta \in [-h, h]$.

Proposition 4.1. *For all $t \in \mathbb{R}$ the function $\varphi_t \in \mathfrak{I}_\mu$.*

Proof. By differentiating the equation

$$\mu u'(\tau) = \frac{u(\tau + h) - u(\tau - h)}{2h}, \quad \tau \in \mathbb{R}, \tag{4.4}$$

and putting $\tau = \theta + t$ we obtain

$$2h\mu u^{(m+1)}(\theta + t) = u^{(m)}(\theta + t + h) - u^{(m)}(\theta + t - h), \tag{4.5}$$

that is,

$$2h\mu\varphi_t^{(m+1)}(\theta) = \varphi_t^{(m)}(\theta + h) - \varphi_t^{(m)}(\theta - h). \tag{4.6}$$

In particular, for $\theta = 0$ we have the relation

$$2h\mu\varphi_t^{(m+1)}(0) = \varphi_t^{(m)}(h) - \varphi_t^{(m)}(-h). \tag{4.7}$$

According to (4.3) $\varphi_t \in \mathfrak{I}_\mu$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let $T(t) : \mathfrak{I}_\mu \longrightarrow \mathfrak{I}_\mu$ be a linear map such that $T(t)\varphi := \varphi_t$, $\varphi \in \mathfrak{I}_\mu$, i.e., $T(t)\varphi(\theta) = u(\theta + t)$, $t \in \mathbb{R}$. It is easily seen that $T(t)$, $t \in \mathbb{R}$, is a one-parameter group of linear transformations of the countably normed space $\mathfrak{I}_\mu$. To be exact, the following relations hold:

1) $T(0) = I$ (the identity operator),
2) $T(t_1 + t_2) = T(t_1)T(t_2)$ for all $t_1, t_2 \in \mathbb{R}$.

Theorem 4.2. *The operator-valued function $T = T(t)$ is strongly differentiable with respect to t and for any $\varphi \in \mathfrak{I}_\mu$ we have*

$$\frac{d}{dt}(T(t)\varphi(\theta)) = T(t)\varphi'(\theta). \tag{4.8}$$

Proof. We first check formula (4.8) for $t = 0$. It is necessary to show that there exists a limit

$$\Phi - \lim_{t \to 0} \frac{T(t)\varphi(\theta) - \varphi(\theta)}{t} = \varphi'(\theta), \tag{4.9}$$

i.e., we need to verify that for any $m \in \mathbb{N}_0$

$$\left\| \frac{T(t)\varphi(\theta) - \varphi(\theta)}{t} - \varphi'(\theta) \right\|_m \longrightarrow 0 \quad \text{as} \quad t \to 0.$$

By (4.1) and (4.2) we have

$$\left\| \frac{T(t)\varphi(\theta) - \varphi(\theta)}{t} - \varphi'(\theta) \right\|_m = \left\| \frac{u(\theta + t) - u(\theta)}{t} - u'(\theta) \right\|_m =$$

$$= \left\| \frac{u^{(m)}(\theta + t) - u^{(m)}(\theta)}{t} - u^{(m+1)}(\theta) \right\|_0.$$

By the Mean Value theorem for some $\delta \in (0, 1)$ (depending on t an θ) we have

$$\frac{u^{(m)}(\theta + t) - u^{(m)}(\theta)}{t} - u^{(m+1)}(\theta) = u^{(m+1)}(\theta + \delta t) - u^{(m+1)}(\theta).$$

Let $|t| < h$, then $\theta + \delta t \in [-2h, 2h]$. Therefore,

$$\left\| \frac{T(t)\varphi(\theta) - \varphi(\theta)}{t} - \varphi'(\theta) \right\|_m = \max_{\theta \in [-h,h]} |u^{(m+1)}(\theta + \delta t) - u^{(m+1)}(\theta)| \leq$$

$$\leq \max \left\{ |u^{(m+1)}(\tilde{\theta}) - u^{(m+1)}(\theta)| : \theta, \tilde{\theta} \in [-2h, 2h], |\theta - \tilde{\theta}| < |t| \right\}.$$

The last expression tends to zero as $t \to 0$ by the uniform continuity of the function $u^{(m+1)}$ on the closed interval $[-2h, 2h]$.

Now to prove (4.8) for $t \neq 0$ we use a group property of $T(t)$:

$$\frac{T(t + \delta)\varphi(\theta) - T(t)\varphi(\theta)}{\delta} = \frac{T(\delta)\varphi_t(\theta) - \varphi_t(\theta)}{\delta}.$$

Since $\varphi_t \in \mathfrak{I}_\mu$, by applying the already considered case of $t = 0$, we get

$$\Phi - \lim_{\delta \to 0} \frac{T(\delta)\varphi_t(\theta) - \varphi_t(\theta)}{\delta} = \varphi'_t(\theta) = u'(\theta + t) = T(t)\varphi'(\theta).$$

Note that the last equality $u'(\theta + t) = T(t)\varphi'(\theta)$ is true because, as it follows from the definition (4.3), the function φ' also belongs to the space $\mathfrak{I}_\mu$. The theorem is proved. $\qquad \square$

Corollary 4.3. *The group $T(t)$ is strongly continuous, i.e.,*

$$\Phi - \lim_{t \to t_0} T(t)\varphi = T(t_0)\varphi$$

for all $\varphi \in \mathfrak{I}_\mu$ and any $t_0 \in \mathbb{R}$.

Proof. In $\mathfrak{I}_\mu$ like in any countably normed space strong differentiability implies strong continuity [6]. □

Corollary 4.4. *A differentiation operator $A : \mathfrak{I}_\mu \to \mathfrak{I}_\mu$ such that $A\varphi(\theta) = \varphi'(\theta)$, $\varphi \in \mathfrak{I}_\mu$, is the generator of the transformation group $T(t)$. The operator A is continuous.*

Proof. Since together with φ its derivative φ' belongs to the space $\mathfrak{I}_\mu$ as well, the operator A maps $\mathfrak{I}_\mu$ to $\mathfrak{I}_\mu$. Consequently, by (4.9) the operator A is the generator of the group $T(t)$. Further, $\|A\varphi\|_m = \|\varphi'\|_m = \|\varphi\|_{m+1}$. So, if $\varphi_n \xrightarrow{\Phi} 0$ as $n \to \infty$, i.e., $\|\varphi_n\|_k \to 0$ as $n \to \infty$ for any $k \in \mathbb{N}_0$, then $\|A\varphi_n\|_m = \|\varphi_n\|_{m+1} \to 0$ as $n \to \infty$ for any $m \in \mathbb{N}_0$ as well. This proves the continuity of the operator A. □

Theorem 4.5. *The point spectrum $\sigma_p(A)$ of the differentiation operator $A : \mathfrak{I}_\mu \to \mathfrak{I}_\mu$ coincides with the set $\Lambda_\mu = \{\lambda \in \mathbb{C} : 2h\mu\lambda = e^{\lambda h} - e^{-\lambda h}\}$ defined in Section 2. The eigenspace $\mathcal{E}_\lambda$ corresponding to the eigenvalue $\lambda \in \Lambda_\mu$ is one-dimensional and generated by a function $u_\lambda(\theta) = e^{\lambda\theta}$, $\theta \in [-h, h]$, i.e., $\mathcal{E}_\lambda = \text{gen}\{e^{\lambda\theta}\}$.*

Proof. A solution of the eigenvalue-eigenvector problem $Au = \lambda u$ or $u'(\theta) = \lambda u(\theta)$ is a smooth function $u_\lambda(\theta) = Ce^{\lambda\theta}$. And $u_\lambda \in \mathfrak{I}_\mu$ if and only if

$$2h\mu u_\lambda^{(k+1)}(0) = u_\lambda^{(k)}(h) - u_\lambda^{(k)}(-h), \quad k \in \mathbb{N}_0,$$

that is, if and only if

$$2h\mu\lambda^{k+1} = \lambda^k(e^{\lambda h} - e^{-\lambda h}), \quad k \in \mathbb{N}_0,$$

or equivalently

$$2h\mu\lambda = e^{\lambda h} - e^{-\lambda h},$$

as the last equation like the previous one holds for $\lambda = 0$ as well. □

Remark 4.6. As shown in Section 2 for $\lambda \in \Lambda_\mu$ the functions $u_\lambda(t) = Ce^{\lambda t}$, $t \in \mathbb{R}$, are solutions to the mixed-type differential-difference equation $2h\mu u'(t) = u(t+h) - u(t-h)$. Therefore, we have

$$T(t)u_\lambda(\theta) := u_\lambda(\theta + t) = Ce^{\lambda(\theta + t)} = e^{\lambda t}(Ce^{\lambda\theta}),$$

i.e., $T(t)u_\lambda(\theta) = e^{\lambda t}u_\lambda(\theta)$. Thus the eigenfunction of the generator A corresponding to the eigenvalue λ is as well an eigenfunction of the operator $T(t)$ corresponding to the eigenvalue $e^{\lambda t}$. It is well known that in view of spectral mapping theorems this assertion is valid in Banach spaces [7].

Let us show that the spectrum of A is a purely point one.

Lemma 4.7. *Let a function $u = u(\theta)$, $\theta \in [-h, h]$, be a solution of the equation*

$$u'(\theta) = \lambda u(\theta) + f(\theta), \tag{4.10}$$

where $\lambda \in \mathbb{C}$ and $f \in \mathfrak{I}_\mu$. If the function $u = u(\theta)$ satisfies the condition

$$2h\mu u'(0) = u(h) - u(-h), \tag{4.11}$$

then $u \in \mathfrak{I}_\mu$, i.e., the equality

$$2h\mu u^{(k+1)}(0) - (u^{(k)}(h) - u^{(k)}(-h)) = 0 \tag{4.12}$$

holds for all $k \in \mathbb{N}_0$.

Proof. Since $f \in C^\infty[-h, h]$, the function $u \in C^\infty[-h, h]$ as well. We prove the assertion by induction. By (4.11) the condition (4.12) is obviously true for $k = 0$. Assume it to be true for $k = n - 1$ with some arbitrary $n > 0$. We need to establish (4.12) for $k = n$. By differentiating equation (4.10), we find that

$$u^{(n+1)}(0) = \lambda u^{(n)}(0) + f^{(n)}(0)$$

and

$$u^{(n)}(\pm h) = \lambda u^{(n-1)}(\pm h) + f^{(n-1)}(\pm h).$$

Consequently,

$$2h\mu u^{(n+1)}(0) - \left(u^{(n)}(h) - u^{(n)}(-h)\right) = 2h\mu\left(\lambda u^{(n)}(0) + f^{(n)}(0)\right) -$$

$$- \left(\lambda u^{(n-1)}(h) + f^{(n-1)}(h)\right) + \left(\lambda u^{(n-1)}(-h) + f^{(n-1)}(-h)\right) =$$

$$= \lambda\left(2h\mu u^{(n)}(0) - u^{(n-1)}(h) + u^{(n-1)}(-h)\right) +$$

$$+ \left(2h\mu f^{(n)}(0) - f^{(n-1)}(h) + f^{(n-1)}(-h)\right) = 0 + 0$$

by the induction hypothesis and the condition $f \in \mathfrak{I}_\mu$. $\qquad\square$

Now suppose that $\lambda \notin \sigma_p(A) = \Lambda_\mu$. Let us compute the resolvent $R(\lambda)$ of A. Given $f \in \mathfrak{I}_\mu$, the solution of the inhomogeneous equation $Au = \lambda u + f$, i.e., of the differential equation $u'(\theta) = \lambda u(\theta) + f(\theta)$ is given by

$$u(\theta) = e^{\lambda\theta} \cdot C_f + e^{\lambda\theta} \int_0^\theta e^{-\lambda\tau} f(\tau) d\tau \tag{4.13}$$

with an arbitrary constant C_f. By Lemma 4.1, for the solution $u = u(\theta)$ to belong to the space $\mathfrak{I}_\mu$ it is sufficient to satisfy the condition (4.11) by choosing the constant C_f. From (4.13) we find that $u'(0) = \lambda C_f + f(0)$. Substituting this expression for $u'(0)$ and the expression for $u(\theta)|_{\pm h}$ also obtained from (4.13) in (4.11) we get

$$(2h\mu\lambda - e^{\lambda h} + e^{-\lambda h})C_f = -2h\mu f(0) +$$

$$+ e^{\lambda h} \int_0^h e^{-\lambda\tau} f(\tau) d\tau - e^{-\lambda h} \int_0^{-h} e^{-\lambda\tau} f(\tau) d\tau. \tag{4.14}$$

Since $\lambda \notin \Lambda_\mu = \sigma_p(A)$, the expression $2h\mu\lambda - e^{\lambda h} + e^{-\lambda h} \neq 0$. Hence it follows from (4.14) that

$$C_f = \frac{-2h\mu f(0) + e^{\lambda h} \int\limits_0^h e^{-\lambda\tau} f(\tau)d\tau - e^{-\lambda h} \int\limits_0^{-h} e^{-\lambda\tau} f(\tau)d\tau}{2h\mu\lambda - e^{\lambda h} + e^{-\lambda h}}. \qquad (4.15)$$

Consequently, by (4.15) and (4.13) for $\lambda \notin \sigma_p(A) = \Lambda_\mu$ the resolvent

$$R(\lambda)f(\theta) = e^{\lambda\theta} \cdot C_f + e^{\lambda\theta} \int\limits_0^\theta e^{-\lambda\tau} f(\tau)d\tau \qquad (4.16)$$

is defined on the whole space $\mathfrak{I}_\mu$. Hence for $\lambda \notin \sigma_p(A)$ the bounded operator $A - \lambda I : \mathfrak{I}_\mu \to \mathfrak{I}_\mu$ is a bijection of the complete countably normed space $\mathfrak{I}_\mu$. By Banach's closed graph theorem [8, 9] the operator $R(\lambda) : \mathfrak{I}_\mu \to \mathfrak{I}_\mu$ is continuous as well. So the following theorem is proved:

Theorem 4.8 (On the spectrum of the generator A)**.** *The spectrum of the linear operator* $A : \mathfrak{I}_\mu \to \mathfrak{I}_\mu$ *is a purely point one. For* $\lambda \notin \sigma_p(A)$ *the resolvent* $R(\lambda)$ *of* A *is given by formulas* (4.16) *and* (4.15)*.*

Remark 4.9. It is shown that the eigenfunctions $u_\lambda(t) = e^{\lambda t}$ of A corresponding to the eigenvalues λ of A different from zero are also eigenfunctions of the Steklov smoothing operator S_h corresponding to the eigenvalue μ. If $\mu = 1$ then $u_0(t) = 1$ is also an eigenfunction of S_h.

Note that the continuity of $R(\lambda)$ can be verified directly without applying Banach's closed graph theorem. Indeed, let us denote by C various constants independent of f. From (4.15) it follows that $|C_f| \leq C\|f\|_0$ and hence for the first term in (4.16) we have $\|C_f \cdot e^{\lambda\theta}\|_m \leq C\|f\|_0$. Further for the second term in (4.16)

$$e^{\lambda\theta} \int\limits_0^\theta e^{-\lambda\tau} f(\tau)d\tau =: g(\theta)$$

we find that $\|g\|_0 \leq C\|f\|_0$. As $g'(\theta) = \lambda g(\theta) + f(\theta)$ and $g''(\theta) = \lambda^2 g(\theta) + \lambda f(\theta) + f'(\theta)$, it is shown by induction that for any $m \in \mathbb{N}_0$

$$g^{(m)}(\theta) = \lambda^m g(\theta) + \sum_{k=0}^{m-1} \lambda^{m-1-k} f^{(k)}(\theta).$$

Whence,

$$\|g\|_m = \|g^{(m)}\|_0 \leq |\lambda|^m \|g\|_0 + \sum_{k=0}^{m-1} |\lambda|^{m-1-k} \|f\|_k \leq C \sum_{k=0}^{m-1} \|f\|_k.$$

Consequently,

$$\|R_\lambda f\|_m \leq \|C_f e^{\lambda\theta}\|_m + \|g\|_m \leq C \sum_{k=0}^{m-1} \|f\|_k,$$

i.e., for any $m \in \mathbb{N}_0$ there exists a constant $C = C_m > 0$ independent of f such that

$$\|R_\lambda f\|_m \leq C \sum_{k=0}^{m-1} \|f\|_k.$$

This settles the continuity of the map $R(\lambda)$.

5. SOME GENERALIZATIONS

The main part of this paper is devoted to studying the differential-difference equation of mixed type

$$\mu u'(t) = \frac{u(t+h) - u(t-h)}{2h}, \tag{5.1}$$

where $\mu \in \mathbb{C} \setminus \{0\}$. In quite a similar way we can consider a more general equation of the form

$$\mu u'(t) = au(t+h) - bu(t-h) + cu(t), \tag{5.2}$$

where $\mu, a, b \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{C}$, $u \in AC(\mathbb{R})$.

An operator S, for example, of the following form

$$Su(t) := a \int_0^{t+h} u(s)ds - b \int_0^{t-h} u(s)ds + c \int_0^t u(s)ds, \quad u \in L^1_{loc}(\mathbb{R}), \tag{5.3}$$

will correspond to equation (5.2).

As in Section 2 it is shown that an absolutely continuous solution of equation (5.2) is smooth, i.e., infinitely differentiable. It is an elementary calculation to show that the function $u_\lambda(t) = e^{\lambda t}$, $\lambda \in \mathbb{C}$, satisfies equation (5.2) if and only if $\lambda \in \Lambda_\mu$, where now the set

$$\Lambda_\mu := \{\lambda \in \mathbb{C} : \mu\lambda = ae^{\lambda h} - be^{-\lambda h} + c\}. \tag{5.4}$$

Analogously an initial function $\varphi \in C^\infty[-h, h]$ will generate a smooth solution to equation (5.2), constructed by the method of steps, if and only if the following conditions are fulfilled

$$\mu\varphi^{(n+1)}(0) = a\varphi^{(n)}(h) - b\varphi^{(n)}(-h) + c\varphi^{(n)}(0), \quad n \in \mathbb{N}_0. \tag{5.5}$$

Respectively the initial space $\mathfrak{J}_\mu$ is defined as follows

$$\mathfrak{J}_\mu := \{\varphi \in \Phi : \varphi \text{ satisfies (5.5)}\}. \tag{5.6}$$

The transformation group $T(t)$ and its generator are defined exactly as before in Section 4. By using the same reasonings it is established that the spectrum of the operator A is a purely point one and coincides with the set Λ_μ, i.e.,

$$\sigma(A) = \sigma_p(A) = \Lambda_\mu. \tag{5.7}$$

And the eigenspace corresponding to the eigenvalue $\lambda \in \Lambda_\mu$ is one-dimensional and generated by a function $u_\lambda(\theta) = e^{\lambda\theta}$, $\theta \in [-h, h]$. For $\lambda \notin \sigma_p(A)$ the resolvent $R(\lambda)$ of A is given by formula (4.16) as before, but obviously with another constant C_f. Now we have

$$C_f = \frac{-\mu f(0) + ae^{\lambda h} \int_0^h e^{-\lambda\tau} f(\tau)d\tau - be^{-\lambda h} \int_0^{-h} e^{-\lambda\tau} f(\tau)d\tau}{(\mu\lambda - ae^{\lambda h} + be^{-\lambda h} - c)}. \tag{5.8}$$

In substance, the only distinction from equation (5.1) is that now we can not affirm in general that the eigenfunctions $u_\lambda(t) = e^{\lambda t}$, $\lambda \in \sigma_p(A) = \Lambda_\mu$, of the generator A are also eigenfunctions of the operator S corresponding to the eigenvalue μ of S. Indeed, the equality $\mu u_\lambda(t) = Su_\lambda(t)$, for example, for $\lambda \neq 0$ means that the equation

$$\mu\lambda = ae^{\lambda h} - be^{-\lambda h} + c + (b - a - c)e^{-\lambda t} \tag{5.9}$$

must be true for all $t \in \mathbb{R}$. As $\lambda \in \Lambda_\mu$, and hence $\mu\lambda = ae^{\lambda h} - be^{-\lambda h} + c$, equality (5.9) is fulfilled if and only if $b - a = c$, that does not clearly take place in a general situation. Setting $c = b - a$ in (5.3) yields

$$Su(t) = a \int_t^{t+h} u(s)ds - b \int_t^{t-h} u(s)ds, \quad u \in L^1_{loc}(\mathbb{R}).$$

So, in particular, if $a = b = 1/(2h)$, then $S = S_h$.

Note too that as is easily seen the value $\lambda = 0$ belongs to the set Λ_μ if and only if $c = b - a$.

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