Opuscula Math. 33, no. 1 (2013), 19–28
http://dx.doi.org/10.7494/OpMath.2013.33.1.19 Opuscula Mathematica

ON VERTEX b-CRITICAL TREES

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Communicated by Mariusz Meszka

Abstract. A b-coloring is a proper coloring of the vertices of a graph such that each color class has a vertex that has neighbors of all other colors. The b-chromatic number of a graph G is the largest k such that G admits a b-coloring with k colors. A graph G is b-critical if the removal of any vertex of G decreases the b-chromatic number. We prove various properties of b-critical trees. In particular, we characterize b-critical trees.

Keywords: b-coloring, b-critical graphs, b-critical trees.

Mathematics Subject Classification: 05C15.

1. INTRODUCTION

Let G be a simple graph with vertex-set $V(G)$ and edge-set $E(G)$. A *coloring* of the vertices of G is any mapping $c : V(G) \to \mathbb{N}$. For every vertex v the integer $c(v)$ is called the *color* of v. A coloring is *proper* if any two adjacent vertices have different colors. The *chromatic number* $\chi(G)$ of graph G is the smallest integer k such that G admits a proper coloring using k colors.

A *b-coloring* of G by k colors is a proper coloring of the vertices of G such that in each color class there exists a vertex that has neighbors in all the other $k-1$ colors classes. We call any such vertex a *b-vertex*. The concept of b-coloring was introduced by Irving and Manlove [3,4]. The b-chromatic number $b(G)$ of a graph G is the largest integer k such that G admits a b-coloring with k colors. It was proved in [3,4] that determining the b-chromatic number of a graph is an NP-complete problem.

A graph G is *edge b-critical* (resp. *vertex b-critical*) if the removal of any edge (resp. vertex) of G decreases the b-chromatic number. Ikhlef Eschouf [2] began the study of edge b-critical graphs. He characterized the edge b-critical P_4-sparse graphs and edge b-critical quasi-line graphs. We propose here to study the effect of removing a vertex of a graph G on the b-chromatic number. From here on, "b-critical" will always mean vertex b-critical. We prove several properties of b-critical trees. In particular, we show that if T is a b-critical tree, then $\Delta(T) \leq b(T) \leq \Delta(T)+1$, where $\Delta(T)$ is the maximum degree in T. Finally, we give a characterization of b-critical trees.

For notation and graph theory terminology we follow [1]. Consider a graph G. For any $A \subset V(G)$, let $G[A]$ denote the subgraph of G induced by A, and let $G \setminus A$ be the subgraph induced by $V(G) \setminus A$. (If x is a vertex, we may write $G \setminus x$ instead of $G \setminus \{x\}$). For any vertex v of G, the *neighborhood* of v is the set $N_G(v) = \{u \in V(G) \mid (u,v) \in E\}$ (or $N(v)$ if there is no confusion). Let $\omega(G)$ denote the size of a maximum clique of G. We let P_k denote the path with k vertices. A vertex of a path P_k distinct from an end-vertex is said to be an internal vertex. The complete bipartite graph with classes of sizes p and q is denoted by $K_{p,q}$, and any graph $K_{1,q}$ is called a *star*.

A *tree* is a connected graph with no induced cycle. A *rooted* tree is a tree T in which one vertex x is distinguished and called the *root*. For every vertex u of $T \setminus x$, the *parent* of u is the neighbor of u on the unique path from u to x, while a *child* of u is any other neighbor of u. A *descendant* of u is defined (recursively) to be either any child of u or any descendant of a child of u. We let $D(u)$ denote the set of *descendants* of u, and we write $D[u] = D(u) \cup \{u\}$. For any set A let $D(A) = \bigcup_{u \in A} D(u)$. The subtree of T induced by $D[u]$ is denoted by T_u. A vertex of degree one is called a *leaf*, and its neighbor is called a *support* vertex. If v is a support vertex, then L_v denotes the set of leaves adjacent to v.

2. PRELIMINARY RESULTS

We will use several definitions and results due to Irving and Manlove [3]. Remark that if a graph G admits a b-coloring with k colors, then G has at least k vertices of degree at least $k-1$. Irving and Manlove define the *m-degree* $m(G)$ of G to be the largest integer ℓ such that G has at least ℓ vertices of degree at least $\ell - 1$. Thus every graph G satisfies $b(G) \leq m(G)$. The difference $m(G) - b(G)$ can be arbitrarily large: for example, $m(K_{p,p}) = p+1$ while $b(K_{p,p}) = 2$. Irving and Manlove [3] proved that $b(T)$ can be computed easily for every tree, as follows. A vertex v is *dense* if $d_G(v) \geq m(G) - 1$.

Definition 2.1 ([3])**.** A tree T is *pivoted* if T has exactly $m(T)$ dense vertices and T contains a vertex v such that v is not dense and every dense vertex is adjacent either to v or to a neighbor of v of degree $m-1$.

Theorem 2.2 ([3])**.** *Let T be a tree. If T is a pivoted tree, then $b(T) = m(T) - 1$; else, $b(T) = m(T)$.*

Now we prove a few general facts about b-critical trees.

Lemma 2.3. *Let T be a b-critical tree and c be a b-coloring of T with $b(T)$ colors. Let B be the set of all b-vertices of c. Then:*

(i) *Every vertex of $V(T) \setminus B$ has a neighbor in B.*
(ii) *If z is a support vertex, then z is in B and is the only b-vertex of color $c(z)$. Moreover, z does not have two neighbors of the same color such that one of them is a leaf.*

Proof. (i) If a vertex u in $V(T) \setminus B$ has no neighbor in B, then c remains a b-coloring of $T \setminus u$ with $b(T)$ colors.

(ii) If any part of (ii) does not hold, then the removal of some leaf adjacent to z does not decrease the b-chromatic number. □

Theorem 2.4. *Let T be a b-critical tree, and let c be a b-coloring of T with $b(T)$ colors. Let B be the set of all b-vertices of c. Then:*

(i) *c does not have two b-vertices of the same color, i.e., $|B| = b(T)$.*
(ii) *Every vertex u in $V(T) \setminus B$ satisfies $d_T(u) \leq b(T) - 1$.*
(iii) *Every vertex x in B satisfies $b(T) - 1 \leq d_T(x) \leq b(T)$.*

Proof. Let $k = b(T)$. If $k = 2$, it is immediate to see that $T = P_2$ and the theorem holds. So we may assume that $k \geq 3$.

(i) Suppose that c has two b-vertices x and y of the same color. If x or y is a support vertex, adjacent to a leaf z, then c remains a b-coloring with k colors of $T \setminus z$, a contradiction. So x and y are not support vertices. Let us root T at vertex x. Let $u_1, \ldots, u_h$ be the neighbors of x. Since x is a b-vertex, we have $h \geq k - 1$. For each i in $\{1, \ldots, h\}$, let T_i be the component of $T \setminus x$ that contains u_i. Since x is not a support vertex, T_i contains a support vertex z_i of T. Lemma 2.3 (ii) implies that z_i is the only b-vertex of color $c(z_i)$ in T; in particular, $c(z_i) \neq c(x)$. Therefore T contains at least $k-1$ support b-vertices $z_1, \ldots, z_h$ of distinct colors. If the number of support vertices is more than $k - 1$, then two of them have the same color, which contradicts Lemma 2.3 (ii). So it must be that $h = k-1$ and each T_i contains exactly one support vertex. If any vertex u of $V(T) \setminus \{x, z_1, \ldots, z_{k-1}\}$ has degree at least 3, then the subgraph T_i that contains u contains two support vertices of T, a contradiction. So $d_T(u) \leq 2$. In particular, $d_T(y) \leq 2$. This implies $k = 3$, $d_T(x) = d_T(y) = 2$, and by Lemma 2.3 (ii) we also have $d_T(z_1) = d_T(z_2) = 2$. Hence T is a path. Since T contains at least four b-vertices such that two of them (x and y) are non-support vertices of the same color, it follows that T is a path of at least 7 vertices. But this is not b-critical, a contradiction. Thus (i) holds.

By (i), we have $B = \{b_1, b_2, \ldots, b_k\}$, where b_i is the unique b-vertex of c of color i, for each i in $\{1, \ldots, k\}$. By Lemma 2.3 (i), we have $V(T) = N[b_1] \cup \cdots \cup N[b_k]$.

(ii) Let u be any vertex in $V(T) \setminus B$ and suppose that $d_T(u) \geq k$. Since $V(T) = N[b_1] \cup \cdots \cup N[b_k]$, we may assume that $u \in N(b_1)$. Vertex u is adjacent to at most $k-2$ b-vertices, for otherwise either u is a b-vertex or there is no available color for u. Thus we may assume that $N(u) \cap B = \{b_1, \ldots, b_r\}$ with $1 \leq r \leq k-2$. Since T is a tree, u has at most one neighbor in $N[b_i]$ for each i in $\{1, \ldots, k\}$. Hence $d_T(u) = k$, vertex u has a neighbor u_j in $N(b_j)$ for every j in $\{r+1, \ldots, k\}$ and $u_{r+1}, \ldots, u_k$ are pairwise distinct. We may assume that $c(u) = r + 1$. For each $j \in \{r+1, \ldots, k-1\}$, let v_j be a vertex of color $j+1$ in $N(b_j)$ (possibly $v_j = u_j$). We define a coloring π of T with k colors obtained from c as follows. For each $j \in \{r+1, \ldots, k-1\}$, if $v_j \neq u_j$ then interchange the colors of u_j and v_j. All other vertices keep their color. We obtain that π is a b-coloring with k colors such that u and b_{r+1} are b-vertices of the same color, which contradicts Theorem 2.4 (i) for π.

(iii) Let x be a b-vertex and $p = d_T(x)$. Clearly, $p \geq k-1$ since x is a b-vertex. Let $T_1, T_2, \ldots, T_p$ be the components of $T \setminus x$. Suppose that $p \geq k+1$. Then $N(x)$ contains a leaf, for otherwise Theorem 2.4 (i) and Lemma 2.3 (ii) imply that $|B| \geq d_T(x) + |\{x\}| \geq k+2$, a contradiction. For each r in $\{1, \ldots, k\}$ let $N^r(x)$ be the set of neighbors of x of color r. Let u be a leaf adjacent to x, and let ℓ be the color of u. Then $|N^\ell(x)| = 1$, for otherwise, c remains a b-coloring of $T \setminus u$ with k colors, a contradiction. Since $p \geq k+1$, there is a color $t \neq \ell$ such that $|N^t(x)| \geq 2$. We distinguish among two cases.
Case 1. $|N^t(x)| \geq 3$. Let x_1, x_2, x_3 be three vertices in $N^t(x)$. We may assume that $x_i \in T_i$, for $i = 1, 2, 3$. Lemma 2.3 (ii) and Theorem 2.4 (i) imply that one of T_1, T_2, T_3, say T_1, does not contain any b-vertex of color t or ℓ. We recolor the vertices of $V(T_1) \cup \{u\}$ by exchanging colors t and ℓ. We obtain a b-coloring where the color of u appears on another vertex of $N(x)$. Hence, c remains a b-coloring of $T \setminus u$ with k colors, a contradiction.
Case 2. For every r in $\{1, \ldots, k\}$, $|N^r(x)| \leq 2$. Since $d_T(x) \geq k+1$, there are two colors that appear exactly twice in $N(x)$. Without loss of generality, we may suppose that $x_1, x_2 \in N^t(x)$ and $x_3, x_4 \in N^h(x)$, with $h \neq t, \ell$. Also we may suppose that $x_i \in T_i$ for each $i \in \{1, 2, 3, 4\}$. By Theorem 2.4 (i) and the pigeonhole principle, there exists a component T_s with $1 \leq s \leq 4$ that contains no b-vertex with color in $\{t, \ell\}$ (or in $\{h, \ell\}$). Without loss of generality, we may suppose that T_1 contains no b-vertex colored t or ℓ. We recolor the vertices $V(T_1) \cup \{u\}$ by exchanging colors t and ℓ and obtain a contradiction as at the end of Case 1. This completes the proof of the theorem. □

An immediate consequence of Theorem 2.4 is the following.

Corollary 2.5. *If T is a b-critical tree, then $\Delta(T) \leq b(T) \leq \Delta(T) + 1$.*

3. CHARACTERIZATION OF b-CRITICAL TREES

In this section, we give a characterization of b-critical trees. By Corollary 2.5, this amounts to characterizing the b-critical trees having a b-chromatic number equal to $\Delta(T)$ or $\Delta(T) + 1$.

3.1. b-CRITICAL TREES WITH $b(T) = \Delta(T)$

In order to characterize the b-critical trees T with $b(T) = \Delta(T)$, we define a family $\mathcal{T}_1$ as follows:

Definition 3.1 (Class $\mathcal{T}_1$)**.** A tree T is in *class* $\mathcal{T}_1$ if, and only if, for some integers k and p with $k \geq 4$ and $2 \leq p \leq k-2$, the vertex-set of T can be partitioned into four sets $\{v\}$, D_1, D_2, X with the following properties:

- $|D_1| = p$, and every vertex of D_1 is adjacent to v;
- $|D_2| = k - p$, and every vertex of D_2 has a neighbor in D_1;
- Every vertex of X has a neighbor in $D_1 \cup D_2$;

– There is a vertex $w \in D_1$ such that w has a neighbor in D_2, w has degree k, and every vertex of $D_1 \cup D_2 \setminus \{w\}$ has degree $k-1$.

Note that there is no other edge than those mentioned in the definition, because T is a tree. The definition implies easily that $|X| = k^2-3k+p+1$. So T has $k^2-2k+p+2$ vertices. Also, $\Delta(T) = k$, $m(T) = k$, the dense vertices are the vertices in $D_1 \cup D_2$, and $b(T) = k$.

Class $\mathcal{T}_1$ may contain several non-isomorphic graphs with the same value of k and p, depending on the adjacency between D_1 and D_2.

Lemma 3.2. *If $T \in \mathcal{T}_1$, then T is b-critical.*

Proof. As observed above, we have $b(T) = k$ and $m(T) = k$. Let $Y = D_1 \cup D_2 \setminus \{w\}$. Consider any vertex x of T. If $x \in N[Y] \cup \{w\}$, then $b(T \setminus x) \leq m(T \setminus x) \leq m(T)-1 = k-1$. If $x \in N(w)$, then $T \setminus x$ is a pivoted tree. By Theorem 2.2, $b(T \setminus x) = m(T)-1 = k-1$. Thus T is b-critical. □

Theorem 3.3. *Let T be a tree with $b(T) = \Delta(T)$. Then T is b-critical if and only if $T \in \mathcal{T}_1$.*

Proof. If $T \in \mathcal{T}_1$, then by Lemma 3.2, T is b-critical. Now let us prove the converse. Let T be a b-critical tree with $b(T) = \Delta(T)$. Let $k = b(T)$. Let c be a b-coloring of T with k colors and let B be the set of all b-vertices of c. By Theorem 2.4, there is a unique b-vertex b_i of color i for each $i \in \{1, \dots, k\}$, and so $B = \{b_1, \dots, b_k\}$.

Pick a vertex x of maximum degree, and root T at x. Let L_x be the set of leaves adjacent to x, let $B_x = B \cap N(x)$ and $Y_x = N(x) \setminus (B_x \cup L_x)$. Put $Y_x = \{y_1, \dots, y_q\}$. By Theorem 2.4, x is a b-vertex. Since $d_T(x) = k$, there are two vertices of the same color in $N(x)$. On the other hand, since x is a b-vertex of degree $b(T)$, all neighbors of x except these two must have different colors. We call these two the repeating pair. By Lemma 2.3 (ii), these two vertices are not in L_x, and by Theorem 2.4 (i), one of them is not in B. So one of them is in Y_x, and so $q \geq 1$.

For each $i \in \{1, \dots, q\}$, let T_i be the component of $T \setminus x$ that contains y_i, and let $B_i = B \cap V(T_i)$. Let $B'_x = B \cap D(B_x)$. The definition of L_x and Y_x implies that T_i contains a support vertex of T, and Lemma 2.3 implies that such a vertex is a b-vertex. Hence $|B_i| \geq 1$ for all i in $\{1, \dots, q\}$. So $|B| \geq q+1+|B_x|$. If $L_x = \emptyset$, this inequality implies $|B| \geq d_T(x)+1$, a contradiction. Therefore we have $L_x \neq \emptyset$. For each $i \in \{1, \dots, q\}$, let $L_i = \{v \in L_x \mid b_{c(v)} \in B_i\}$ and $L' = \{v \in L_x \mid b_{c(v)} \in B'_x\}$. Note that for any vertex z in L_x, the color $c(z)$ does not appear in $N(x) \setminus z$, by Lemma 2.3. So $L_x = L_1 \cup \dots \cup L_q \cup L'$.

We observe that the following fact holds:

Let $b_i, b_j \in B$ and $y \in N(x)$. Suppose that $c(x) \neq i, j$ and b_i and b_j are either both in $D(y)$ or not in $D(y)$. Then interchanging colors i and j in $G[D(y)]$ produces a b-coloring of T with k colors. (3.1)

Indeed, after the interchange the coloring is proper (because $c(x) \neq i, j$), every b-vertex b_h with $h \neq i, j$ is still a b-vertex of color h, and b_i and b_j are either unchanged or b-vertices of color j and i respectively. Thus (3.1) holds.

All colors that appear in Y_x are different. (3.2)

Suppose on the contrary that two vertices y_1, y_2 in Y_x have the same color h (so they form the unique repeating pair). Up to symmetry, we may assume that $b_h \notin B_1$. Recall that $L_x \neq \emptyset$. Pick any vertex $z \in L_x$ and let $\ell = c(z)$. By Lemma 2.3, we have $\ell \neq h$. If b_ℓ is not in T_1, then we interchange colors h and ℓ in T_1. By (3.1), this produces a b-coloring π of T with k colors such that z is a leaf of a repeated color in $N(x)$. Hence, π remains a b-coloring of $T \setminus z$ with k colors, a contradiction. Therefore every color that appears in L_x has its b-vertex in T_1, i.e., $L_x = L_1$, and $L_2 = \cdots = L_q = L' = \emptyset$. Then $b_h \in T_2$, for otherwise we should also have $L_x = L_2$. The set $B_3 \cup \cdots \cup B_q$ (if $q \geq 3$) must contain $q-2$ b-vertices (because $B_j \neq \emptyset$ for all $j \in \{1, \ldots, q\}$), and these must be the b-vertices whose colors are in $\{y_3, \ldots, y_q\}$ (because all other colors have their b-vertices in $B_x \cup B_1 \cup \{b_h\}$). By the pigeonhole principle, we have $|B_i| = 1$ for all i in $\{3, \ldots, q\}$ and also $|B_2| = 1$, i.e., $B_2 = \{b_h\}$. If T_2 has a vertex of degree at least 3 other than b_h, then there are at least two support vertices in T_2, and these are b-vertices, a contradiction. Therefore T_2 consists of a path between y_2 and b_h plus leaves attached to b_h. It is easy to recolor the vertices of T_2 in such a way that the coloring is proper, y_2 gets color ℓ and b_h remains a b-vertex of color h. This produces a b-coloring π of T with k colors such that z is a leaf of a repeated color in $N(x)$. Hence, π remains a b-coloring of $T \setminus z$ with k colors, a contradiction. Thus (3.2) holds.

Claim (3.2) implies that the repeating pair can be written as $\{y_1, b_t\}$ with $y_1 \in Y_x$ and $b_t \in B_x$. Moreover we claim that

$$Y_x = \{y_1\}. \tag{3.3}$$

Suppose on the contrary that $|Y_x| \geq 2$. Then $D(y_2)$ contains a support vertex of T, and by Lemma 2.3, such a vertex is a b-vertex b_r. Note that $r \neq t$ and T_1 does not contain a b-vertex of color r or t. We interchange colors t and r in $G[T_1]$. By (3.1), this produces a b-coloring π with k colors. Vertex x has a neighbor x^r of color r, and we have $x^r \in L_x \cup Y_x$. Suppose that $x^r \in Y_x$. Then $x^r \neq y_1$, and Y_x contains two vertices of color r (in π), a contradiction to (3.2). So $x^r \in L_x$. Then x^r is a vertex with a repeated color in $N(x)$. Thus π remains a b-coloring of $T \setminus x^r$ with k colors, a contradiction. Thus (3.3) holds.

$$\text{Every child of a vertex in } B_x \text{ is a leaf.} \tag{3.4}$$

Suppose the contrary. Then, for some vertex $\beta \in B_x$ the set $D(\beta)$ contains a support vertex of T, and by Lemma 2.3 such a vertex is a b-vertex b_r. Clearly $r \neq t$. Now T_1 contains no b-vertex colored t or r. Since x is a b-vertex, and by (3.3), L_x contains a vertex x^r of color r. We interchange colors t and r in T_1. By (3.1), this produces a b-coloring π of T with k colors such that x^r is a leaf of repeated color in $N(x)$. Hence, π remains a b-coloring of $T \setminus x^r$ with k colors, a contradiction. Thus (3.4) holds.

$$\text{Every child of } y_1 \text{ is a b-vertex.} \tag{3.5}$$

Suppose that some child u of y_1 is not a b-vertex. By Lemma 2.3 (i), u is adjacent to a b-vertex b_r. Clearly, $r \neq t$ and $c(u) \neq r, t$. Since x is a b-vertex, and by (3.3), L_x contains a vertex x^r of color r. Note that $D(y_1) \setminus D[u]$ contains no b-vertex of color r or t. We interchange colors t and r in $G[T_1 \setminus D[u]]$. By (3.1), this produces a b-coloring π of T with k colors such that x^r is vertex with a repeated color in $N(x)$. Thus π remains a b-coloring of $T \setminus x^r$ with k colors, a contradiction. So (3.5) holds.

$$L_x = L_1. \tag{3.6}$$

Pick any vertex $z \in L_x$ and let $\ell = c(z)$. Suppose that b_ℓ is not in T_1. Recall that $b_t \in B_x$. So b_t and b_ℓ are not in T_1. We interchange colors t and ℓ in T_1. By (3.1), this produces a b-coloring π of T with k colors such that z is a leaf of a repeated color in $N(x)$. Hence, π remains a b-coloring of $T \setminus z$ with k colors, a contradiction. Therefore every color that appears in L_x has its b-vertex in T_1. Thus (3.6) holds.

Note that the preceding claims imply that $B = \{x\} \cup B_x \cup B_1$.

The *distance* between two vertices x and y, denoted by $\text{dist}(x, y)$, is the length of a shortest path from x to y in T.

$$\text{Every b-vertex } b_r \text{ in } B_1 \text{ satisfies } \text{dist}(y_1, b_r) \leq 2. \tag{3.7}$$

Suppose there exists a b-vertex $b_r \in B_1$ such that $\text{dist}(y_1, b_r) \geq 3$. Without loss of generality, we may suppose that $\text{dist}(y_1, b_r) = max\{\text{dist}(y_1, v) \mid v \in B_1\}$. This and Lemma 2.3 (ii) imply that b_r is a support vertex. Since x is a b-vertex, and by (3.3), L_x contains a vertex u of color r. Let z_0 be the parent of b_r and z_1 be the parent of z_0. Note that z_0 is not adjacent to y_1 (in particular, $z_1 \neq y_1$), for otherwise $\text{dist}(y_1, b_r) < 3$. Then there are two cases to consider.

Case 1. $c(z_0) \neq t$. By Theorem 2.4 (i), $D(y_1) \setminus D[z_0]$ contains no b-vertex of color r and t. Thus, interchanging colors t and r in $G[T_1 \setminus D[z_0]]$ produces a b-coloring π of T with k colors such that u is a vertex with a repeated color in $N(x)$. Thus, π remains a b-coloring of $T \setminus u$ with k colors, a contradiction.

Case 2. $c(z_0) = t$. Then z_0 is not a b-vertex. Also, $D(z_0)$ contains no b-vertices of colors $c(x)$ and t. If $c(z_1) \neq c(x)$, then we interchange the color t and $c(x)$ in $D[z_0]$. Hence, $c(z_0) = c(x) \neq t$. Therefore, an exchange of colors as described in Case 1 produces a new b-coloring π of T with k colors such that u is vertex with a repeated color in $N(x)$. Thus, π remains a b-coloring of $T \setminus u$ with k colors, a contradiction. If $c(z_1) = c(x)$, then z_1 is not a b-vertex. In this case, we can interchange colors t and r in $T_1 \setminus D[z_1]$. This is possible since $T_1 \setminus D[z_1]$ contains no b-vertices of color t and r. We obtain a b-coloring π of T with k colors such that u is a vertex with a repeated color in $N(x)$. Thus, π remains a b-coloring of $T \setminus u$ with k colors, a contradiction. So (3.7) holds.

$$\text{Every vertex } v \text{ in } B \setminus \{x\} \text{ satisfies } d_T(v) = \Delta(T) - 1. \tag{3.8}$$

Recall that $B = \{x\} \cup B_x \cup B_1$. First suppose that $v \in B_x$. By (3.4), every child of v is a leaf. By Lemma 2.3 (ii), $d_T(v) = \Delta(T) - 1$. Now suppose that $v \in B_1$. If $\text{dist}(v, y_1) = 2$, then all children of v are leaves, for otherwise, by Lemma 2.3 (ii), $D(y_1)$ contains a b-vertex that lies at distance at least three from y_1, which contradicts (3.7). By Lemma 2.3 (ii), we have $d_T(v) = \Delta(T) - 1$. If $\text{dist}(v, y_1) = 1$ (i.e., v is a child of y_1), then by Theorem 2.4 (iii), we have $d_T(v) \geq \Delta(T) - 1$. If $d_T(v) = \Delta(T)$, then v can serve the same role as x with $Y(v) = \{y_1\}$. Then the analogue of (3.3) implies that B_v contains a b-vertex of color t different from b_t. This contradicts Theorem 2.4 (i). So $d_T(v) = \Delta(T) - 1$. Thus (3.8) holds.

Claims (3.3)–(3.8) imply that $T \in \mathcal{T}_1$ (where y_1 plays the role of v and x plays the role of w). This completes the proof of Theorem 3.3. □

3.2. b-CRITICAL TREES WITH $b(T) = \Delta(T) + 1$

Let $k = \Delta(T) + 1$. For the purpose of characterizing b-critical trees with $b(T) = k$, we define a family $\mathcal{T}_2$ of trees as follows. A tree T is in $\mathcal{T}_2$ if there is a sequence $T_1, T_2, \ldots, T_k$ of trees, with $T = T_k$, where T_1 is a star of order k, and, for each i in $\{1, \ldots, k-1\}$, T_{i+1} can be obtained from T_i by one of the operations listed below.

- Operation O_1: Identify the center of a star of order $k-1$ with one leaf of a support vertex of degree $k-1$ of T_i.
- Operation O_2: Attach a star of order $k-1$ of center x by joining x to any vertex u of T_i such that $1 \leq d_{T_i}(u) \leq k-3$.
- Operation O_3: Attach a star of order k by joining one of its leaves to any vertex u of T_i such that $1 \leq d_{T_i}(u) \leq k-3$.

Let $\mathcal{P}$ be the class of pivoted trees.

Lemma 3.4. *If $T \in \mathcal{P}$, then T is not b-critical.*

Proof. Since T is pivoted, we have $b(T) = m(T) - 1$. Let z be any leaf of T. Then it is easy to see that $B(T \setminus z) = b(T)$. So T is not b-critical. □

Lemma 3.5. *If $T \in \mathcal{T}_2 \setminus \mathcal{P}$, then T is b-critical with $b(T) = \Delta(T) + 1$.*

Proof. Since T is in $\mathcal{T}_2$, it is easy to check that $\Delta(T) = m(T) - 1$. Since T is not pivoted, we have $b(T) = m(T)$. Hence, $b(T) = \Delta(T) + 1$. Consider any vertex x in T. By the definition of $\mathcal{T}_2$, we have $m(T \setminus w) \leq m(T) - 1$, and consequently $b(T \setminus w) \leq \Delta(T)$. Thus, T is b-critical. □

Theorem 3.6. *Let T be a tree with $b(T) = \Delta(T) + 1$. Then T is b-critical if and only if $T \in \mathcal{T}_2 \setminus \mathcal{P}$.*

Proof. Let $k = \Delta(T) + 1$. Lemma 3.5 implies the sufficiency. To prove the necessity, let T be a b-critical tree with $b(T) = k$. We first show that T belongs to $\mathcal{T}_2$. Since $b(T) = k = \Delta(T) + 1$, Theorem 2.4 implies that T has a unique b-vertex b_i of color i, for each $i \in \{1, \ldots, k\}$, and that $d_T(b_i) = k - 1$. Let $B = \{b_1, b_2, \ldots, b_k\}$. For each i, let $S_i = T[N[b_i]]$. Then S_i is a star of order k. Root T at b_1 and assume without loss of generality that $\text{dist}(b_1, b_2) \leq \text{dist}(b_1, b_3) \leq \cdots \leq \text{dist}(b_1, b_k)$. Let $T_1 = S_1$. For $i = 2, \ldots, k$, let T_i be the subgraph of T induced by $V(S_1) \cup \cdots \cup V(S_i)$. Assume that $i \leq k - 1$. Let $r \in \{1, \ldots, i\}$ be such that $\text{dist}(b_r, b_{i+1}) = \min\{\text{dist}(b_s, b_{i+1}) \mid 1 \leq s \leq i\}$. Since T is a tree, there is a unique path P connecting b_r to b_{i+1}. The choice of b_r implies that any internal vertex of P is not a b-vertex.

Suppose that the length of P is at least 4. Let u be a vertex of P that is not adjacent to b_r or b_{i+1}. The choice of b_r and b_{i+1} implies that u has no neighbor in B, so $b(T \setminus u) \geq b(T)$, a contradiction.

Now suppose that P has length 3. Let $P = b_r$-u-v-b_{i+1}. Then u and v are not b-vertices and we have $u \in V(T_i)$ and $v \notin V(T_i)$. Thus T_{i+1} is obtained from T_i by the third operation applied with star S_{i+1}.

Now suppose that P has length 2. Let $P = b_r$-u-b_{i+1}. Then u is not a b-vertex, and $u \in V(T_i)$. Thus T_{i+1} is obtained from T_i by the second operation applied to star $S_{i+1} \setminus \{u\}$.

Finally, suppose that P has length 1, that is, b_r is adjacent to b_{i+1}. Then T_{i+1} is obtained from T_i by the first operation applied to star $S_{i+1} \setminus \{b_r\}$.

At the end of the procedure, we have $T = T_k$, so T is obtained after $k - 1$ steps by one of the three operations O_1, O_2 or O_3, from a star of order k. Thus $T \in \mathcal{T}_2$. By Lemma 3.4, $T \notin \mathcal{P}$. This completes the proof. □

We can now summarize our results as follows.

Theorem 3.7. *A tree is b-critical if and only if it belongs to* $\mathcal{T}_1 \cup \mathcal{T}_2 \setminus \mathcal{P}$.

Acknowledgments

We thank the referee for many helpful comments.

This work was supported by Programmes Nationaux de Recherche: Code 8/u09/510.

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Received: May 9, 2011.
Revised: March 8, 2012.
Accepted: September 18, 2012.