MECHANICS AND CONTROL
Vol. 30 No. 4 2011

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# ORTHOTROPIC YIELD CRITERIA IN A MATERIAL MODEL FOR TIMBER STRUCTURES

*SUMMARY*

*The continuum structural model for the failure analysis of timber structures in the plane stress state is discussed in the paper. Constitutive relations are established in the framework of the mathematical multi-surface elasto-plasticity theory with three orthotropic strength criteria that have been incorporated in the model as the plasticity conditions. The invariant form of the criteria is given based on the representation theory of scalar-valued functions of the orthotropic invariants. The model is implemented into a commercial finite element code by means of user-defined subroutines. Introductory implementation tests of the proposed numerical algorithm are presented in the paper.*

**Keywords**: *structural mechanics, timber structures, biaxial loading, orthotropic material, multi-surface plasticity, finite element method*

*ORTOTROPOWE WARUNKI PLASTYCZNOŚCI W MATERIAŁOWYM MODELU DO KONSTRUKCJI DREWNIANYCH*

*W pracy omawiane są podstawowe zależności konstytutywne materiałowego modelu do analizy zniszczenia w konstrukcjach drewnianych w płaskim stanie naprężenia. Zależności konstytutywne sformułowano w ramach matematycznej teorii sprężysto-plastyczności z trzema ortotropowymi kryteriami wytrzymałościowymi, które wprowadzono do modelu jako warunki plastyczności. Podano postać niezmienniczą kryteriów na podstawie teorii reprezentacji skalarnych funkcji ortotropowych argumentów tensorowych. Model zaimplementowano do komercyjnego pakietu MES jako tzw. program użytkownika. Zaprezentowano wstępne testy implementacji proponowanego algorytmu numerycznego.*

**Słowa kluczowe**: *mechanika konstrukcji, konstrukcje drewniane, obciążenie dwuosiowe, materiał ortotropowy, plastyczność wielopowierzchniowa, metoda elementów skończonych*

## 1. INTRODUCTION

Timber is often a sustainable alternative to other construction materials like concrete and steel although it has distinct directional mechanical characteristics with large ratios of Young's modulus or the strength between the values parallel and transverse to the grain direction. For uniaxial loading states the behaviour and characteristics of timber and its engineered products as a structural material are known in great detail with the well-known equation developed from tests to quantify the angle of a load effect in the from

$$\frac{1}{N}=\frac{(\sin\varphi)^2}{Q}+\frac{(\cos\varphi)^2}{P} \tag{1}$$

where:

- $N$ – mechanical characteristic at an angle of load to grain of $\varphi$ degrees,
- $P$ and $Q$ – mechanical characteristic when loaded at zero and 90 degrees, respectively.

For the biaxial behaviour the research necessity remains comprehensive. The general effect of multiaxial states of stress on strength was hardly known until the mid 1980s when some experiments under mixed loading conditions were preformed including a unique testing device (Eberhardsteiner 2002) which in combination with a contactless optical in-plane deformation measurement system helped to build a basis for constitutive modeling of orthotropic timber and its products behaviour. The material can be assumed homogeneous but obviously an anisotropic. Although its constitutive behaviour is subjected to various influencing factors, mostly resulting from its complicated and organic microstructure, for the viable analysis and the prediction of the global structural behaviour, a macroscopic phenomenological approach is required. A number of constitutive models that are relied on of the Finite Element Method are widely used in the last decades with a different degree of complexity and idealizations – from initial models with simply linear elastic behaviour to recent advanced models with non-linear post peak behaviour although timber structures are usually designed using linear-elastic analysis for the calculation of both the internal forces and the ultimate load.

Some basic constitutive relations of a continuum structural model for the failure analysis of timber structures in the plane stress state are discussed in the paper. Constitutive relations are established in the framework of the mathematical multi-surface elastoplasticity theory. The orthotropic

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strength criteria that were proposed by Geniev (1981) and that were later discussed by Małyszko (Geniev and Małyszko 2002) have been incorporated in the model as the plasticity conditions. The possibility of the model implementation into a commercial finite element code at the integration point level by means of user-defined subroutines is also of interest within a framework of an incremental-iterative algorithm. An implementation test of the proposed numerical algorithm for an anisotropic continuum is presented in the paper.

The discussion has been focused on the formulation of failure criteria in structural mechanics of a timber material based on the representation theory of orthotropic tensor functions. In the paper this theory is used for the invariant formulation of the criteria. Since the failure criteria are scalar valued functions of the stress tensor then their invariant representations are dependent only on orthotropic invariants of the stress tensors.

## 2. ORTHOTROPIC CRITERIA WITH DIFFERENT FAILURE MECHANISMS

The first anisotropic failure condition was given by Mises in 1928 introducing altogether the concept of the plastic potential (Mises 1928). The anisotropic Mises criterion had the form of a quadratic function of the Cartesian components of the symmetric stress tensor. In the same year Burzyński proposed the hypothesis of material effort for isotropic bodies in his doctoral dissertation (Burzyński 1928). His hypothesis surprises by its clear energy-based interpretation, variety of classes of materials it can be applied to and simplicity in formulation of the failure criterion, which can be determined only in terms of limit stresses under simple loads: uniaxial tension, compression and pure shear. However, it seems to be almost forgotten, especially abroad Poland. Extension of the hypothesis accounting for anisotropy is even less known (see Szeptyński 2011 for more information). Both papers were published a few decades before other similar propositions and presented a basic and fruitful idea for the future research given mostly by Hill (1948), Olszak and Urbanowski (1956), Hoffman (1967), Tsai and Wu (1971) and Rychlewski (1984) among others.

The Mises anisotropic yield condition was successfully specialized by Hill for the case of orthotropic polycrystalline metals. The Hill orthotropic yield condition is dependent on six material parameters and implies the same yield stress in tension and compression. Hoffman generalized the Hill limit condition attempting to formulate a fracture condition for orthotropic brittle materials with the different tensile and compressive strengths. Nowadays, the most popular anisotropic failure criteria with the field of composite structural design have the general form of the tensor polynomial failure function of stress tensor and symmetric failure tensors. In the literature, the proposed failure criteria have different polynomial orders or are distinct in the way of determining the Cartesian components of failure tensors but the practical considerations are often limited to the quadratic function and to special types of anisotropic materials, such as orthotropic or transversely isotropic ones.

The more detailed formulation of orthotropic strength criteria with different failure mechanisms that are presented here have been discussed in the papers (Geniev *at al.* 1993, Geniev and Małyszko 2002). Three basic failure mechanisms were distinguished in the plane stress state, such as failure caused by tensile stresses, compressive stresses and shearing stresses. In order to capture orthotropic behaviour, an isotropic maximum principal stress criterion of Rankine was extended to model the anisotropic tension and compression failure. The orthotropic generalization of the Coulomb-Mohr shear failure criterion and the Tresca criterion as its special case were also demonstrated. The conditions at failure were described in terms of traction components of the traction vector $\mathbf{t}(\mathbf{n}) = \boldsymbol{\sigma}\mathbf{n}$ acting on a material plane determined by the unit normal vector $\mathbf{n} = (n_1, n_2, n_3)$ and of distribution functions specifying the directional variation of the material strengths. The failure criteria are represented by three independent analytical expressions, each being the condition of limit equilibrium of the material with the determination of the critical plane direction, along which failure under a complex stress state would occur.

### 2.1. The orthotropic Rankine-type strength criteria

The condition at the tension failure with the normal component of the traction vector $\mathbf{t}^n = (\mathbf{n} \cdot \boldsymbol{\sigma}\mathbf{n})\mathbf{n}$ may be described by the maximization procedure

$$\underbrace{\max}_{n}\left[\mathbf{t}^n \cdot \mathbf{n} - Y_t(\mathbf{n})\right] = 0 \quad \text{subjected to the constraint } \mathbf{n} \cdot \mathbf{n} = 1 \tag{2}$$

where the strength function $Y_t(\mathbf{n})$ has the following form in the reference system of the orthotropy axes

$$Y_t(\mathbf{n}) = Y_{t1}(n_1)^2 + Y_{t2}(n_2)^2 + Y_{t3}(n_3)^2 \tag{3}$$

with three positive uniaxial strength parameters $Y_{t1}, Y_{t2}$ and $Y_{t3}$ that are obtained from the three tensile tests in the direction of the first, second and third axis of orthotropy, respectively.

After applying the maximization procedure (2) with the strength distribution function (3) one can obtain the analytical form of the Rankine-type strength criterion from which we have the following expression for the directional strength $Y_{tn}$ in the uniaxial tension test in arbitrary direction **n** when referred to axes of orthotropy

$$\frac{1}{Y_{tn}}=\frac{(n_1)^2}{Y_{t1}}+\frac{(n_2)^2}{Y_{t2}}+\frac{(n_3)^2}{Y_{t3}} \tag{4}$$

That expression is the same as the expression of the equation (1) from the experimental tests.

In the case of the plane state of stresses the criterion for the tension failure mechanism (2) has the following form

$$(Y_{t1}-\sigma_{11})(Y_{t2}-\sigma_{22})-\tau_{12}^2=0 \tag{5}$$

when referred to the axes of orthotropy and

$$\begin{gathered}\left(\frac{\cos^2\varphi}{Y_{t1}}+\frac{\sin^2\varphi}{Y_{t2}}\right)\sigma_1-\frac{\sigma_1\sigma_2}{Y_{t1}Y_{t2}}+\\+\left(\frac{\cos^2\varphi}{Y_{t2}}+\frac{\sin^2\varphi}{Y_{t1}}\right)\sigma_2=1\end{gathered} \tag{6}$$

when referred to the axes of principal stresses. The angle $\varphi$ denotes the angle between the first principal stress axis ($\sigma_1$) and the first axis of orthotropy.

The formulation of the strength criterion based on the compression failure mechanism can be obtained using the similar solution as in equations (2) and (3) if the tensile strengths are replaced by the compressive strengths with positive values, i.e. by the strengths $-Y_{c1}, -Y_{c2}$ and $-Y_{c2}$. In the plane state of stresses the criterion for the compression failure mechanism has the following form

$$(Y_{c1}+\sigma_{11})(Y_{c2}+\sigma_{22})-\tau_{12}^2=0 \tag{7}$$

when referred to the axes of orthotropy and

$$\begin{gathered}\left(\frac{\cos^2\varphi}{Y_{c1}}+\frac{\sin^2\varphi}{Y_{c2}}\right)\sigma_1+\frac{\sigma_1\sigma_2}{Y_{t1}Y_{t2}}+\\+\left(\frac{\cos^2\varphi}{Y_{c2}}+\frac{\sin^2\varphi}{Y_{c1}}\right)\sigma_2=-1\end{gathered} \tag{8}$$

when referred to the axes of principal stresses.

For numerical purposes, if the frame of reference is coincided with the principal axes of orthotropy, the tension and compression criterion (5) and (7) may be written in the following matrix equation

$$\frac{1}{2}\boldsymbol{\sigma}^T\mathbf{P}\boldsymbol{\sigma}+\boldsymbol{\sigma}^T\mathbf{p}-1=0 \tag{9}$$

where the matrix **P** and **p** have then the form

$$\begin{gathered}\mathbf{p}_t=\begin{bmatrix}\frac{1}{Y_{t1}}\\ \frac{1}{Y_{t2}}\\ 0\end{bmatrix},\quad \mathbf{p}_c=\begin{bmatrix}-\frac{1}{Y_{c1}}\\ -\frac{1}{Y_{c2}}\\ 0\end{bmatrix}\\ \mathbf{P}_\Delta=\begin{bmatrix}0 & \frac{-1}{Y_{\Delta1}Y_{\Delta2}} & 0\\ \frac{-1}{Y_{\Delta1}Y_{\Delta2}} & 0 & 0\\ 0 & 0 & \frac{2}{Y_{\Delta1}Y_{\Delta2}}\end{bmatrix}\end{gathered} \tag{10}$$

Here $\Delta=t$ for tension and $\Delta=c$ for compression and we have the same the four positive uniaxial strength parameters $Y_{\Delta1}$ and $Y_{\Delta2}$ that are obtained from the two tensile tests and two compressive tests in the direction of the first and second axis of orthotropy, respectively.

### 2.2. The orthotropic Mohr-Coulomb-type strength criteria

The condition of the shear failure can be described by the following maximization procedure

$$\begin{gathered}\underbrace{\max}_{n}\left[\tau+\mu\sigma-C(\mathbf{n})\right]=0\\ \text{subjected to the constraint } \mathbf{n}\cdot\mathbf{n}=1\end{gathered} \tag{11}$$

Here $\tau$ and $\sigma$ are shear and normal stresses on the material plane **n**. The cohesion $C(\mathbf{n})$ is orientation-dependent and at the same time the coefficient of internal friction $\mu$ is assumed to be orientation-independent. The normal stress is $\sigma = |\mathbf{t}^n|$ and the shear stress is $\tau = |\mathbf{t}^s|$, where the tangential component of the traction vector is $\mathbf{t}^s=(\mathbf{1}-\mathbf{n}\otimes\mathbf{n})\boldsymbol{\sigma}\mathbf{n}$ with **1** denoting the unit tensor. When referred to the principal material axes of the orthotropic medium, the variation of the strength parameter $C(\mathbf{n})$ is described similar to the equation (3) by the following distribution function

$$C(\mathbf{n})=C_{11}(n_1)^2+C_{22}(n_2)^2+C_{33}(n_3)^2 \tag{12}$$

where the shear strength parameters $C_{11}$, $C_{22}$ and $C_{33}$ are obtained from the tests with the predetermined shear failure plane, i.e. in the $C_{11}$– test, the normal to the failure plane is predetermined in the direction of the first axis of orthotropy.

In the case of the plane state of stresses the criterion for the shear failure mechanism (11) has the following form

$$(\sigma_{11}-\sigma_{22})^2=4(C_{11}+\tau_{12})(C_{22}-\tau_{12}) \tag{13}$$

when referred to the axes of orthotropy and if the parameter of internal friction μ equals to zero. In that case it is Tresca-type strength criterion. In the matrix equation (9) the matrix **P** and **p** of the orthotropic Mohr-Coulomb-type strength criterion have the form

$$\mathbf{p} = \begin{bmatrix} \dfrac{\mu}{C_{11}} \\ \dfrac{\mu}{C_{22}} \\ \dfrac{(C_{11}-C_{22})}{C_{11}C_{22}}\operatorname{sign}(\tau_{12}) \end{bmatrix}$$
$$\mathbf{P} = \begin{bmatrix} \dfrac{1}{2C_{11}C_{22}} & \dfrac{-(1+2\mu^2)}{2C_{11}C_{22}} & 0 \\ \dfrac{-(1+2\mu^2)}{2C_{11}C_{22}} & \dfrac{1}{2C_{11}C_{22}} & 0 \\ 0 & 0 & \dfrac{2(1+\mu^2)}{C_{11}C_{22}} \end{bmatrix} \tag{14}$$

from which the Tresca-type criterion (13) can be recovered if the parameter of internal friction μ equals to zero.

## 3. INVARIANT FORMS OF THE CRITERIA IN THE PLANE STATE OF STRESS

The theory of tensor functions together with the theorems on their representations has been recognized to be an efficient mathematical tool for the formulation of constitutive relationships with both the desirable analytical clarity and required generality. It also allows accounting in a straightforward manner the invariance requirements of the principle of the space isotropy and the material symmetries so that the orientation of the material in the space has no effect on its constitutive relation. Using this theory with Boehler's results (Boehler *et al.* 1987), we can assume that each of the orthotropic criteria is a particular case of the more general scalar-valued orthotropic function of three invariants $tr\mathbf{M}_1\boldsymbol{\sigma}$, $tr\mathbf{M}_2\boldsymbol{\sigma}$, $tr\boldsymbol{\sigma}^2$ of the following form

$$f\left(tr\mathbf{M}_1\boldsymbol{\sigma}, tr\mathbf{M}_2\boldsymbol{\sigma}, tr\boldsymbol{\sigma}^2\right) - 1 = 0 \tag{15}$$

where $\boldsymbol{\sigma}$ is the symmetric plane stress tensor $(\boldsymbol{\sigma} \in T_2^s, \dim T_2^s = 3)$ and **M** are the parametric (structural) tensors defined as the dyad of the vectors $\mathbf{M}_\alpha$ by

$$\mathbf{M}_1 = \mathbf{m}_1 \otimes \mathbf{m}_1, \quad \mathbf{M}_2 = \mathbf{m}_2 \otimes \mathbf{m}_2 \tag{16}$$

The unit vectors $\mathbf{m}_\alpha$ are the privileged directions of the orthotropic material – axes of orthotropy. The symbol "*tr*" denotes the trace of a second order tensor $(tr\mathbf{AB} \equiv tr(\mathbf{AB}))$. The invariants are very useful for interpretation of the failure surface in any coordinate systems of the plane stress tensor that are different from the principal axes of orthotropy.

Following the papers (Mackenzie-Helnwein *et al.* 2003, Jemioło and Małyszko 2008) we can choose another set of invariants $K_p$ in the form

$$K_1 = tr\mathbf{M}_1\boldsymbol{\sigma}, \quad K_2 = tr\mathbf{M}_2\boldsymbol{\sigma}, \quad K_3 = 2tr\mathbf{M}_1\boldsymbol{\sigma}\mathbf{M}_2\boldsymbol{\sigma} \tag{17}$$

which is very convenient because in the $\{\mathbf{m}_\alpha\}$ axes the invariants are

$$K_1 = \sigma_{11}, \quad K_2 = \sigma_{22}, \quad K_3 = \left(\sqrt{2}\sigma_{12}\right)^2 \tag{18}$$

Using the invariants (17) the Rankine-type criteria (5) and (7) may be written in the following invariant form

$$a_\alpha K_\alpha + b_{\alpha\beta} K_\alpha K_\beta + cK_3 - 1 = 0 \tag{19}$$

where α, β = 1.2 and $b_{11} = b_{22} = 0$.The other material constants are defined for a tension regime as

$$b_{12} = -\frac{1}{2Y_{t1}Y_{t2}}, \quad a_\alpha = \frac{1}{Y_{t\alpha}}, \quad c = \frac{1}{2Y_{t1}Y_{t2}} \tag{20}$$

and for compression as

$$b_{12} = -\frac{1}{2Y_{c1}Y_{c2}}, \quad a_\alpha = -\frac{1}{Y_{c\alpha}}, \quad c = \frac{1}{2Y_{c1}Y_{c2}} \tag{21}$$

Using the invariants (17) the Mohr-Coulomb-type criterion can be written in the following invariant form

$$a_\alpha K_\alpha + b_{\alpha\beta} K_\alpha K_\beta + cK_3 + d\sqrt{K_3} - 1 = 0 \tag{22}$$

where the material constants are defined as

$$a_1 = \frac{\mu}{C_{11}}, a_2 = \frac{\mu}{C_{22}}, b_{12} = -\frac{1+2\mu^2}{4C_{11}C_{22}}, c = \frac{1+\mu^2}{2C_{11}C_{22}} \tag{23}$$

$$b_{11} = b_{22} = \frac{1}{4C_{11}C_{22}}, d = \frac{\sqrt{2}}{2}\left(\frac{1}{C_{22}} - \frac{1}{C_{11}}\right) \tag{24}$$

The parameter of internal friction μ in the expression (23) equals to zero for the criterion of Tresca.

The invariants can be also expressed using the second order tensor **N** and fourth order tensor **M** as follows

$$\begin{gathered} K_1 = tr\mathbf{M}_1\boldsymbol{\sigma} = \boldsymbol{\sigma} : \mathbf{M}_1, \ K_2 = tr\mathbf{M}_2\boldsymbol{\sigma} = \boldsymbol{\sigma} : \mathbf{M}_2 \\ K_3 = 2tr\mathbf{M}_1\boldsymbol{\sigma}\mathbf{M}_2\boldsymbol{\sigma} = 2[\boldsymbol{\sigma} : \mathbf{N}]^2 = \tfrac{1}{2}\boldsymbol{\sigma} : \mathbb{M} : \boldsymbol{\sigma} \end{gathered} \tag{25}$$

where a double contraction of two tensors is denoted by two dots and

$$\mathbf{N} = \frac{1}{2}\left(\mathbf{m}_1 \otimes \mathbf{m}_2 + \mathbf{m}_2 \otimes \mathbf{m}_1\right), \mathbf{M} = 4\mathbf{N} \otimes \mathbf{N} \tag{26}$$

The criteria can be then written in the following invariant form

$$\frac{1}{2}\boldsymbol{\sigma} : \mathbf{P} : \boldsymbol{\sigma} + \mathbf{p} : \boldsymbol{\sigma} - 1 = 0 \tag{27}$$

where **p** is the symmetric tensor function of the second order

$$\mathbf{p} = a_1\mathbf{M}_1 + a_2\mathbf{M}_2 \tag{28}$$

and **P** is the double symmetric tensor function of the fourth order

$$\begin{gathered}\mathbf{P} = 2b_{11}\mathbf{M}_1 \otimes \mathbf{M}_1 + 2b_{22}\mathbf{M}_2 \otimes \mathbf{M}_2 + \\ +2b_{12}\left(\mathbf{M}_1 \otimes \mathbf{M}_2 + \mathbf{M}_2 \otimes \mathbf{M}_1\right) + c\mathbf{M}\end{gathered} \tag{29}$$

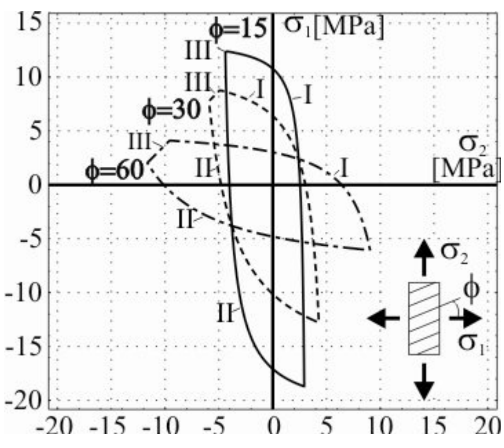
Several criteria proposed in literature for orthotropic materials are special cases of the quadratic limit surface (27) including the Hoffman criterion (Hoffman 1967) and an elliptic failure surface according to Tsai and Wu (Tsai and Wu 1971) and also the criteria discussed recently in (Małyszk*o et al.* 2009). It should be noted that a phenomenological single-surface model may give an insufficient description of the mechanical behaviour. It does not permit easy identification of failure modes and thus renders the description of different post-failure mechanisms very difficult. At least two failure criteria should be taken into consideration, the one for the compression regime and the second for tension regime, each of them may be of the form (27) as proposed in (Jemioło and Małyszko 2008) where the failure criterion for an orthotropic materials in the spatial stress state is represented by two quadratic functions of the six invariants of the stress tensor and parametric tensors. It may more accurately describe the failure data distribution then classical limit surfaces, although it may include fifteen independent material parameters for the description of failure surfaces. In the case of clear spruce wood as reported in (Mackenzie-Helnwein *et al.* 2003), the experimentally observed mechanical behaviour has to be described by means of a multi-surface model that consists of four surfaces representing four basic failure modes. On the other hand, the concept of a smooth single-surface description seems to be attractive from numerical point of view and also allows for modelling of different inelastic behaviour by a changing size, shape and location of a quadratic state function according to the form (27) in the orthotropic stress space. The return of the authors (Mackenzie-Helnwein *et al.* 2003) to the initial concept of a smooth single-surface description is reported in (Mackenzie-Helnwein *et al.* 2005).

Figure 1 shows the graphical representation of the strength criteria. Predictions according to the hyperbolic lines I and II correspond to the Rankine-type criteria (6 and 8) for the tension and the compression failure, respectively. Straight lines III are drawn according to the shear failure criterion of Tresca (13). Different representations of the criteria are found according to different values of the angle between the first axis of orthotropy and the axis of the first principal stress φ (15°, 30°, 60°).

## 4. CONSTITUTIVE RELATIONS AND IMPLEMENTATION OF THE MODEL

It should be emphasized that timber structures are usually designed using linear-elastic analysis for the calculation of both the internal forces and the ultimate load. This is the only design method currently recommended in the Euro-pean timber norm, whilst for other materials such as reinforced concrete, steel and composite steel–concrete

a)




b)

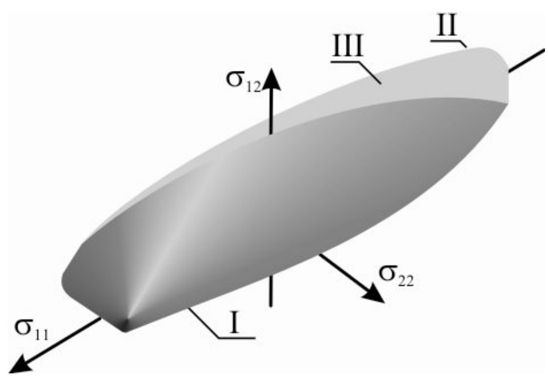



**Fig. 1.** Contours of strength criteria in the biaxial stress state as proposed in (Geniev and Małyszko 2002): a) in the principal stress space; b) in the axes of orthotropy $Y_{t1} = 14.4, Y_{t2} = 2.4, Y_{c1} = 22.8, Y_{c2} = 3.8, C_{11} = 19.2, C_{22} = 6.4$ MPa

structures plastic analysis is generally used to calculate the strength capacity even when an elastic analysis is used to evaluate internal forces. Unreinforced timber elements loaded in bending usually fail in a brittle manner whilst some plastic behaviour can be exhibited in compression or when subjected to coupled bending and compression. Both cases can be described by the mathematical theory of elasto-plasticity. This theory proved to be helpful not only for materials with ductility but also for brittle materials if modern hardening/softening evolution laws are used.

Numerical simulations of wooden structures using own models are presented in (Schmidt and Kaliske 2009) in the framework of the large deformation theory for the three-dimensional continuum. The model presented here is based on the theory of small displacements and small deformations. The more detailed discussion of the constitutive equations of the similar models and their numerical implementation into commercial FEM system has been recently presented in (Małyszko *et al.* 2010). The elastic-plastic orthotropic material is considered assuming an additive decomposition of the strain tensor into the elastic part $\varepsilon^e$ and the plastic part $\varepsilon^p$. The elastic part is defined by the orthotropic Hooke's law. The plastic part of the strain tensor is defined by a flow rule associated with the yield function given by the plasticity (failure) criterion written in the following dimensionless form:

$$f(\boldsymbol{\sigma},\alpha) = \tfrac{1}{2}\boldsymbol{\sigma} : \mathbf{P} : \boldsymbol{\sigma} + \mathbf{p} : \boldsymbol{\sigma} - K(\alpha) = 0 \tag{30}$$

$K(\alpha)$ is a given function with the real value from a closed interval [0, 1] that describes the type of hardening/softening and $\alpha$ is an internal hardening variable, hence $\mathbf{P}$ and $\mathbf{p}$ are symmetric tensor functions of the fourth and second order, respectively.

The flow rule defines the sign (direction) of plastic-strain increment in the following form:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\left.\frac{\partial f(\boldsymbol{\sigma},\alpha)}{\partial \boldsymbol{\sigma}}\right|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}^T} = \dot{\gamma}(\mathbf{P} : \boldsymbol{\sigma} + \mathbf{p}) \equiv \dot{\gamma}\mathbf{r} \tag{31}$$

where $\dot{\gamma} > 0$ is a plastic multiplier. After applying differentiation to orthotropic Hooke's elastic law with the respect to the time and after substituting (31) we obtain:

$$\dot{\boldsymbol{\sigma}} = \mathbf{C} : (\dot{\boldsymbol{\varepsilon}} - \dot{\gamma}\mathbf{r}) \equiv \mathbf{C}_{ep} : \dot{\boldsymbol{\varepsilon}} \tag{32}$$

where $\mathbf{C}$ has the interpretation of the positive definite, doubly symmetric stiffness tensor of order four. The operator $\mathbf{C}_{ep}$ can be calculated after the parameter $\dot{\gamma}$ is known. Assuming that

$$\dot{\alpha} = \dot{\gamma}\sqrt{(\mathbf{r} : \mathbf{r})} \equiv \dot{\gamma}\|\mathbf{r}\| \tag{33}$$

one can compute from the consistency condition

$$\dot{\gamma}\dot{f}(\boldsymbol{\sigma},\alpha) = 0, \quad \dot{\gamma} > 0 \tag{34}$$

the plastic multiplier

$$\dot{\gamma} = \frac{\langle \mathbf{r} : \mathbf{C} : \dot{\boldsymbol{\varepsilon}} \rangle}{\mathbf{r} : \mathbf{C} : \mathbf{r} + \partial_{\alpha} K \|\mathbf{r}\|} \tag{35}$$

and the operator $\mathbf{C}_{ep}$ in the following form

$$\mathbf{C}_{ep} = \mathbf{C} - \frac{\mathbf{C} : \mathbf{r} \otimes \mathbf{C} : \mathbf{r}}{\mathbf{r} : \mathbf{C} : \mathbf{r} + \partial_{\alpha} K \|\mathbf{r}\|} \tag{36}$$

According to the above relations, the model has been coded in the programming language FORTRAN and next implemented into a commercial finite element code DIANA (Diana 2009). There are standard Newton-Raphson and Riks algorithms for the solving nonlinear equilibrium equations in the program. Because of this, implementation of the model into a finite element code is done at the integration point level by means of user-defined subroutine *usrmat*. The subroutine lets the user specify a general nonlinear material behavior by updating the state variables over the equilibrium step $n \to n+1$ within a framework of an incremental-iterative algorithm of finite element method. Both the return-mapping algorithm allowing the stresses to be returned to the yield surface and a consistent tangent stiffness operator have been coded. The use of a commercial code as the development environment for computational materials research has some advantages. The new models can be applied to a variety of structural problems, not only to single element tests but also to simulate physical experiments, using different elements types and using standard features of the program such as advanced solution procedures, for instance indirect displacement control with full Newton-Raphson. Also, the use of the post-processing capabilities of the program is an advantage. The single element test of the implementation of the model is presented below.

It should be noted that three yield criteria are combined in the model into a composite yield surface and the intersection of different yield surfaces defines corners that require special attention in a numerical algorithm according to Koiter's generalization. The implementation test is presented here as introductory examples in the context of hardening/softening evolution laws which may significant affect the implementation algorithm. In the test the most important step of the implementation is based on the single quadratic equation of the algorithmic value $\Delta\gamma$ of the variable $\dot{\gamma}$. More general approach is based on the elastic predictor – plastic corrector algorithm and the implicit Euler backward algorithm as described in (Simo and Hughes 1998). Although that algorithm is applied there to the isotropic material it can be used to small orthotropic plasticity without changes in the procedure.

### 4.1. Single-element test

For the testing of the directional mechanical response of the 2-D model with *Diana* several tests have been performed.

**Table 1**

Inelastic material parameters of the model in the test [MPa]

| Compression | | Tension | | Shear | |
|---|---|---|---|---|---|
| $Y_{c1}$ | $Y_{c2}$ | $Y_{t1}$ | $Y_{t2}$ | $C_{11}$ | $C_{22}$ |
| 22.8 | 3.8 | 14.4 | 2.4 | 19.2 | 6.4 |

For the presented single-element test under displacement control the following elastic material parameters are used: $G_{12}$ = 0.9 GPa, ν = 0.1, $E_1$ = 11.0 GPa, $E_2$ = 8.5 GPa. The inelastic parameters are shown in Table 1 for the Tresca-type criterion.

Figure 2 shows numerical responses of the model for perfect plasticity, i.e. $K(\alpha)$ = 1 for all three criteria. Three different responses are found according to different values of the angle between the first axis of orthotropy and the axis of the displacement loading θ ($0^o$, $45^o$, $90^o$).

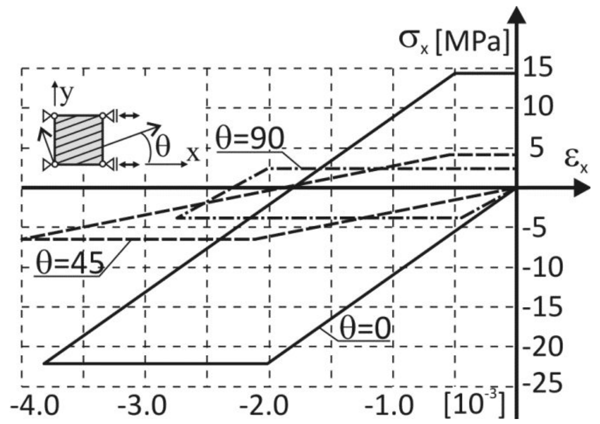



**Fig. 2.** Stress-strain response in the element test under displacement control for different values of the angle between the first axis of orthotropy and the axis of the displacement loading θ ($0^o$, $45^o$, $90^o$)

The anticipated constitutive behaviour is exactly reproduced. For the uniaxial loading along the first axis of orthotropy, i.e. θ = $0^o$, the obtained yield tensile strength is 14.4 MPa and the yield compressive strength is 22.2 MPa which means that the criterion of Tresca is active instead of the Rankine criterion for which the yield strength is 22.8 MPa (see Tab. 1). For the uniaxial loading along the second axis of orthotropy, i.e. θ = $90^o$, the obtained yield tensile strength is 2.4 MPa and the yield compressive strength is 3.8 MPa which means that in this case the criterion of Rankine is active both for tension and compression regime.

## 5. CONCLUSIONS

A new constitutive model for the analysis of timber structures in biaxial states of plane stress has been presented. The macroscopic phenomenological multi-surface plasticity model consists of three surfaces representing three failure modes and resulting from the orthotropic strength criteria that were proposed by Geniev (Geniev 1981) and were later discussed by Małyszko (Geniev and Małyszko 2002). The criteria have been incorporated in the model as the plasticity conditions. The discussion has been focused on the invariant formulation of the failure criteria based on the representation theory of orthotropic tensor functions. The analytical forms of the criteria are presented both in the system of orthotropy axes and in the system coaxial with the principal stress axes. Moreover, invariant forms of the criteria that are dependent only on orthotropic invariants of the stress tensors are also presented. The model can be used to describe failure locations in stress space and to fit to available biaxial test data. Additionally, the multi-surface model enables the identification of the relevant macroscopic failure modes. The separated description of the three modes easily enables the modeling of their respective post-failure behavior.

The possibility of the model implementation into a finite element code has been also of interest. The commercial version of Diana (Diana 2009) with the so-called user supplied subroutine mechanism has been used as the development environment for computational materials research. The model has been implemented at the integration point level by means of user-defined subroutines within a framework of an incremental-iterative algorithm. An implementation test of the proposed model of orthotropic continuum that confirmed correctness of the implementation is presented in the paper for the case of perfect plasticity. The implementation of the other hardening/softening evolution laws requires some more additional works and is intended to discuss in a separated paper.

**Acknowledgments**

*The author gratefully acknowledges the financial support of the Polish Ministry of Science and Higher Education as a personal research project N506 N 396435.*

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