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Heorhiy SULYM*, Iaroslav PASTERNAK*

MODELLING OF STRESS STATE OF ELASTIC MEDIUM CONTAINING PERFECTLY AND IMPERFECTLY BONDED THIN INCLUSIONS AND OVERLAYS

SUMMARY

This study considers modelling of two-dimensional stress state of solids containing thin elastic inclusions. In modeling the coupling principle for continua of different dimension is utilized. Basing on the model of inclusion under the perfect contact three other models of imperfect contact are developed. The simplest one is a model of thin inclusion, which is completely delaminated at certain segments. Two other models take into account a smooth contact between inclusion and a solid, and also a contact with friction. The developed models are easy to introduce into the used boundary element approach. The model of inclusion, completely debonded at one face, is also used in modeling of solids with thin elastic overlays or stringers.

Keywords: *thin inclusion, delaminating, overlay, imperfect contact, friction, boundary element method*

MODELOWANIE STANU NAPRĘŻENIA DWUWYMIAROWEGO OŚRODKA SPRĘŻYSTEGO ZAWIERAJĄCEGO DOSKONAŁE I NIEDOSKONAŁE KONTAKTUJĄCE CIENKIE INKLUZJE ORAZ NAKŁADKI

W pracy przy użyciu zasady sprzężenia kontinuów różnowymiarowych rozważane są sposoby matematycznego modelowania zagadnienia płaskiego dla ośrodków sprężystych zawierających cienkie inkluzje. Zasadniczy model matematyczny dla cienkiej inkluzji połączonej w doskonały sposób z tarczą uwzględnia możliwość sprężystego odkształcenia w kierunkach: poprzecznym i wzdłużnym względem osi inkluzji oraz efekty jej zginania. Zostały sprecyzowane trzy szczególne modele niedoskonałego kontaktu inkluzji. Najprostszym z nich jest model cienkiego wtrącenia, które wzdłuż pewnych segmentów jest odseparowane od macierzy. Dwa inne modele uwzględniają gładki kontakt inkluzji z ciałem oraz kontakt z tarciem. Opracowane modele łatwo łączy się ze schematem metody elementów brzegowych. Model cienkiej inkluzji zupełnie odseparowanej wzdłuż jednej strony nadaje się również do badania ośrodków z cienkim wzmocnieniem powierzchniowym (nakładką).

Słowa kluczowe: *cienka inkluzja, odseparowanie, nakładka, kontakt niedoskonały, tarcie, metoda elementów brzegowych*

1. INTRODUCTION

Thin inhomogeneities of material structure are not only the wide-spread type of technological defects (those are cracks, different voids filled with a foreign material etc.), but they are often intentionally introduced into the medium for obtaining new functional properties of materials (those are materials reinforced with fibers, plates, stringers etc.). Due to the imperfect processing of such materials, thin elements introduced in a material can often be in the imperfect mechanical contact (fully or partially debonded) that can cause the wear and even mechanical fracture of the key structural elements, which are produced of these materials. On the other hand, due to the friction the heat is generated on the surfaces of imperfect contact, and consequently mechanical energy dissipates. Thus, these composites can be a good damping materials and absorbers of a sound and mechanical vibrations.

The review of the basic works concerning models of solids' contact is provided in Ref. [10]. The study of thin elements of material structure under the perfect contact with the medium is held in the works, which consider thin stringers and overlays [1]; fibers of the reinforced composite materials [2, 15]; thin lamellar reinforcing elements [5, 6]; methods of calculation of piles in a ground [8]; elastic fastenings of console beams and anchors [14]; approaches concerning strengthening of walls of underground buildings and mines, methods for reinforcement of holes by thin elements [17] etc. Less attention is paid to the questions of imperfect interaction of thin inclusions with a medium. Mainly thin absolutely rigid inclusions with a fully debonded single side are considered under the smooth contact with a medium [1, 3, 12], and also with an account of friction [4, 12]. The elastic inclusions are studied only in a few works, for example, a thin completely debonded at one side flexible inclusion is considered in Ref. [7].

Thus, the construction of different possible models of contact interaction of thin elastic inclusion with a medium, in particular, which are based on the models of inclusion with a perfect mechanical contact, seems to be expedient. As a basic one, the model of a thin elastic inclusion [9] is considered, which is constructed basing on the coupling principle for continua of different dimension [16] and intro-

* Katedra Mechaniki i Informatyki Stosowanej, Politechnika Białostocka, Chair of Technical Mechanics, Lutsk National Technical University, Ukraine; sulym@pb.edu.pl; pasternak@ukrpost.ua

duced into the boundary element method. The constructed models together with the general approach [9] will allow considering a wide range of problems of imperfect interaction of thin elastic inhomogeneities with a medium, which is the first step to the study of wear of structures, produced from the structurally inhomogeneous materials and energy dissipation in them.

2. PROBLEM STATEMENT AND ANALYSIS

For the modeling of solids with thin inhomogeneities, a coupling principle for continua of different dimension is often used [16]. This principle involves the replacement of a thin inclusion with a surface of a discontinuity of stress and displacement fields. Frequently, a median surface of a thin inhomogeneity is chosen as the discontinuity surface. The inclusion is thus removed from consideration as a geometrical object, and it is assumed that inclusion's mechanical influence is reduced to the influence of the above-mentioned discontinuity surface (a line for 2D problems). Thus, according to a discontinuity function method [16] the study of a stress state of a solid (an exterior problem) is reduced to the study of the influence of unknown discontinuity functions and is considered without account of the inclusion's material properties. It is clear that the stress state of the solid depends on these discontinuity functions, the solid's material properties, the geometrical features of the problem, the contact conditions at the thin inhomogeneity interface, and the external load.

On the other hand, owing to a small thickness of the inclusion, the tractions and displacements at the faces of the inclusion are related with each other. Corresponding relations, which include mechanical properties of the inclusion and its thickness, are called the mathematical model of a thin inclusion. This model does not depend on the solid's properties, and it can be considered as an interior problem. There are only three basic requirements for the mathematical model of a thin inclusion [16]: (1) the number of equations should be equal to the number of the unknown discontinuity functions; (2) the model should be simple for further solution of the obtained system of equations; and (3) the model should simulate the essential features of the inclusion's behavior.

Since the coupling principle and a discontinuity function method consider exterior and interior problems independently, several inclusion models, which simulate different features of the inhomogeneity, can be developed for the same exterior problem, and using the same inclusion model, one can solve different exterior problems.

Consider a plane problem of elasticity for an isotropic solid with a boundary Γ containing a thin internal inclusion, which is modeled based on the coupling principle for continua of different dimension by the line Γ_C of field discontinuities. In Ref. [9] according to the approaches [13, 16] the equations of elastic equilibrium of a solid (under the conditions of plane problem of elasticity) are presented in the following form:

– for the collocation point $\mathbf{y}$ placed at a smooth surface Γ of a solid –

$$\begin{aligned}
&\frac{1}{2}u_i(\mathbf{y}) = \\
&= \int_{\Gamma_C^+}\left[U_{ij}(\mathbf{x},\mathbf{y})\Sigma t_j(\mathbf{x}) - T_{ij}(\mathbf{x},\mathbf{y})\Delta u_j(\mathbf{x})\right]d\Gamma(\mathbf{x}) + \\
&+\mathrm{RPV}\int_{\Gamma}U_{ij}(\mathbf{x},\mathbf{y})t_j(\mathbf{x})d\Gamma(\mathbf{x}) - \\
&-\mathrm{CPV}\int_{\Gamma}T_{ij}(\mathbf{x},\mathbf{y})u_j(\mathbf{x})d\Gamma(\mathbf{x});
\end{aligned} \tag{1}$$

– for the collocation point $\mathbf{y}$ placed at a smooth surface Γ_C^+ of field discontinuities line –

$$\begin{aligned}
&F_i^u\left(\mathbf{y},\Sigma t_j,\Delta u_j\right)/2 = \\
&= \mathrm{RPV}\int_{\Gamma_C^+}U_{ij}(\mathbf{x},\mathbf{y})\Sigma t_j(\mathbf{x})d\Gamma(\mathbf{x}) \\
&-\mathrm{CPV}\int_{\Gamma_C^+}T_{ij}(\mathbf{x},\mathbf{y})\Delta u_k(\mathbf{x})d\Gamma(\mathbf{x}) \\
&+\int_{\Gamma}\left[U_{ij}(\mathbf{x},\mathbf{y})t_j(\mathbf{x}) - T_{ij}(\mathbf{x},\mathbf{y})u_j(\mathbf{x})\right]d\Gamma(\mathbf{x}), \\
&F_i^t\left(\mathbf{y},\Sigma t_j,\Delta u_j\right)/2 = \\
&= n_j^+(\mathbf{y})\Big[\mathrm{CPV}\int_{\Gamma_C^+}D_{ijk}(\mathbf{x},\mathbf{y})\Sigma t_k(\mathbf{x})d\Gamma(\mathbf{x}) \\
&-\mathrm{HPV}\int_{\Gamma_C^+}S_{ijk}(\mathbf{x},\mathbf{y})\Delta u_k(\mathbf{x})d\Gamma(\mathbf{x}) \\
&+\int_{\Gamma}\left[D_{ijk}(\mathbf{x},\mathbf{y})t_k(\mathbf{x}) - S_{ijk}(\mathbf{x},\mathbf{y})u_k(\mathbf{x})\right]d\Gamma(\mathbf{x})\Big],
\end{aligned} \tag{1'}$$

and the model of thin elastic inclusion (actually its interaction conditions with a medium) is reduced to the functional dependences

$$\begin{aligned}
&\Sigma u_i(\mathbf{y}) = F_i^u\left(\mathbf{y},\Sigma t_j,\Delta u_j\right), \\
&\Delta t_i(\mathbf{y}) = F_i^t\left(\mathbf{y},\Sigma t_j,\Delta u_j\right) \quad (i,j=1,2)
\end{aligned} \tag{2}$$

Here u_i and t_i are the displacement and traction vectors; $\Delta(\bullet)=(\bullet)^+-(\bullet)^-$, $\Sigma(\bullet)=(\bullet)^++(\bullet)^-$; superscripts «+» and «–» denote values, which correspond to the surfaces Γ_C^+ and Γ_C^- formed with the line Γ_C; n_j is a unit normal vector; RPV stands for the Riemann Principal Value, CPV for the Cauchy Principal Value, and HPV for the Hadamard Principal Value of an integral. Here and further the Einstein summation convention is assumed. The kernels of the integral equations are derived in Refs. [13, 16]:

$$U_{ij}(\mathbf{x},\xi) = C_1\left(C_2\delta_{ij}\ln r - r_i r_j / r^2\right),$$

$$T_{ij}(\mathbf{x},\xi) = \left(C_3/r^2\right)\cdot$$

$$\cdot\left[C_4\left(n_i r_j - n_j r_i\right) + \left(C_4\delta_{ij} + 2r_i r_j / r^2\right) r_k n_k\right],$$

$$D_{ijk}(\mathbf{x},\xi) = C_{ijpm}\frac{\partial U_{pk}(\mathbf{x},\xi)}{\partial \xi_m},$$

$$S_{ijk}(\mathbf{x},\xi) = C_{ijpm}\frac{\partial T_{pk}(\mathbf{x},\xi)}{\partial \xi_m},$$

where:

$$r_i = x_i - \xi_i,$$

$$r = \sqrt{r_k r_k},$$

$$C_1 = -1/\left[2\pi G(1+\kappa)\right],$$

$$C_2 = \kappa,$$

$$C_3 = -1/\left[\pi(1+\kappa)\right],$$

$$C_4 = (\kappa - 1)/2$$

G – shear modulus,
κ – Muskhelishvili constant, which equals $\kappa = (3-\nu)/(1+\nu)$ for plane stress and $\kappa = 3 - 4\nu$ for plane strain;
ν – Poisson ratio;
C_{ijkm} – components of elasticity tensor.

In the case of a perfect mechanical interaction of thin inclusion with a medium in Ref. [9] the explicit relations for the functionals (2) are obtained

$$\Sigma u_i(\mathbf{y}) = \tilde{F}_i^u\left(\mathbf{y}, \Sigma t_j, \Delta u_j\right),$$
$$\Delta t_i(\mathbf{y}) = \tilde{F}_i^t\left(\mathbf{y}, \Sigma t_j, \Delta u_j\right) \quad (i, j = 1, 2) \tag{3}$$

which account the transverse and longitudinal tension and shear, and also bending of the inhomogeneity. Besides, Ref. [9] presents the numerical solution approach, which is based on the dual boundary element method [13]. The conditions of a perfect mechanical interaction (3) allow constructing of corresponding dependences for different cases of imperfect contact between thin inclusion and a medium.

2.1. Inclusion debonded at certain sections

Assume, that thin inclusion is debonded from the surface Γ_C^- at the section $\gamma_C^- \subset \Gamma_C^-$. Thus, there is no mechanical interaction at $\gamma_C^- \subset \Gamma_C^-$. Then at a collocation point $\mathbf{y} \in \gamma_C^-$ the following boundary conditions hold:

$$t_i^-(\mathbf{y}) = 0, \quad u_i^+(\mathbf{y}) = u_i^{\mathrm{i}+}(\mathbf{y}) \quad (i = 1, 2) \tag{4}$$

Here the nonitalic superscript «i» denotes values, which correspond to the inclusion.

With the account of notations $\Sigma t_i = t_i^+ + t_i^-$ and $\Delta t_i = t_i^+ - t_i^-$ the first of the conditions (4) allows obtaining the first equation of the model of thin inclusion debonded at $\gamma_C^- \subset \Gamma_C^-$:

$$F_i^{t-}(\mathbf{y}) \equiv \Delta t_i(\mathbf{y}) = \Sigma t_i(\mathbf{y}) \tag{5}$$

As thin inclusion adjoins one face (Γ_C^+) of the discontinuity surface it is necessary to consider its longitudinal tension and shear, and also bending. For this purpose one can use the second equation of a model (3) of a perfectly bonded inclusion. Due to the small thickness of inclusion, the displacements of the median surface are close to the displacements $u_i^{\mathrm{i}+}$ of the face Γ_C^+, thus one obtains the second equation for debonding at γ_C^- :

$$F_i^{u-}(\mathbf{y}) \equiv \Sigma u_i(\mathbf{y}) = \tilde{F}_i^u(\mathbf{y}) - \Delta u_i(\mathbf{y}) \tag{6}$$

Similarly, if inclusion is debonded from the face Γ_C^+ at the section $\gamma_C^+ \subset \Gamma_C^+$, then at the collocation points $\mathbf{y} \in \gamma_C^+$ the following boundary conditions hold

$$t_i^+(\mathbf{y}) = 0, \quad u_i^-(\mathbf{y}) = u_i^{\mathrm{i}-}(\mathbf{y}) \quad (i = 1, 2) \tag{7}$$

Thus, one can obtain the equation of inclusion debonded at γ_C^+:

$$F_i^{t+}(\mathbf{y}) \equiv \Delta t_i(\mathbf{y}) = -\Sigma t_i(\mathbf{y});$$
$$F_i^{u+}(\mathbf{y}) \equiv \Sigma u_i(\mathbf{y}) = \tilde{F}_i^u(\mathbf{y}) + \Delta u_i(\mathbf{y}) \tag{8}$$

Model (5), (6) of inclusion completely debonded at a face Γ_C^- with the account of additional conditions of the absence of a material in the domain, which borders Γ_C^-, can be used for studying of thin elastic overlays. Thus, for the problem solution it is possible to use the dual equations (1'), or a single equation (1) together with the model (6) of an overlay.

2.2. Thin inclusion under a smooth contact

Now consider that thin inclusion is under a smooth contact (tangential tractions are zero) at the section $\gamma_C^- \subset \Gamma_C^-$ of the discontinuity line Γ_C. At the collocation points $\mathbf{y} \in \gamma_C^-$ the following boundary conditions should be satisfied

$$t_\tau^-(\mathbf{y}) = 0, \quad u_n^\pm(\mathbf{y}) = u_n^{\mathrm{i}\pm}(\mathbf{y}), \quad u_\tau^+(\mathbf{y}) = u_\tau^{\mathrm{i}+}(\mathbf{y}) \tag{9}$$

Proceeding from the first of the conditions (9) and a model (3) of a perfect contact one can obtain

$$\Delta t_\tau(\mathbf{y}) = \Sigma t_\tau(\mathbf{y}), \quad \Delta t_n(\mathbf{y}) = \tilde{F}_n^t(\mathbf{y}) \tag{10}$$

With the account of relations

$$t_i' = \alpha_{ij} t_j, \quad t_i = \alpha_{ji} t_j' \tag{11}$$

where $t_1' = t_n$, $t_2' = t_\tau$, and the components of rotation tensor $\boldsymbol{\alpha}$ equal $\alpha_{11} = n_1$, $\alpha_{12} = n_2$, $\alpha_{21} = -n_2$, $\alpha_{22} = n_1$, from Eq. (10) one receives

$$F_i^{t-n}(\mathbf{y}) \equiv \Delta t_i(\mathbf{y}) = \alpha_{1i}\alpha_{1j}\tilde{F}_j^t(\mathbf{y}) + \alpha_{2i}\alpha_{2j}\Sigma t_j(\mathbf{y}) \tag{12}$$

Similarly with (6), using the second and the third of contact conditions (9) one can obtain

$$\begin{aligned} &\Sigma u_n(\mathbf{y}) = \tilde{F}_n^u(\mathbf{y}) \\ &\Sigma u_\tau(\mathbf{y}) = \tilde{F}_\tau^u(\mathbf{y}) - \Delta u_\tau(\mathbf{y}) \end{aligned} \tag{13}$$

Taking into account the coordinate transformation (11) and identity $\alpha_{ki}\alpha_{kj} = \delta_{ij}$ from Eq. (13) one can receive the second equation of a smooth contact at γ_C^-:

$$F_i^{u-n}(\mathbf{y}) \equiv \Sigma u_i(\mathbf{y}) = \tilde{F}_i^u(\mathbf{y}) - \alpha_{2i}\alpha_{2j}\Delta u_j(\mathbf{y}) \tag{14}$$

If thin inclusion is under a smooth contact at section $\gamma_C^+ \subset \Gamma_C^+$, similarly with (12) and (14) taking into account Eq. (8) at the collocation points $\mathbf{y} \in \gamma_C^+$ the following equations hold

$$\begin{aligned} &F_i^{t+n}(\mathbf{y}) = \alpha_{1i}\alpha_{1j}\tilde{F}_j^t(\mathbf{y}) - \alpha_{2i}\alpha_{2j}\Sigma t_j(\mathbf{y}) \\ &F_i^{u+n}(\mathbf{y}) = \tilde{F}_i^u(\mathbf{y}) + \alpha_{2i}\alpha_{2j}\Delta u_j(\mathbf{y}) \end{aligned} \tag{15}$$

2.3. Thin inclusion under the contact with friction

Consider the case, when besides the normal tractions the forces of friction governed by the Coulomb law exist on the sections of imperfect contact between inclusion and a medium. In the case, when the force of friction is less than the maximal one, the conditions of perfect mechanical contact are satisfied, and Eq. (3) is used. Therefore, only the limit case can be considered.

Assume that inclusion is under the contact with friction at section $\gamma_C^- \subset \Gamma_C^-$ of its face Γ_C^-. Then at the collocation points $\mathbf{y} \in \gamma_C^-$ the following boundary conditions should be satisfied:

$$\begin{aligned} &t_\tau^-(\mathbf{y}) = sft_n^-(\mathbf{y}) \\ &u_n^\pm(\mathbf{y}) = u_n^{i\pm}(\mathbf{y}), \quad u_\tau^+(\mathbf{y}) = u_\tau^{i+}(\mathbf{y}) \end{aligned} \tag{16}$$

where f is a coefficient of a dry friction; the numerical multiplier s, which can get values of 1 or –1, specifies the direction of a friction.

Two last conditions in (16) are the same as corresponding conditions in Eq. (9), therefore, one of the equations of a model of imperfect contact with friction are the same as given by Eq. (14). The first condition in Eq. (16) gives the following relations

$$\begin{aligned} &\Delta t_n(\mathbf{y}) = \tilde{F}_n^t(\mathbf{y}) \\ &\Delta t_\tau(\mathbf{y}) = \Sigma t_\tau - 2t_\tau^- = \Sigma t_\tau - sf\left[\Sigma t_n - \Delta t_n\right] = \\ &= \Sigma t_\tau - sf\left[\Sigma t_n - \tilde{F}_n^t(\mathbf{y})\right] \end{aligned} \tag{17}$$

With the account of dependences (11) this allows to obtain

$$\begin{aligned} &F_i^{t-f}(\mathbf{y}) \equiv \Delta t_i(\mathbf{y}) = (\alpha_{1i} + sf\alpha_{2i})\alpha_{1j}\tilde{F}_j^t(\mathbf{y}) + \\ &+\alpha_{2i}\left(\alpha_{2j} - sf\alpha_{1j}\right)\Sigma t_j(\mathbf{y}) \end{aligned} \tag{18}$$

In the case of imperfect contact with friction at section $\gamma_C^+ \subset \Gamma_C^+$ at the collocation points $\mathbf{y} \in \gamma_C^+$ the following conditions of imperfect contact hold

$$\begin{aligned} &t_\tau^+(\mathbf{y}) = sft_n^+(\mathbf{y}), \quad u_n^\pm(\mathbf{y}) = u_n^{i\pm}(\mathbf{y}), \\ &u_\tau^-(\mathbf{y}) = u_\tau^{i-}(\mathbf{y}) \end{aligned} \tag{19}$$

Therefore, one of the equations of a contact with friction according to (15) is as follows:

$$F_i^{u+f}(\mathbf{y}) = \tilde{F}_i^u(\mathbf{y}) + \alpha_{2i}\alpha_{2j}\Delta u_j(\mathbf{y}) \tag{20}$$

The second equation is constructed basing on the first of conditions (19):

$$\begin{aligned} &\Delta t_n(\mathbf{y}) = \tilde{F}_n^t(\mathbf{y}), \\ &\Delta t_\tau(\mathbf{y}) = 2t_\tau^+ - \Sigma t_\tau = sf\left[\Sigma t_n + \Delta t_n\right] - \Sigma t_\tau = \\ &= sf\left[\Sigma t_n + \tilde{F}_n^t(\mathbf{y})\right] - \Sigma t_\tau \end{aligned} \tag{21}$$

Hence, one can obtain

$$\begin{aligned} &F_i^{t+f}(\mathbf{y}) \equiv \Delta t_i(\mathbf{y}) = \\ &= (\alpha_{1i} + sf\alpha_{2i})\alpha_{1j}\tilde{F}_j^t(\mathbf{y}) - \\ &-\alpha_{2i}\left(\alpha_{2j} - sf\alpha_{1j}\right)\Sigma t_j(\mathbf{y}) \end{aligned} \tag{22}$$

The constructed conditions of imperfect interaction of thin inclusion with a medium can be easily introduced into the boundary element method code of Ref. [9].

2.4. Determination of contact conditions

Consider that the location and the length of a zone of possible delamination of thin inclusion are known. However, the contact conditions between inclusion and a medium are still unknown, because inclusion can be in a smooth contact if $t_n < 0$ (normal stress are considered negative at contact of solids due to the exerted pressure), or to be completely debonded (when inclusion is debonded the surfaces of the latter and the medium should not be constrained, thus, $\Delta u_n > 0$). The account of friction is also related with certain complications, because the Coulomb law sets only the limit value of friction. Therefore, for determination of the contact conditions one should use the iterative procedures provided in Ref. [11].

3. NUMERICAL EXAMPLES

3.1. Delaminated inclusion

Consider the plane strain of the infinite elastic medium containing a thin elastic inclusion from another material. The thickness of inclusion $2h$ equals 0.1 of its lengths $2a$. Relative rigidity of inclusion equals $k = E^i/E^m = 10^2$ and the Poisson ratios of inclusion and a medium are identical and equal 0.3. Here E^i is an elastic modulus of inclusion and E^m of the medium. Inclusion at one face (Γ_C^+) is under the perfect contact with the medium, and at another (Γ_C^-) it is under the perfect or a smooth contact or contact with friction. In the latter case it is assumed that the friction coefficient f equals 0.7. At the infinity the medium is loaded by tractions $\sigma_{xx} = p$, $\sigma_{yy} = -p$ $(p > 0)$. The problem scheme is depicted in Figure 1. The values of tangential contact traction $t_x^{\pm}$ acting in the medium at the surfaces Γ_C^+ and Γ_C^- for the case of perfect (curve 1) and smooth (curve 2) contact or contact with friction (curve 3) are plotted in Figure 1. The dashed curves represent the displacement discontinuity and the ratio of tangential and normal tractions at the surface Γ_C^-. 30 quadratic discontinuous boundary elements are used to mesh the inclusion (when the contact with friction is considered, 50 boundary elements are used). When the number of elements is increased from 30 to 50, the deviation in the obtained values of contact tractions is less than 0.5%.

One can see in Figure 1 that delamination of a thin inclusion at Γ_C^- increases the values of contact tractions near the ends of inclusion at Γ_C^+. The smaller is friction the higher are tangential tractions, which reach their maximal values in the limit case of $f = 0$. Under the contact with friction at Γ_C^- the inclusion is sliding near its ends. It should be also noted that inclusion is completely debonded from a medium at its tips. At the section of $(-0.36a; 0.36a)$ thin inclusion is under the perfect contact with the medium and does not slide.

The calculated values of friction together with displacements of inclusion and medium faces can be used for studying of the energy dissipation at cyclic load of a composite with deboded inclusion.

3.2. Partially enforced hole

In Ref. [17] the practical importance of study of holes enforced with thin overlays is shown and the approach for the study of curvilinear holes, which can be mapped onto the unit circle with the function $\omega(\zeta) = R_0\left(\zeta + \varepsilon/\zeta^{N+1}\right)$ is proposed. This approach allows studying symmetrically enforced holes with the shape of regular polygon with rounded vertices. The model of an overlay proposed in this article is free of these restrictions and allows studying partially enforced holes of arbitrary shape, and also solids with surface overlays and stringers.

As an example, consider the distribution of hoop stress $\sigma_{\theta\theta}$ at the circular hole, which is symmetrically enforced along the half of the circle with the opened stiffening rib.

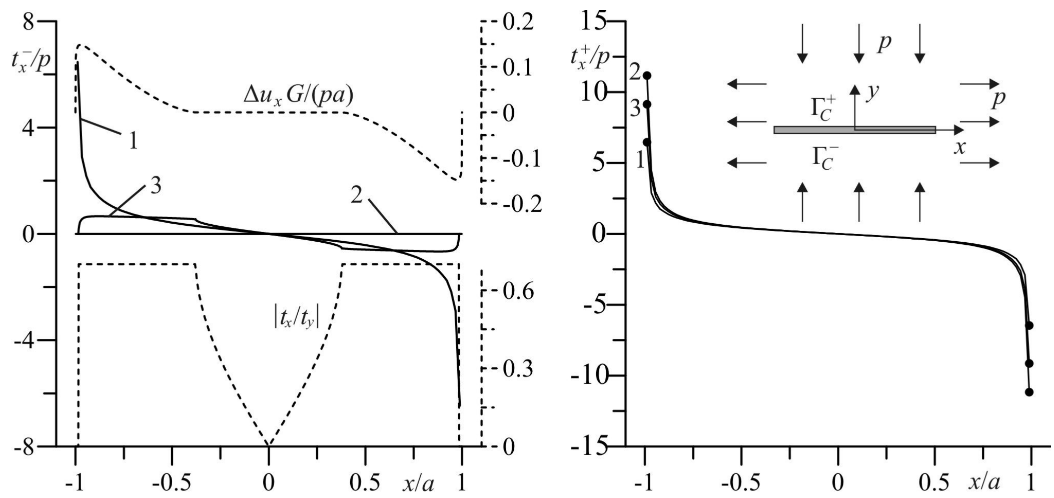

Fig. 1. Tangential contact tractions for the different contact conditions of thin inclusion

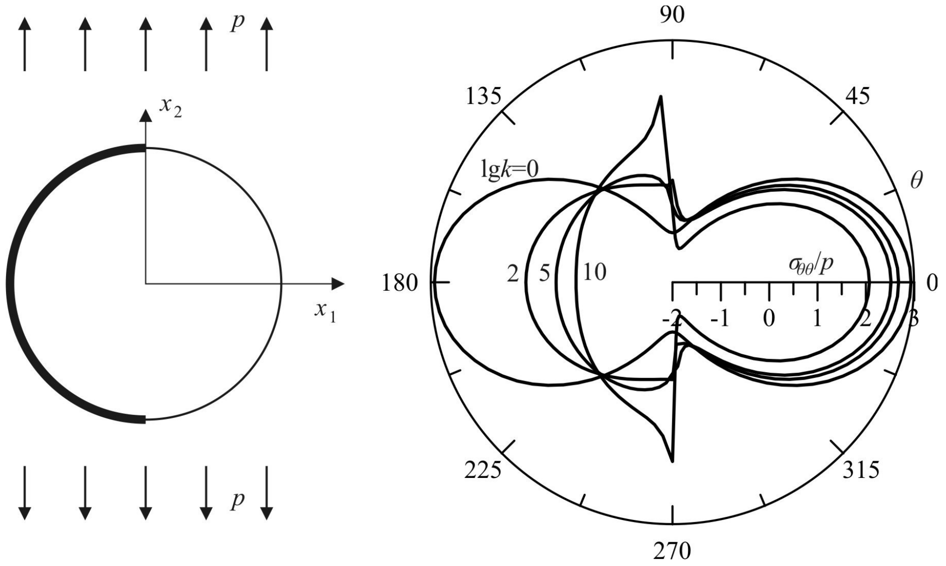

Fig. 2. Hoop stress $\sigma_{\theta\theta}$ at a hole enforced with a thin overlay

The scheme of the problem and the dependence of hoop stresses on a polar angle θ and overlay's relative rigidity k are depicted in Figure 2. This figure shows that the increase in enforcement's relative rigidity causes the reduction of stress concentration, given by the Kirsch problem solution; and mostly at the enforced part of a hole. The desired notable effect of stress concentration reduction is provided with the overlay's relative rigidity of $k = 10^2–10^5$ (for its half-thickness $h = 0.01R$, where R is a radius of a hole). However, it is necessary to simultaneously take into account that with the increase in overlay's relative rigidity k the increase in hoop stress concentration is observed under an overlay near its end faces. This is explained with the occurrence of the square root singularity of stress field, the same as for the problems of rigid stamps indention.

4. CONCLUSION

Basing on the integral equation method and a coupling principle for continua of different dimension the equations for analysis of the plane problem of elasticity of solids containing thin inclusions are obtained. Proceeding from the inclusion model under the perfect contact with a medium the models of debonded and partially delaminated inclusions are constructed. The smooth contact and contact with friction along with the full delamination are considered. The numerical analysis shows the correctness and efficiency of the proposed models of thin overlays and the models of thin inclusions under the imperfect contact.

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