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ON CONTROLLABILITY FOR FRACTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACES

JinRong Wang, XueZhu Li, and Wei Wei

Abstract. In this paper, we investigate the controllability for systems governed by fractional differential inclusions in Banach spaces. The techniques rely on fractional calculus, multivalue mapping on a bounded set and Bohnenblust-Karlin's fixed point theorem.

Keywords: controllability, fractional differential inclusions, Bohnenblust-Karlin's fixed point theorem.

Mathematics Subject Classification: 34A60, 93B05, 93C25.

1. INTRODUCTION

Fractional order models can be found to be more adequate than integer order models in some real world problems as fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electro dynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order. As a consequence, the subject of fractional differential equations is gaining more importance and attention. There has been significant development in ordinary and partial differential equations involving both Riemann-Liouville and Caputo fractional derivatives. For details and examples, one can see the monographs of Kilbas *et al.* [22], Miller and Ross [27], Podlubny [32], Lakshmikantham *et al.* [23], the survey of Agarwal *et al.* [1, 2]. In particular, we investigated some fractional functional differential equations [4, 42–44], fractional evolution equations and optimal controls [34–39, 45, 46] and introduced an appropriate definition for mild solutions based on the well known theory of the Laplace transform and probability density functions.

During the past decades, differential inclusions arise in the mathematical modeling of certain problems in economics, engineering, optimal control, etc. and are widely studied by many authors, see [7–10, 12, 13, 15, 28–30, 33, 40, 41] and the references therein. For some recent developments on fractional differential inclusions, we refer the reader to the references [1–3, 5, 6, 14, 16, 20, 25, 31].

Recently, Agarwal, Benchohra, Hamani [1] proved the existence of solutions for the following fractional boundary problem in *finite dimensional spaces* by means of a nonlinear alternative of Leray-Schauder type and a fixed point theorem for contraction multivalued maps

$$\begin{cases} {}^C D_t^q x(t) \in F\left(t, x(t)\right), \ t \in J = [0, b], \ 0 < q < 1, \\ a_1 x(0) + a_2 x(b) = a_3, \end{cases}$$

where ${}^C D_t^q$ is the *Caputo* fractional derivative of order q, $b > 0$ is a finite number, $F : J \times R \to \mathcal{P}(R)$ is a multivalued map, where $\mathcal{P}(R)$ is the family of all nonempty subsets of R, a_1, a_2, a_3 are real constants with $a_1 + a_2 \neq 0$.

In this paper, we extend the above work to study the controllability for system governed fractional differential inclusions in *infinite dimensional spaces* of the type

$$\begin{cases} {}^C D_t^q x(t) \in F\left(t, x(t)\right) + Bu(t), \ t \in J = [0, b], \ 0 < q < 1, \\ x(0) = x_0 \in X, \end{cases} \tag{1.1}$$

where ${}^C D_t^q$ is the *Caputo* fractional derivative, the state $x(\cdot)$ takes values in a Banach space X, $F : J \times X \to 2^X \setminus \{\varnothing\}$ is a nonempty, bounded, closed, convex multivalued map (not necessary compact). Also the control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions, with U being a Banach space. Finally, B is a bounded linear operator from U into X.

To establish the controllability result for the system (1.1), the main idea used here is to verify that $\mathscr{P}$ defined by (3.3) (see Section 3) is a compact multivalued map, upper semicontinuous with convex, closed values which guarantee the Bohnenblust-Karlin's fixed point theorem can be applied. For this purpose, we subdivide the proof into five steps. The key step is to check that operator $\mathscr{F}$ defined by (3.4) (see Section 3) satisfies the conditions of Lasota-Opial's result (see Lemma 2.7). More technical problems have to be overcome in our proof when $\mathscr{F}$ is a continuous mapping. Both our method and the conditions on the multivalued map F (see [HF1]–[HF4]) are different from [1].

The rest of this paper is organized as follows. In Section 2, we give the necessary preliminaries from the fields of fractional integral and derivative, multivalued maps. Finally, we build up the controllability result for the system (1.1).

2. PRELIMINARIES

We denote by X a Banach space with the norm $\|\cdot\|$. Let Y be another Banach space, $L_b(X, Y)$ denote the space of bounded linear operators from X to Y. For measurable functions $m : J \to R$, we define the norm

$$\|m\|_{L^p(J,R)} := \left(\int_J |m(t)|^p dt \right)^{\frac{1}{p}}, \ 1 \leq p < \infty.$$

Let $L^p(J,R)(1 \leq p < \infty)$ be the Banach space of all Lebesgue measurable functions m from $J \to R$ with $\|m\|_{L^p(J,R)} < \infty$. Let $L^p(J,X)$ be the Banach space of functions f: $J \to X$ which are Bochner integrable normed by $\|f\|_{L^p(J,X)}$. We denote by $\mathcal{C}$, the Banach space $C(J,X)$ endowed with supnorm given by $\|x\|_{\mathcal{C}} \equiv \sup_{t\in J} \|x(t)\|$, for $x \in \mathcal{C}$.

Let us recall the following known definitions. For more details, see [22].

Definition 2.1. The fractional integral of order γ with the lower limit zero for a function $f : [0,\infty) \to R$ is defined as

$$I^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \ t > 0, \ \gamma > 0,$$

provided the right hand side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f : [0,\infty) \to R$ can be written as

$$^L D^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} ds, \ t > 0, \ n-1 < \gamma < n.$$

Definition 2.3. The Caputo derivative of order γ for a function $f : [0,\infty) \to R$ can be written as

$$^C D^\gamma f(t) = \ ^L D^\gamma \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \ t > 0, \ n-1 < \gamma < n.$$

Remark 2.4. (1) If $f(t) \in C^n[0,\infty)$, then

$$^C D^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}} ds = I^{n-\gamma} f^{(n)}(t), \ t > 0, \ n-1 < \gamma < n.$$

(2) The Caputo derivative of a constant is equal to zero.

(3) If f is an abstract function with values in X, then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

We also introduce some basic definitions and results of multivalued maps. For more details on multivalued maps see the books of Deimling [18] and Hu and Papageorgious [21].

A multivalued map $G : X \to 2^X \setminus \{\varnothing\}$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(C) = \bigcup_{x\in C} G(x)$ is bounded in X for any bounded set C of X, i.e., $\sup_{x\in C} \left\{ \sup\{\|y\| : y \in G(x)\} \right\} < \infty$.

Definition 2.5. G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set C of X containing $G(x_0)$, there exists an open neighborhood V of x_0 such that $G(V) \subseteq C$.

Definition 2.6. G is called completely continuous if $G(C)$ is relatively compact for every bounded subset C of X.

If the multivalued map G is completely continuous with nonempty values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in Gx_n$ imply $y_* \in Gx_*$. G has a fixed point if there is a $x \in X$ such that $x \in G(x)$.

The following lemmas are of great of importance in the proof of our main results.

Lemma 2.7 (Lasota and Opial [24])**.** *Let J be a compact real interval, $BCC(X)$ be the set of all nonempty, bounded, closed and convex subset of X and F be a multivalued map satisfying $F : J \times X \to BCC(X)$ is measurable to t for each fixed $x \in X$, u.s.c. to x for each $t \in J$, and for each $x \in \mathcal{C}$ the set $S_{F,x} = \{f \in L^1(J, X) : f(t) \in F(t, x(t)),\ t \in J\}$ is nonempty. Let $\mathscr{F}$ be linear continuous from $L^1(J, X)$ to $\mathcal{C}$. Then the operator*

$$\mathscr{F} \circ S_F : \mathcal{C} \to BCC(\mathcal{C}),\ x \mapsto (\mathscr{F} \circ S_F)(x) = \mathscr{F}(S_{F,x}),$$

is a closed graph operator in $\mathcal{C} \times \mathcal{C}$, where $\mathcal{C} = C(J, X)$.

Lemma 2.8 (Bohnenblust and Karlin [11])**.** *Let $\mathcal{D}$ be a nonempty subset of X, which is bounded, closed, and convex. Suppose $G : \mathcal{D} \to 2^X \setminus \{\varnothing\}$ is u.s.c. with closed, convex values, such that $G(\mathcal{D}) \subseteq \mathcal{D}$ and $G(\mathcal{D})$ is compact. Then G has a fixed point.*

3. CONTROLLABILITY RESULTS

To set the framework for the controllability results, we need the following definitions.

Definition 3.1. A function $x \in \mathcal{C}$ is said to be a solution of the system (1.1) if $x(0) = x_0$ and there exists a function $f \in L^1(J, X)$ such that $f(t) \in F(t, x(t))$ on $t \in J$ and

$$^C D_t^q x(t) = f(t) + Bu(t),\ t \in J,\ 0 < q < 1.$$

Definition 3.2. The system (1.1) is said to be controllable on the interval J if for every x_0, $x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that a solution x of system (1.1) satisfies $x(b) = x_1$.

We assume the following hypothesis:

[HW] The linear operator $B : L^2(J, U) \to L^1(J, X)$ is bounded, $W : L^2(J, U) \to X$ defined by

$$Wu = \frac{1}{\Gamma(q)} \int_0^b (b - s)^{q-1} Bu(s) ds$$

has an inverse operator W^{-1} which takes values in $L^2(J, U)/\ker W$, where the kernel space of W is defined by $\ker W = \{x \in L^2(J, U) : Wx = 0\}$ and there exist constants M_1, $M_2 > 0$ such that $\|B\| \le M_1$ and $\|W^{-1}\| \le M_2$.

[HF1] F is a multivalued map satisfying $F : J \times X \to BCC(X)$ is measurable to t for each fixed $x \in X$, u.s.c. to x for each $t \in J$, and for each $x \in \mathcal{C}$ the set

$$S_{F,x} = \{f \in L^1(J, X) : f(t) \in F(t, x(t)),\ t \in J\}$$

is nonempty.

[HF2] For each positive number r and $x \in \mathcal{C}$ with $\|x\|_{\mathcal{C}} \leq r$, there exists a constant $q_1 \in (0, q)$ and $L_{f,r}(\cdot) \in L^{\frac{1}{q_1}}(J, R^+)$ such that

$$\sup\left\{\|f\| : f(t) \in F(t, x(t))\right\} \leq L_{f,r}(t)$$

for a.e. $t \in J$.
[HF3] The function $s \mapsto (t-s)^{q-1}L_{f,r}(s) \in L^1([0,t], R^+)$ and there exists $\gamma > 0$ such that

$$\lim_{r\to\infty} \inf \frac{\int_0^t (t-s)^{q-1}L_{f,r}(s)ds}{r} = \gamma < +\infty.$$

[HF4] The function $f : J \to X$ is compact, where $f \in S_{F,x}$ and $S_{F,x}$ takes the same notation as in Lemma 2.7.

Now, we are ready to present and prove our main results.

Theorem 3.3. *Suppose that* [HW], [HF1]–[HF4] *are satisfied and* $q \in (\frac{1}{2}, 1)$. *Then system* (1.1) *is controllable on* J *provided that*

$$\frac{(q + M_2M_1b^q)\gamma}{\Gamma(1+q)} < 1. \tag{3.1}$$

Proof. Using hypothesis [HW], [HF1] and [HF2], for an arbitrary function $x(\cdot) \in \mathcal{C}$, we can define the control $u_x(t)$ by

$$u_x(t) = W^{-1}\left[x_1 - x_0 - \frac{1}{\Gamma(q)}\int_0^b (b-s)^{q-1}f(s)ds\right](t), \ t \in J, \tag{3.2}$$

where $f \in S_{F,x}$.

We show that, using this control, the operator $\mathscr{P}: \mathcal{C} \to 2^{\mathcal{C}}$ defined by

$$\mathscr{P}(x) = \left\{x \in \mathcal{C} : \varphi(t) = x_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f(s) + Bu_x(s)]ds, f \in S_{F,x}\right\} \tag{3.3}$$

has a fixed point x, which is a solution of the system (1.1). We observe that $x_1 \in (\mathscr{P}x)(b)$ which means that u_x steers the system (1.1) from x_0 to x_1 in a finite time b. This implies that the system (1.1) is controllable on J.

We now show that $\mathscr{P}$ satisfies all the conditions of Lemma 2.8. For the sake of convenience, we subdivide the proof into several steps.

Step 1. $\mathscr{P}$ is convex for each $x \in \mathcal{C}$.

In fact, if φ_1 and φ_2 belong to $\mathscr{P}(x)$, then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in J$, we have

$$\begin{aligned}
&\varphi_i(t) = \\
&= x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_i(s)\, ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} B u_x(s) ds = \\
&= x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_i(s)\, ds + \\
&\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} B W^{-1} \left[x_1 - x_0 - \frac{1}{\Gamma(q)} \int_0^b (b-\eta)^{q-1} f_i(\eta)\, d\eta \right] (s) ds, \ i = 1, 2.
\end{aligned}$$

Let $\lambda \in [0,1]$. Then for each $t \in J$, we get

$$\begin{aligned}
&\lambda\varphi_1(t) + (1-\lambda)\varphi_2(t) = \\
&= x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [\lambda f_1(s) + (1-\lambda) f_2(s)] ds + \\
&+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} B W^{-1} \Bigg[x_1 - x_0 - \\
&\qquad - \frac{1}{\Gamma(q)} \int_0^b (b-\eta)^{q-1} [\lambda f_1(\eta) + (1-\lambda) f_2(\eta)] d\eta \Bigg] (s) ds.
\end{aligned}$$

Since $S_{F,x}$ is convex (because F has convex values), $\lambda f_1 + (1-\lambda) f_2 \in S_{F,x}$. Thus,

$$\lambda\varphi_1 + (1-\lambda)\varphi_2 \in \mathscr{P}(x).$$

Step 2. For each positive number $r > 0$, let $\mathfrak{B}_r = \{x \in \mathcal{C} : \|x\|_{\mathcal{C}} \le r\}$. Obviously, $\mathfrak{B}_r$ is a bounded, closed and convex set of $\mathcal{C}$. We claim that there exists a positive number r such that $\mathscr{P}(\mathfrak{B}_r) \subseteq \mathfrak{B}_r$.

If this is not true, then for each positive number r, there exists a function $x^r \in \mathfrak{B}_r$, but $\mathscr{P}(x^r)$ does not belong to $\mathfrak{B}_r$, i.e.,

$$\|\mathscr{P}(x^r)\|_{\mathcal{C}} \equiv \sup \left\{ \|\varphi^r\|_{\mathcal{C}} : \varphi^r \in (\mathscr{P} x^r) \right\} > r,$$

and

$$\varphi^r(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f^r(s)\, ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} B u_{x^r}(s) ds,$$

for some $f^r \in S_{F,x^r}$.

Taking into account (3.2), using [HW] and [HF2], we have

$$\|u_{x^r}(t)\| \leq \|W^{-1}\| \left\{ \|x_1\| + \|x_0\| + \frac{1}{\Gamma(q)} \int_0^b (b-s)^{q-1} \|f^r(s)\| \, ds \right\} \leq$$
$$\leq M_2(\|x_1\| + \|x_0\|) + M_2 \int_0^b (b-s)^{q-1} L_{f,r}(s) ds.$$

On the other hand,

$$\begin{aligned}
r &< \|(\mathscr{P}x^r)(t)\| \leq \\
&\leq \|x_0\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f^r(s)\| \, ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|Bu_{x^r}(s)\| ds \leq \\
&\leq \|x_0\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} L_{f,r}(s) ds + \\
&\quad + \frac{M_1 b^q}{\Gamma(1+q)} \left[M_2(\|x_1\| + \|x_0\|) + M_2 \int_0^b (b-s)^{q-1} L_{f,r}(s) ds \right] = \\
&= \|x_0\| + \frac{M_2 M_1 b^q (\|x_1\| + \|x_0\|)}{\Gamma(1+q)} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} L_{f,r}(s) ds = \\
&\quad + \frac{M_2 M_1 b^q}{\Gamma(1+q)} \int_0^b (b-s)^{q-1} L_{f,r}(s) ds = \\
&= a + c_1 \int_0^t (t-s)^{q-1} L_{f,r}(s) ds + c_2 \int_0^b (b-s)^{q-1} L_{f,r}(s) ds,
\end{aligned}$$

where

$$a = \|x_0\| + \frac{M_2 M_1 b^q (\|x_1\| + \|x_0\|)}{\Gamma(1+q)},$$
$$c_1 = \frac{1}{\Gamma(q)}, \quad c_2 = \frac{M_2 M_1 b^q}{\Gamma(1+q)}.$$

Dividing both sides of the above inequality by r and taking the limit as $r \to \infty$, using [HF3], we get

$$(c_1 + c_2)\gamma \geq 1.$$

This contradicts with condition (3.1). Hence, for some $r > 0$, $\mathscr{P}(\mathfrak{B}_r) \subseteq \mathfrak{B}_r$.
Step 3. $\mathscr{P}$ sends bounded sets into equicontinuous sets of $\mathcal{C}$.

Let $0 < s < t < t+h \leq b$ and $\varepsilon > 0$. For each $x \in \mathfrak{B}_r$, $\varphi \in \mathscr{P}(x)$, there exists a $f \in S_{F,x}$ such that

$$\varphi(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) \, ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Bu_x(s) ds.$$

Clearly,

$$\begin{aligned}\varphi(t+h)-\varphi(t) &= \frac{1}{\Gamma(q)}\int_t^{t+h}(t+h-s)^{q-1}f(s)ds+\\ &+\frac{1}{\Gamma(q)}\int_0^t[(t+h-s)^{q-1}-(t-s)^{q-1}]f(s)ds+\\ &+\frac{1}{\Gamma(q)}\int_t^{t+h}(t+h-s)^{q-1}Bu_x(s)ds+\\ &+\frac{1}{\Gamma(q)}\int_0^t[(t+h-s)^{q-1}-(t-s)^{q-1}]Bu_x(s)ds.\end{aligned}$$

Let

$$\begin{aligned}I_1 &= \frac{1}{\Gamma(q)}\int_t^{t+h}(t+h-s)^{q-1}f(s)ds,\\ I_2 &= \frac{1}{\Gamma(q)}\int_0^t[(t+h-s)^{q-1}-(t-s)^{q-1}]f(s)ds,\\ I_3 &= \frac{1}{\Gamma(q)}\int_t^{t+h}(t+h-s)^{q-1}Bu_x(s)ds,\\ I_4 &= \frac{1}{\Gamma(q)}\int_0^t[(t+h-s)^{q-1}-(t-s)^{q-1}]Bu_x(s)ds.\end{aligned}$$

It is obvious that

$$\|\varphi(t+h)-\varphi(t)\| \ \leq \ \sum_{i=1}^{4}\|I_i\|.$$

Now, we only need to check that $\|I_i\|\to 0$ as $h\to 0$, $i=1,2,3,4$.

For I_1, by the Hölder inequality and [HF2],

$$\begin{aligned}\|I_1\| &\leq \int_t^{t+h}(t+h-s)^{q-1}\,\|f(s)\|\,ds\leq\\ &\leq \frac{1}{\Gamma(q)}\int_t^{t+h}(t+h-s)^{q-1}L_{f,r}(s)ds\leq\\ &\leq \frac{1}{\Gamma(q)}\left[\left(\frac{1-q_1}{q-q_1}\right)h^{\frac{q-q_1}{1-q_1}}\right]^{1-q_1}\|L_{f,r}\|_{L^{\frac{1}{q_1}}(J,R^+)}\to 0 \text{ as } h\to 0.\end{aligned}$$

For I_2, let $\beta = \frac{q-1}{1-q_1} \in (-1, 0)$, after some calculation, we have

$$\begin{aligned}
\|I_2\| &\le \int_{t-\gamma}^{t} [(t+h-s)^{q-1} - (t-s)^{q-1}] L_{f,r}(s)ds \le \\
&\le \left(\int_{t-\gamma}^{t} [(t+h-s)^{q-1} - (t-s)^{q-1}]^{\frac{1}{1-q_1}} ds \right)^{1-q_1} \|L_{f,r}\|_{L^{\frac{1}{q_1}}(J,R^+)} \le \\
&\le \left(\int_{t-\gamma}^{t} [(t+h-s)^{\beta} - (t-s)^{\beta}] ds \right)^{1-q_1} \|L_{f,r}\|_{L^{\frac{1}{q_1}}(J,R^+)} \le \\
&\le \frac{1}{(1+\beta)^{1-q_1}} \left[(h+\gamma)^{1+\beta} - h^{1+\beta} - \gamma^{1+\beta} \right]^{1-q_1} \|L_{f,r}\|_{L^{\frac{1}{q_1}}(J,R^+)} \le \\
&\le \frac{(2h)^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \|L_{f,r}\|_{L^{\frac{1}{q_1}}(J,R^+)} \to 0 \text{ as } h \to 0.
\end{aligned}$$

For I_3, I_4, repeating the same process of checking as in the case of I_1, I_2, and nothing that

$$\|u_x(t)\| \le M_2(\|x_1\| + \|x_0\|) + \frac{M_2 b^q}{\Gamma(1+q)} \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{1-q_1} \equiv M,$$

one can verify that $\|I_3\|$ and $\|I_4\|$ tend to zero as $h \to 0$.

As a result, we immediately obtain that

$$\|\varphi(t+h) - \varphi(t)\| \to 0 \text{ as } h \to 0,$$

for all $x \in \mathfrak{B}_r$. Therefore, $\mathscr{P}(\mathfrak{B}_r) \subset \mathcal{C}$ is equicontinuous.

Step 4. The set $\Pi(t) = \mathscr{P}(\mathfrak{B}_r)(t) = \{\varphi(t) : \varphi \in \mathscr{P}(\mathfrak{B}_r)\} \subset X$ is relatively compact for any $t \in J$.

By [HF4], we know that $(\cdot - s)^{q-1} f(\cdot)$ is compact, and then the set $\mathcal{S} = \{(t-s)^{q-1} f(s) : t \in J, s \in [0,t]\} \subset X$ is relatively compact. So for any $t \in J$,

$$\mathcal{S}'_t = \left\{ \int_0^t (t-s)^{q-1} f(s) ds \right\} \subset t\overline{conv}\mathcal{S}$$

is relatively compact in X, where $\overline{conv}\mathcal{S}$ means the closure of the convex hull of $\mathcal{S}$ in X. By [HW], we obtain that

$$\mathcal{S}'' = \left\{ u_x = W^{-1} \left[x_1 - x_0 - \frac{1}{\Gamma(q)} \int_0^b (b-s)^{q-1} f(s) ds \right] : x \in \mathfrak{B}_r \right\}$$

is relatively compact in $L^2(J,U)$. As $B : L^2(J,U) \to L^1(J,X)$ is bounded, it implies that $B\mathcal{S}'' \subset L^1(J,X)$ is relatively compact. So we know that

$$\mathcal{S}''' = \left\{ \int_0^t (t-s)^{q-1} B u_x(s) ds : u_x \in \mathcal{S}'' \right\} \subset X$$

is relatively compact as the map $y \mapsto \int_0^t (t-s)^{q-1} y(s) ds$, $L^1(J,X) \to X$ is continuous. For any $t \in J$,

$$\mathscr{P}(\mathfrak{B}_r)(t) \subset \{x_0\} + \mathcal{S}_t' + \mathcal{S}''',$$

we have that $\mathscr{P}(\mathfrak{B}_r)(t)$ is relatively compact in X for every $t \in J$. Thus, $\Pi(t)$ is relatively compact in X for every $t \in J$.

Step 5. $\mathscr{P}$ has a closed graph.

Let $x_n \to x_*$ as $n \to \infty$, $\varphi_n \in \mathscr{P}(x_n)$, and $\varphi_n \to \varphi_*$ as $n \to \infty$. We shall show that $\varphi_* \in \mathscr{P}(x_*)$. Since $\varphi_n \in \mathscr{P}(x_n)$, there exists $f_n \in S_{F,x_n}$ such that

$$\begin{aligned}\varphi_n(t) &= x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_n(s)\, ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} B u_{x_n}(s) ds = \\ &= x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_n(s)\, ds + \\ &\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} B W^{-1} \left[x_1 - x_0 - \frac{1}{\Gamma(q)} \int_0^b (b-\eta)^{q-1} f_n(\eta)\, d\eta \right](s) ds.\end{aligned}$$

We must prove that there exists $f_* \in S_{F,x_*}$ such that

$$\begin{aligned}\varphi_*(t) &= x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_*(s)\, ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} B u_{x_*}(s) ds = \\ &= x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_*(s)\, ds + \\ &\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} B W^{-1} \left[x_1 - x_0 - \frac{1}{\Gamma(q)} \int_0^b (b-\eta)^{q-1} f_*(\eta)\, d\eta \right](s) ds.\end{aligned}$$

Set

$$\bar{u}_x(t) = W^{-1}(x_1 - x_0)(t).$$

Since W^{-1} is continuous, then

$$\bar{u}_{x_n}(t) \to \bar{u}_{x_*}(t), \text{ for } t \in J, \text{ as } n \to \infty.$$

Clearly,

$$\begin{aligned}&\Bigg\| \left(\varphi_n - x_0 - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} B \bar{u}_{x_n}(s) ds \right) - \\ &\quad - \left(\varphi_* - x_0 - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} B \bar{u}_{x_*}(s) ds \right) \Bigg\|_{\mathcal{C}} \leq \\ &\leq \|\varphi_n - \varphi_*\|_{\mathcal{C}} + \frac{M_1 b^q}{\Gamma(1+q)} \|\bar{u}_{x_n} - \bar{u}_{x_*}\|_{\mathcal{C}} \to 0 \text{ as } n \to \infty.\end{aligned}$$

Consider the linear operator $\mathscr{F} : L^1(J,X) \to \mathcal{C}$,

$$(\mathscr{F} f)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[f(s) - B W^{-1} \left(\frac{1}{\Gamma(q)} \int_0^b (b-\eta)^{q-1} f(\eta)\, d\eta \right)(s) \right] ds. \quad (3.4)$$

We can prove that the operator $\mathscr{F}$ is continuous. In fact, for any $0 < t < t+\delta < b$, $\delta > 0$,

$$\|(\mathscr{F}f)(t+\delta) - (\mathscr{F}f)(t)\| \le J_1 + J_2,$$

where

$$J_1 = \frac{1}{\Gamma(q)} \int_0^t \left\| [(t+\delta-s)^{q-1} - (t-s)^{q-1}] f(s) \right\| ds + \frac{1}{\Gamma(q)} \int_t^{t+\delta} \left\| (t+\delta-s)^{q-1} f(s) \right\| ds,$$

$$J_2 = \frac{1}{\Gamma(q)} \int_0^t \left\| [(t+\delta-s)^{q-1} - (t-s)^{q-1}] BW^{-1} \left(\frac{1}{\Gamma(q)} \int_0^b (b-\eta)^{q-1} f(\eta)\, d\eta \right)(s) \right\| ds +$$
$$+ \frac{1}{\Gamma(q)} \int_t^{t+\delta} \left\| (t+\delta-s)^{q-1} BW^{-1} \left(\frac{1}{\Gamma(q)} \int_0^b (b-\eta)^{q-1} f(\eta)\, d\eta \right)(s) \right\| ds.$$

By [HF2] and the Hölder inequality, we obtain

$$\begin{aligned}
J_1 &\le \frac{1}{\Gamma(q)} \int_0^t [(t-s)^{q-1} - (t+\delta-s)^{q-1}] L_{f,r}(s) ds + \\
&\quad + \frac{1}{\Gamma(q)} \int_t^{t+\delta} (t+\delta-s)^{q-1} L_{f,r}(s) ds \le \\
&\le \frac{1}{\Gamma(q)} \left(\int_0^t [(t-s)^{q-1} - (t+\delta-s)^{q-1}]^{\frac{1}{1-q_1}} ds \right)^{1-q_1} \left(\int_0^t (L_{f,r}(s))^{\frac{1}{q_1}} ds \right)^{q_1} + \\
&\quad + \frac{1}{\Gamma(q)} \left(\int_t^{t+\delta} ((t+\delta-s)^{q-1})^{\frac{1}{1-q_1}} ds \right)^{1-q_1} \left(\int_t^{t+\delta} (L_{f,r}(s))^{\frac{1}{q_1}} ds \right)^{q_1} \le \\
&\le \frac{1}{\Gamma(q)} \left(\int_0^t [(t-s)^{\beta} - (t+\delta-s)^{\beta}] ds \right)^{1-q_1} \|L_{f,r}\|_{L^{\frac{1}{q_1}}(J,R^+)} + \\
&\quad + \frac{1}{\Gamma(q)} \left(\int_t^{t+\delta} (t+\delta-s)^{\beta} ds \right)^{1-q_1} \|L_{f,r}\|_{L^{\frac{1}{q_1}}(J,R^+)} \le \\
&\le \frac{\|L_{f,r}\|_{L^{\frac{1}{q_1}}(J,R^+)}}{\Gamma(q)(1+\beta)^{1-q_1}} \left(t^{1+\beta} - (t+\delta)^{1+\beta} + \delta^{1+\beta} \right)^{1-q_1} + \\
&\quad + \frac{\|L_{f,r}\|_{L^{\frac{1}{q_1}}(J,R^+)}}{\Gamma(q)(1+\beta)^{1-q_1}} \delta^{(1+\beta)(1-q_1)} \le \\
&\le 2 \frac{\|L_{f,r}\|_{L^{\frac{1}{q_1}}(J,R^+)}}{\Gamma(q)(1+\beta)^{1-q_1}} \delta^{(1+\beta)(1-q_1)}.
\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
J_2 &\leq \\
&\leq \frac{M_1}{\Gamma(q)} \int_0^t [(t-s)^{q-1} - (t+\delta-s)^{q-1}] \left\| W^{-1}\left(\frac{1}{\Gamma(q)} \int_0^b (b-\eta)^{q-1} f(\eta)\, d\eta\right)(s) \right\| ds+ \\
&\quad + \frac{M_1}{\Gamma(q)} \int_t^{t+\delta} (t+\delta-s)^{q-1} \left\| W^{-1}\left(\frac{1}{\Gamma(q)} \int_0^b (b-\eta)^{q-1} f(\eta)\, d\eta\right)(s) \right\| ds \leq \\
&\leq \frac{M_1}{\Gamma(q)} \left(\int_0^t [(t-s)^{q-1} - (t+\delta-s)^{q-1}]^2 ds \right)^{\frac{1}{2}} \times \\
&\quad \times \left(\int_0^t \left\| W^{-1}\left(\frac{1}{\Gamma(q)} \int_0^b (b-\eta)^{q-1} f(\eta)\, d\eta\right)(s) \right\|^2 ds \right)^{\frac{1}{2}} + \\
&\quad + \frac{M_1}{\Gamma(q)} \left(\int_t^{t+\delta} (t+\delta-s)^{2(q-1)} ds \right) \times \\
&\quad \times \left(\int_t^{t+\delta} \left\| W^{-1}\left(\frac{1}{\Gamma(q)} \int_0^b (b-\eta)^{q-1} f(\eta)\, d\eta\right)(s) \right\|^2 ds \right)^{\frac{1}{2}} \leq \\
&\leq \frac{M_1}{\Gamma(q)} \left(\int_0^t [(t-s)^{q-1} - (t+\delta-s)^{q-1}]^2 ds \right)^{\frac{1}{2}} \times \\
&\quad \times \left(\int_0^b \left\| W^{-1}\left(\frac{1}{\Gamma(q)} \int_0^b (b-\eta)^{q-1} f(\eta)\, d\eta\right)(s) \right\|^2 ds \right)^{\frac{1}{2}} + \\
&\quad + \frac{M_1}{\Gamma(q)} \left(\int_t^{t+\delta} (t+\delta-s)^{2(q-1)} ds \right) \times \\
&\quad \times \left(\int_0^b \left\| W^{-1}\left(\frac{1}{\Gamma(q)} \int_0^b (b-\eta)^{q-1} f(\eta)\, d\eta\right)(s) \right\|^2 ds \right)^{\frac{1}{2}} \leq \\
&\leq \frac{M_1 M_2}{\Gamma(q)} \left(\int_0^t [(t-s)^{q-1} - (t+\delta-s)^{q-1}]^2 ds \right)^{\frac{1}{2}} \left\| \frac{1}{\Gamma(q)} \int_0^b (b-\eta)^{q-1} f(\eta)\, d\eta \right\| + \\
&\quad + \frac{M_1 M_2}{\Gamma(q)} \left(\int_t^{t+\delta} (t+\delta-s)^{2(q-1)} ds \right)^{\frac{1}{2}} \left\| \frac{1}{\Gamma(q)} \int_0^b (b-\eta)^{q-1} f(\eta)\, d\eta \right\| \leq \\
&\leq \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{1-q_1} \|L_{f,r}\|_{L^{\frac{1}{q_1}}(J,R^+)} \frac{M_1 M_2}{(\Gamma(q))^2} \times \frac{1}{\sqrt{2q-1}} \left(t^{2q} - (t+\delta)^{2q} + \delta^{2q} \right)^{\frac{1}{2}} + \\
&\quad + \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{1-q_1} \|L_{f,r}\|_{L^{\frac{1}{q_1}}(J,R^+)} \frac{M_1 M_2}{(\Gamma(q))^2} \times \frac{1}{\sqrt{2q-1}} \times \delta^{2q \times \frac{1}{2}} \leq \\
&\leq 2 \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{1-q_1} \|L_{f,r}\|_{L^{\frac{1}{q_1}}(J,R^+)} \frac{M_1 M_2}{(\Gamma(q))^2} \times \frac{\delta^q}{\sqrt{2q-1}}.
\end{aligned}
$$

As $\delta \to 0$, one can see that J_1 and J_2 tend to zero. Therefore, $\mathscr{F}(f) \in \mathcal{C}$.

Moreover, one has

$$\|\mathscr{F} f\|_{\mathcal{C}} \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s)\| ds + \frac{M_2 M_1 b^q}{q(\Gamma(q))^2} \int_0^b (b-\eta)^{q-1} \|f(\eta)\| d\eta \leq$$
$$\leq \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{1-q_1} \|L_{f,r}\|_{L^{\frac{1}{q_1}}(J,X)} \left[\frac{1}{\Gamma(q)} + \frac{M_2 M_1 b^q}{q(\Gamma(q))^2} \right].$$

From [HF1] and Lemma 2.7, it follows that $\mathscr{F} \circ S_F$ is a closed graph operator. Also, from the definition of $\mathscr{F}$, we have that

$$\varphi_n - x_0 - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} B \bar{u}_{x_n}(s) ds \in \mathscr{F}(S_{F,x_n}).$$

In view of $x_n \to x_*$ as $n \to \infty$, it follows again from Lemma 2.7 that

$$\varphi_* - x_0 - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} B \bar{u}_{x_*}(s) ds =$$
$$= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[f_*(s) - BW^{-1} \left(\frac{1}{\Gamma(q)} \int_0^b (b-\eta)^{q-1} f_*(\eta) \, d\eta \right)(s) \right] ds$$

for some $f_* \in S_{F,x_*}$. This implies that $\varphi_* \in \mathscr{P}(x_*)$.

As a consequence of Steps 1–5 with the well known Arzela-Ascoli theorem, we conclude that $\mathscr{P}$ is a compact multivalued map, u.s.c. with convex closed values. As a consequence of Lemma 2.8, we can deduce that $\mathscr{P}$ has a fixed point x, which is a mild solution of the system (1.1). Thus, the system (1.1) is controllable on J. □

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JinRong Wang
wjr9668@126.com

Guizhou University
Department of Mathematics
Guiyang, Guizhou 550025, P.R. China

XueZhu Li
lixuezhu1988@126.com

Guizhou University
Department of Mathematics
Guiyang, Guizhou 550025, P.R. China

Wei Wei
wwei@gzu.edu.cn

Guizhou University
College of Science
Guiyang, Guizhou 550025, P.R. China

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