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IMPLICIT RANDOM ITERATION PROCESS WITH ERRORS FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE IN THE INTERMEDIATE SENSE RANDOM OPERATORS

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Abstract. In this paper, we give a necessary and sufficient condition for the strong convergence of an implicit random iteration process with errors to a common fixed point for a finite family of asymptotically quasi-nonexpansive in the intermediate sense random operators and also prove some strong convergence theorems using condition $(\overline{C})$ and the semi-compact condition for said iteration scheme and operators. The results presented in this paper extend and improve the recent ones obtained by S. Plubtieng, P. Kumam and R. Wangkeeree, and also by the author.

Keywords: asymptotically quasi-nonexpansive in the intermediate sense random operator, implicit random iteration process with errors, common random fixed point, strong convergence, separable uniformly convex Banach space.

Mathematics Subject Classification: 47H10, 47J25.

1. INTRODUCTION

Random nonlinear analysis is an important mathematical discipline which is mainly concerned with the study of random nonlinear operators and their properties and is needed for the study of various classes of random equations. The study of random fixed point theory was initiated by the Prague school of Probabilities in the 1950s [11,12,27]. Common random fixed point theorems are a stochastic generalization of classical common fixed point theorems. The machinery of random fixed point theory provides a convenient way of modeling many problems arising from economic theory (see e.g. [19]) and references mentioned therein. Random methods have revolutionized the financial markets. The survey article by Bharucha-Reid [6] attracted the attention of several mathematicians and gave wings to the theory. Itoh [14] extended Spacek's and Hans's theorem to multivalued contraction mappings. Now this theory has become a full fledged research area and various ideas associated with random fixed point theory are used to obtain the solution of nonlinear random system (see [2–5, 13, 15, 23, 24, 28]). In 2001, Xu and Ori [29] introduced an implicit iteration process to approximate common fixed points of a finite family of nonexpansive mappings. This process proved to be the main tool to approximate common fixed points of various class of mappings in deterministic operator theory. Zhao and Chang [31] studied convergence of the modified implicit iteration process to common fixed point of a finite family of asymptotically nonexpansive mappings. Sun [26] proved necessary and sufficient conditions for convergence of an implicit iteration process to a common fixed point of asymptotically quasi-nonexpansive mappings. Latter on Chen *et al.* [8], Osilike [16] and Osilike and Akuchu [17] studied an implicit iteration process to approximate common fixed points of continuous pseudocontractive mappings, asymptotically pseudocontractive mappings and strictly pseudocontractive mappings respectively (see also [7, 10, 30]).

In 2007, Plubteing *et al.* [20] studied the implicit random iteration process with errors which converges strongly to a common fixed point of a finite family of asymptotically quasi-nonexpansive random operators on an unbounded set in uniformly convex Banach spaces. They gave necessary and sufficient condition of the said scheme and mappings and also they proved some strong convergence theorems.

The purpose of this paper to study an implicit random iteration process with errors and to give necessary and sufficient conditions for strong convergence of this iteration process to a common random fixed point of a finite family of asymptotically quasi-nonexpansive in the intermediate sense random operators in separable Banach spaces and also prove some strong convergence theorems for said iteration scheme and operators in separable uniformly convex Banach spaces. The results presented in this paper extend and improve the recent ones announced by Plubtieng *et al.* [20] and Saluja [21].

2. PRELIMINARIES

Let (Ω, Σ) be a measurable space (Σ-sigma algebra) and let C be a nonempty subset of a separable Banach space X. A mapping $\xi\colon \Omega \to X$ is measurable if $\xi^{-1}(U) \in \Sigma$ for each open subset U of X. The mapping $T\colon \Omega \times C \to C$ is a random map if and only if for each fixed $x \in C$ the mapping $T(\cdot, x)\colon \Omega \to C$ is measurable and it is continuous if for each $\omega \in \Omega$ the mapping $T(\omega, \cdot)\colon C \to X$ is continuous. A measurable mapping $\xi\colon \Omega \to X$ is a random fixed point of a random map $T\colon \Omega \times C \to X$ if and only if $T(\omega, \xi(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$. We denote the set of random fixed points of a random map T by $RF(T)$.

Let $B(x_0, r)$ denote the spherical ball centered at x_0 with radius r defined as the set $\{x \in X : \|x - x_0\| \le r\}$.

We denote the n-th iterate $T(\omega, T(\omega, T(\omega, \dots, T(\omega, x) \dots,)))$ of T by $T^n(\omega, x)$. The letter I denotes the random mapping $I\colon \Omega \times C \to C$ defined by $I(\omega, x) = x$ and $T^0 = I$.

Let C be a closed and convex subset of a separable Banach space X and the sequence of functions $\{\xi_n\}$ is pointwise convergent, that is, $\xi_n(\omega) \to q := \xi(\omega)$. Then the closedness of C implies that ξ is a mapping from Ω to C. Since C is a subset of a separable Banach space X, if T is a continuous random operator then, by [1, Lemma 8.2.3], the mapping $\omega \to T(\omega, f(\omega))$ is a measurable function for any measurable function f from Ω to C. Thus $\{\xi_n\}$ is a sequence of measurable functions. Hence $\xi\colon \Omega \to C$, being the limit of the sequence of measurable functions, is also measurable [3, Remark 2.3].

Let $T\colon \Omega \times C \to C$ be a random operator, where C is a nonempty convex subset of a separable Banach space X.

Definition 2.1. (1) Mapping T is said to be an asymptotically nonexpansive random operator if there exists a sequence of measurable mappings $h_n\colon \Omega \to [1, \infty)$ with $\lim\limits_{n\to\infty} h_n(\omega) = 1$ for each $\omega \in \Omega$, such that for $x, y \in C$, we have

$$\|T^n(\omega, x) - T^n(\omega, y)\| \quad \leq \quad h_n(\omega)\,\|x - y\| \ \text{ for each } \omega \in \Omega. \tag{2.1}$$

(2) T is said to be asymptotically quasi-nonexpansive random operator if for each $\omega \in \Omega$, $G(\omega) = \{x \in C : x = T(\omega, x)\} \neq \emptyset$ and there exists a sequence of measurable mappings $h_n\colon \Omega \to [1, \infty)$ with $\lim\limits_{n\to\infty} h_n(\omega) = 1$, for each $\omega \in \Omega$, such that for $x \in C$ and $y \in G(\omega)$, the following inequality holds:

$$\|T^n(\omega, x) - y\| \quad \leq \quad h_n(\omega)\,\|x - y\| \ \text{ for each } \omega \in \Omega. \tag{2.2}$$

(3) T is said to be asymptotically quasi-nonexpansive in the intermediate sense random operator provided that T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x\in C,\ y\in G(\omega)} \Big(\|T^n(\omega, x) - y\| - \|x - y\| \Big) \leq 0 \ \text{ for each } \omega \in \Omega, \tag{2.3}$$

where $G(\omega) = \{x \in C : x = T(\omega, x)\} \neq \emptyset$.

Definition 2.2. The modified random Mann iteration scheme is a sequence of functions $\{\xi_n\}$ defined by

$$\xi_{n+1}(\omega) \quad = \quad (1 - \alpha_n)\xi_n(\omega) + \alpha_n T^n(\omega, \xi_n(\omega)) \ \text{ for each } \omega \in \Omega, \tag{2.4}$$

where $0 \leq \alpha_n \leq 1$, $n = 1, 2, \ldots$, and $\xi_0\colon \Omega \to C$ is an arbitrary measurable mapping.

Since C is a convex set, it follows that for each n, ξ_n is a mapping from Ω to C.

Definition 2.3. The modified random Ishikawa iteration scheme is the sequences of functions $\{\xi_n\}$ and $\{\eta_n\}$ defined by

$$\begin{aligned} \xi_{n+1}(\omega) &= (1 - \alpha_n)\xi_n(\omega) + \alpha_n T^n(\omega, \eta_n(\omega)), \\ \eta_n(\omega) &= (1 - \beta_n)\xi_n(\omega) + \beta_n T^n(\omega, \xi_n(\omega)) \ \text{ for each } \omega \in \Omega, \end{aligned} \tag{2.5}$$

where $0 \leq \alpha_n, \beta_n \leq 1$, $n = 1, 2, \ldots$ and $\xi_0\colon \Omega \to C$ is an arbitrary measurable mapping. Also $\{\xi_n\}$ and $\{\eta_n\}$ are sequences of functions from Ω to C.

Now we define an implicit random iteration process with errors.

Definition 2.4. Let $\{T_1, T_2, \dots, T_N\}$ be a family of random asymptotically quasi-nonexpansive in the intermediate sense operators from $\Omega \times C \to C$, where C is a closed, convex subset of a separable Banach space E. Let $F = \bigcap_{i=1}^{N} RF(T_i) \neq \emptyset$, where $RF(T_i)$ is the set of all random fixed points of a random operator T_i for each $i \in \{1, 2, \dots, N\}$. Let $\xi_0 \colon \Omega \to C$ be any fixed measurable map, then the sequence of function $\{\xi_n\}$ defined by

$$\begin{aligned}
\xi_1(\omega) &= \alpha_1 \xi_0(\omega) + \beta_1 T_1(\omega, \xi_1(\omega)) + \gamma_1 f_1(\omega),\\
\xi_2(\omega) &= \alpha_2 \xi_1(\omega) + \beta_2 T_2(\omega, \xi_2(\omega)) + \gamma_2 f_2(\omega),\\
&\vdots\\
\xi_N(\omega) &= \alpha_N \xi_{N-1}(\omega) + \beta_N T_N(\omega, \xi_N(\omega)) + \gamma_N f_N(\omega),\\
\xi_{N+1}(\omega) &= \alpha_{N+1} \xi_N(\omega) + \beta_{N+1} T_1^2(\omega, \xi_{N+1}(\omega)) + \gamma_{N+1} f_{N+1}(\omega),\\
&\vdots\\
\xi_{2N}(\omega) &= \alpha_{2N} \xi_{2N-1}(\omega) + \beta_{2N} T_N^2(\omega, \xi_{2N}(\omega)) + \gamma_{2N} f_{2N}(\omega),\\
\xi_{2N+1}(\omega) &= \alpha_{2N+1} \xi_{2N}(\omega) + \beta_{2N+1} T_1^3(\omega, \xi_{2N+1}(\omega)) + \gamma_{2N+1} f_{2N+1}(\omega),\\
&\vdots
\end{aligned} \tag{2.6}$$

is called the implicit random iteration process with errors for a finite family of asymptotically quasi-nonexpansive in the intermediate sense random operators $\{T_1, T_2, \dots, T_N\}$.

Since each $n \geq 1$ can be written as $n = (k-1)N + i$, where $i = i(n) \in \{1, 2, \dots, N\}$, $k = k(n) \geq 1$ is a positive integer and $k(n) \to \infty$ as $n \to \infty$. Hence the above iteration process can be written in the following compact form:

$$\xi_n(\omega) \;=\; \alpha_n \xi_{n-1}(\omega) + \beta_n T_{i(n)}^{k(n)}(\omega, \xi_n(\omega)) + \gamma_n f_n(\omega) \quad \text{for each} \quad n \geq 1, \tag{2.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three appropriate real sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for $n = 1, 2, \dots$ and each $\{f_n(\omega)\}$ is a bounded sequence in C.

In the sequel we need the following lemma to prove our main results.

Lemma 2.5 ([18])**.** *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n) a_n + b_n, \qquad n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 ([22])**.** *Let E be a uniformly convex Banach space and $0 < a \leq t_n \leq b < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in E satisfying*

$$\limsup_{n\to\infty} \|x_n\| \leq r, \quad \limsup_{n\to\infty} \|y_n\| \leq r, \quad \lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r,$$

for some $r \geq 0$*. Then*

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

3. MAIN RESULTS

Theorem 3.1. *Let* E *be a real separable uniformly convex Banach space,* C *be a nonempty closed convex subset of* E*. Let* $\{T_i\}_{i=1}^{N}$ *be* N *uniformly L-Lipschitzian asymptotically quasi-nonexpansive in the intermediate sense random operators from* $\Omega \times C \to C$*. Let* $F = \bigcap_{i=1}^{N} RF(T_i) \neq \emptyset$*. Let* $\{\xi_n(\omega)\}$ *be the sequence defined by* (2.7)*. Put*

$$\begin{aligned} A_n(\omega) \quad = \quad \max\Big\{ \sup_{\xi(\omega)\in F,\ n\geq 1} & \Big(\|T_i^n(\omega, \xi_n(\omega)) - \xi(\omega)\| - \|\xi_n(\omega) - \xi(\omega)\| \Big) \\ & \vee 0 : i \in I \Big\}. \end{aligned} \tag{3.1}$$

Assume that $\sum_{n=1}^{\infty} A_n(\omega) < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\{\alpha_n\}$ *is a sequence of real numbers in* $(s, 1-s)$ *for some* $s \in (0,1)$*. Then:*

(a) $\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\|$ *exists for all* $\omega \in \Omega$*,*
(b) $\lim_{n\to\infty} d(\xi_n(\omega), F)$ *exists, where* $d(\xi_n(\omega), F) = \inf_{\xi(\omega)\in F} \|\xi_n(\omega) - \xi(\omega)\|$*,*
(c) $\lim_{n\to\infty} \|\xi_n(\omega) - T_n(\omega, \xi_n(\omega))\| = 0$ *for each* $\omega \in \Omega$*.*

Proof. Let $\xi(\omega) \in F$, where ξ is any measurable mapping from Ω to C. Using (2.7) and (3.1), we have

$$\|\xi_n(\omega) - \xi(\omega)\| = \|\alpha_n \xi_{n-1}(\omega) + \beta_n T_n^k(\omega, \xi_n(\omega)) + \gamma_n f_n(\omega) - \xi(\omega)\|, \tag{3.2}$$

where $n = (k-1)N + i$, $k = k(n)$, and $T_n = T_i(mod\ N) = T_i$. This implies that

$$\begin{aligned} \|\xi_n(\omega) - \xi(\omega)\| \leq{} & \alpha_n \|\xi_{n-1}(\omega) - \xi(\omega)\| + \beta_n \left\|T_i^k(\omega, \xi_n(\omega)) - \xi(\omega)\right\| + \\ & + \gamma_n \|f_n(\omega) - \xi(\omega)\| \leq \\ \leq{} & \alpha_n \|\xi_{n-1}(\omega) - \xi(\omega)\| + \beta_n \Big(\|\xi_n(\omega) - \xi(\omega)\| + A_n(\omega) \Big) + \\ & + \gamma_n \|f_n(\omega) - \xi(\omega)\| = \\ ={} & \alpha_n \|\xi_{n-1}(\omega) - \xi(\omega)\| + (1 - \alpha_n - \gamma_n) \|\xi_n(\omega) - \xi(\omega)\| + \\ & + \beta_n A_n(\omega) + \gamma_n \|f_n(\omega) - \xi(\omega)\| \leq \\ \leq{} & \alpha_n \|\xi_{n-1}(\omega) - \xi(\omega)\| + (1 - \alpha_n) \|\xi_n(\omega) - \xi(\omega)\| + \\ & + A_n(\omega) + \gamma_n \|f_n(\omega) - \xi(\omega)\|, \end{aligned} \tag{3.3}$$

which on simplifying, we have

$$
\begin{aligned}
\|\xi_n(\omega)-\xi(\omega)\| \le \|\xi_{n-1}(\omega)-\xi(\omega)\| + \frac{1}{\alpha_n}A_n(\omega)+ \\
+\frac{\gamma_n}{\alpha_n}\|f_n(\omega)-\xi(\omega)\| \le \\
\le \|\xi_{n-1}(\omega)-\xi(\omega)\| + \frac{1}{\alpha_n}A_n(\omega)+\frac{\gamma_n}{\alpha_n}M,
\end{aligned}
\tag{3.4}
$$

where $M=\sup_{n\ge 1}\|f_n(\omega)-\xi(\omega)\|$, since $\{f_n(\omega)\}$ is bounded sequence in C. Since $0<s<\alpha_n<1-s<1$, it follows from (3.4) that

$$
\|\xi_n(\omega)-\xi(\omega)\| \quad \le \quad \|\xi_{n-1}(\omega)-\xi(\omega)\| + \left\{\frac{1}{s}A_n(\omega)+\frac{M}{s}\gamma_n\right\}. \tag{3.5}
$$

Since, by hypothesis,

$$
\sum_{n=1}^{\infty}A_n(\omega)<\infty \quad \text{and} \quad \sum_{n=1}^{\infty}\gamma_n<\infty,
$$

we have

$$
\frac{1}{s}\sum_{n=1}^{\infty}A_n(\omega)+\frac{M}{s}\sum_{n=1}^{\infty}\gamma_n<\infty.
$$

In (3.5) taking the infimum over all $\xi(\omega)\in F$, we have

$$
d(\xi_n(\omega),F) \quad \le \quad d(\xi_{n-1}(\omega),F)+\left\{\frac{1}{s}A_n(\omega)+\frac{M}{s}\gamma_n\right\}. \tag{3.6}
$$

It follows from Lemma 2.5 that

$$
\lim_{n\to\infty}\|\xi_n(\omega)-\xi(\omega)\| \quad \text{and} \quad \lim_{n\to\infty}d(\xi_n(\omega),F) \quad \text{exist.}
$$

Without loss of generality we can assume that

$$
\lim_{n\to\infty}\|\xi_n(\omega)-\xi(\omega)\|=d, \tag{3.7}
$$

where $d\ge 0$ is some number. Since $\{\|\xi_n(\omega)-\xi(\omega)\|\}$ is a convergent sequence and so $\{\xi_n(\omega)\}$ is a bounded sequence in C.

Observe that

$$
\begin{aligned}
\|\xi_n(\omega)-\xi(\omega)\| =&\|\alpha_n\xi_{n-1}(\omega)+\beta_nT_i^k(\omega,\xi_n(\omega))+\gamma_nf_n(\omega)-\xi(\omega)\|= \\
=&\|\beta_n[T_i^k(\omega,\xi_n(\omega))-\xi(\omega)+\gamma_n(f_n(\omega)-\xi_{n-1}(\omega))]+ \\
&+(1-\beta_n)[\xi_{n-1}(\omega)-\xi(\omega)+\gamma_n(f_n(\omega)-\xi_{n-1}(\omega))]\|.
\end{aligned}
\tag{3.8}
$$

From $\sum_{n=1}^{\infty} \gamma_n < \infty$ and (3.7) it follows that

$$\begin{aligned} &\limsup_{n\to\infty} \|\xi_{n-1}(\omega) - \xi(\omega) + \gamma_n(f_n(\omega) - \xi_{n-1}(\omega))\| \leq \\ &\leq \limsup_{n\to\infty} \Big(\|\xi_{n-1}(\omega) - \xi(\omega)\| + \\ &\quad + \gamma_n \|f_n(\omega) - \xi_{n-1}(\omega)\| \Big) \leq \\ &\leq d, \end{aligned} \tag{3.9}$$

and hence

$$\begin{aligned} \limsup_{n\to\infty} &\left\|T_n^k(\omega, \xi_n(\omega)) - \xi(\omega) + \gamma_n(f_n(\omega) - \xi_{n-1}(\omega))\right\| \leq \\ &\leq \limsup_{n\to\infty} \Big(\left\|T_n^k(\omega, \xi_n(\omega)) - \xi(\omega)\right\| + \\ &\quad + \gamma_n \|f_n(\omega) - \xi_{n-1}(\omega)\| \Big) \leq \\ &\leq \limsup_{n\to\infty} \Big(\|\xi_n(\omega)) - \xi(\omega)\| + A_n(\omega)\Big) + \\ &\quad + \limsup_{n\to\infty} \Big(\gamma_n \|f_n(\omega) - \xi_{n-1}(\omega)\| \Big) \leq \\ &\leq d, \end{aligned} \tag{3.10}$$

where $n = (k-1)N + i$.

Therefore, from (3.7)–(3.10) and Lemma 2.6 we have that

$$\lim_{n\to\infty} \left\|T_n^k(\omega, \xi_n(\omega)) - \xi_{n-1}(\omega)\right\| = 0 \tag{3.11}$$

for each $\omega \in \Omega$.

Moreover, since

$$\begin{aligned} &\|\xi_n(\omega) - \xi_{n-1}(\omega)\| = \\ &= \left\|\alpha_n \xi_{n-1}(\omega) + \beta_n T_n^k(\omega, \xi_n(\omega)) + \gamma_n f_n(\omega) - \xi_{n-1}(\omega)\right\| = \\ &= \left\|\beta_n [T_n^k(\omega, \xi_n(\omega)) - \xi_{n-1}(\omega)] + \gamma_n [f_n(\omega) - \xi_{n-1}(\omega)]\right\| \leq \\ &\leq \beta_n \left\|T_n^k(\omega, \xi_n(\omega)) - \xi_{n-1}(\omega)\right\| + \\ &\quad + \gamma_n \|f_n(\omega) - \xi_{n-1}(\omega)\|, \end{aligned} \tag{3.12}$$

hence by (3.11), we obtain

$$\lim_{n\to\infty} \|\xi_n(\omega) - \xi_{n-1}(\omega)\| = 0 \tag{3.13}$$

for each $\omega \in \Omega$ and $\|\xi_n(\omega) - \xi_{n+l}(\omega)\| \to 0$ as $n \to \infty$ and for each $\omega \in \Omega$ and $l < 2N$. Now, for $n > N$, we have

$$\begin{aligned} \|\xi_{n-1}(\omega) - T_n(\omega, \xi_n(\omega))\| &\leq \left\|\xi_{n-1}(\omega) - T_n^k(\omega, \xi_n(\omega))\right\| + \\ &\quad + \left\|T_n^k(\omega, \xi_n(\omega)) - T_n(\omega, \xi_n(\omega))\right\| \leq \\ &\leq \left\|\xi_{n-1}(\omega) - T_n^k(\omega, \xi_n(\omega))\right\| + \\ &\quad + L\left\|T_n^{k-1}(\omega, \xi_n(\omega)) - \xi_n(\omega)\right\| \leq \\ &\leq \left\|\xi_{n-1}(\omega) - T_n^k(\omega, \xi_n(\omega))\right\| + \\ &\quad + L\left\|T_n^{k-1}(\omega, \xi_n(\omega)) - T_{n-N}^{k-1}(\omega, \xi_{n-N}(\omega))\right\| + \\ &\quad + L\Big[\left\|T_{n-N}^{k-1}(\omega, \xi_{n-N}(\omega)) - \xi_{(n-N)-1}(\omega)\right\| + \\ &\quad + \left\|\xi_{(n-N)-1}(\omega) - \xi_n(\omega)\right\|\Big]. \end{aligned} \tag{3.14}$$

Since for each $n > N$, $n \equiv (n-N) \ mod \ N$. Thus $T_n = T_{n-N}$, therefore

$$\begin{aligned} &\|\xi_{n-1}(\omega) - T_n(\omega, \xi_n(\omega))\| \leq \\ &\leq \left\|\xi_{n-1}(\omega) - T_n^k(\omega, \xi_n(\omega))\right\| + L^2\|\xi_n(\omega) - \xi_{n-N}(\omega)\| + \\ &\quad + L\left\|T_{n-N}^{k-1}(\omega, \xi_{n-N}(\omega)) - \xi_{(n-N)-1}(\omega)\right\| + \\ &\quad + L\left\|\xi_{(n-N)-1}(\omega) - \xi_n(\omega)\right\|. \end{aligned} \tag{3.15}$$

Using (3.11) and (3.13), we obtain

$$\lim_{n\to\infty} \|\xi_{n-1}(\omega) - T_n(\omega, \xi_n(\omega))\| = 0 \tag{3.16}$$

for each $\omega \in \Omega$. Now

$$\|\xi_n(\omega) - T_n(\omega, \xi_n(\omega))\| \leq \|\xi_{n-1}(\omega) - \xi_n(\omega)\| + \|\xi_{n-1}(\omega) - T_n(\omega, \xi_n(\omega))\|. \tag{3.17}$$

By (3.13), (3.16) and (3.17), we have

$$\lim_{n\to\infty} \|\xi_n(\omega) - T_n(\omega, \xi_n(\omega))\| = 0 \tag{3.18}$$

for each $\omega \in \Omega$. This completes the proof. □

Theorem 3.2. *Let E be a real separable uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N asymptotically quasi-nonexpansive in the intermediate sense random operators from $\Omega \times C \to C$. Let $F = \bigcap_{i=1}^N RF(T_i) \neq \emptyset$. Let $\{\xi_n(\omega)\}$ be the sequence defined by (2.7). Put*

$$\begin{aligned} A_n(\omega) \quad = \quad \max\Big\{ \sup_{\xi(\omega)\in F,\ n\geq 1} \Big(\|T_i^n(\omega, \xi_n(\omega)) - \xi(\omega)\| - \|\xi_n(\omega) - \xi(\omega)\| \Big) \\ \vee 0 : i \in I\Big\}. \end{aligned}$$

Assume that $\sum_{n=1}^{\infty} A_n(\omega) < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\{\alpha_n\}$ *is a sequence of real numbers in* $(s, 1-s)$ *for some* $s \in (0,1)$. *Then the sequence* $\{\xi_n(\omega)\}$ *converges to a common random fixed point of random operators* $\{T_i : i = 1, 2, \dots, N\}$ *if and only if* $\liminf_{n\to\infty} d(\xi_n(\omega), F) = 0$.

Proof. If for some $\xi \in F$, $\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\| = 0$ for each $\omega \in \Omega$, then obviously $\liminf_{n\to\infty} d(\xi_n(\omega), F) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(\xi_n(\omega), F) = 0$. Then we have

$$\lim_{n\to\infty} d(\xi_n(\omega), F) = 0 \quad \text{for each } \omega \in \Omega.$$

Thus for any $\varepsilon > 0$ there exists a positive integer N_1 such that for $n \geq N_1$,

$$d(\xi_n(\omega), F) < \frac{\varepsilon}{6} \quad \text{for each} \quad \omega \in \Omega. \tag{3.19}$$

Again since $\sum_{n=1}^{\infty} A_n(\omega) < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$ imply that there exists positive integers N_2 and N_3 such that

$$\sum_{j=n}^{\infty} A_j(\omega) < \frac{s\varepsilon}{6} \quad \text{for each} \quad n \geq N_2 \tag{3.20}$$

and

$$\sum_{j=n}^{\infty} \gamma_j < \frac{s\varepsilon}{6M} \quad \text{for each} \quad n \geq N_3. \tag{3.21}$$

Let $N = \max\{N_1, N_2, N_3\}$. It follows from (3.5) that

$$\|\xi_n(\omega) - \xi(\omega)\| \leq \|\xi_{n-1}(\omega) - \xi(\omega)\| + \frac{1}{s} A_n(\omega) + \frac{M}{s}\gamma_n. \tag{3.22}$$

Now, for each $m, n \geq N$ and each $\omega \in \Omega$, we have

$$\begin{aligned}
\|\xi_n(\omega) - \xi_m(\omega)\| &\leq \|\xi_n(\omega) - \xi(\omega)\| + \|\xi_m(\omega) - \xi(\omega)\| \leq \\
&\leq \|\xi_N(\omega) - \xi(\omega)\| + \frac{1}{s}\sum_{j=N+1}^{n} A_j(\omega) + \frac{M}{s}\sum_{j=N+1}^{n} \gamma_j + \\
&\quad + \|\xi_N(\omega) - \xi(\omega)\| + \frac{1}{s}\sum_{j=N+1}^{n} A_j(\omega) + \frac{M}{s}\sum_{j=N+1}^{n} \gamma_j \leq \\
&\leq 2\,\|\xi_N(\omega) - \xi(\omega)\| + \frac{2}{s}\sum_{j=N+1}^{n} A_j(\omega) + \frac{2M}{s}\sum_{j=N+1}^{n} \gamma_j < \\
&< 2 \cdot \frac{\varepsilon}{6} + \frac{2}{s} \cdot \frac{s\varepsilon}{6} + \frac{2M}{s} \cdot \frac{s\varepsilon}{6M} < \varepsilon.
\end{aligned} \tag{3.23}$$

This implies that $\{\xi_n(\omega)\}$ is a Cauchy sequence for each $\omega \in \Omega$. Therefore $\xi_n(\omega) \to p(\omega)$ for each $\omega \in \Omega$, and $p\colon \Omega \to C$, being the limit of the sequence of measurable functions, is also measurable. Now, $\lim_{n\to\infty} d(\xi_n(\omega), F) = 0$ for each $\omega \in \Omega$, and the set F is closed, we have $p(\omega) \in F$, that is, p is a common random fixed point of the mappings $\{T_i : i = 1, 2, \dots, N\}$. This completes the proof. □

Recall the following definitions.

A mapping $T\colon C \to C$ where C is a subset of a Banach space E with $F(T) \neq \emptyset$ is said to satisfy *condition* (A) (see [25]) if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that for all $x \in C$

$$\|x - Tx\| \geq f(d(x, F(T))),$$

where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

A family $\{T_i\}_{i=1}^N$ of N self-mappings of C with $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ is said to satisfy:

(1) *condition* (B) on C ([10]) if there is a nondecreasing function $f\colon [0,1] \to [0,1]$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ and all $x \in C$ such that

$$\max_{1\leq l\leq N} \|x - T_l x\| \geq f(d(x, \mathcal{F})),$$

(2) *condition* $(\overline{C})$ on C ([9]) if there is a nondecreasing function $f\colon [0,1] \to [0,1]$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ and all $x \in C$ such that

$$\|x - T_l x\| \geq f(d(x, \mathcal{F}))$$

for at least one T_l, $l = 1, 2, \dots, N$; or in other words at least one of the $T_l's$ satisfies *condition* (A).

Condition (B) reduces to condition (A) when all but one of the $T_i's$ are identities. Also conditions (B) and $(\overline{C})$ are equivalent (see [9]).

A random operator $T\colon \Omega \times C \to C$ is said to satisfy condition (A), condition (B), condition $(\overline{C})$, if the map $T(\omega, .)\colon C \to C$ is so, for each $\omega \in \Omega$.

Let $T\colon \Omega \times C \to C$ be a random map. Then T is said to be:

(i) a completely continuous random operator if for a sequence of measurable mappings ξ_n from $\Omega \to C$ such that $\{\xi_n(\omega)\}$ is bounded for each $\omega \in \Omega$ then $T(\omega, \xi_n(\omega))$ has convergent subsequence for each $\omega \in \Omega$.

(ii) demicompact random operator if for a sequence of measurable mappings ξ_n from $\Omega \to C$ such that $\{\xi_n(\omega) - T(\omega, \xi_n(\omega))\}$ converges, there exists a subsequence say $\{\xi_{n_j}(\omega)\}$ of $\{\xi_n(\omega)\}$ that converges strongly to some $\xi(\omega)$ for each $\omega \in \Omega$, where ξ is a measurable mapping from Ω to C.

(iii) semi-compact random operator if for a sequence of measurable mappings ξ_n from $\Omega \to C$ such that $\lim_{n\to\infty} \|\xi_n(\omega) - T(\omega, \xi_n(\omega))\|\, 0$, for every $\omega \in \Omega$, there exists a subsequence say $\{\xi_{n_j}(\omega)\}$ of $\{\xi_n(\omega)\}$ that converges strongly to some $\xi(\omega)$ for each $\omega \in \Omega$, where ξ is a measurable mapping from Ω to C.

Senter and Dotson [25] established a relation between condition (A) and demicompactness. They actually showed that condition (A) is weaker than demicompactness for a nonexpansive mapping.

Every compact operator is demicompact. Since every completely continuous mapping $T\colon C \to C$ is continuous and demicompact, so it satisfies *condition* (A).

Therefore to study strong convergence of $\{x_n\}$ defined by (3.1) we use *condition* $(\overline{C})$ instead of the complete continuity of the mappings $T_1, T_2, \ldots, T_N$.

Theorem 3.3. *Let E be a real separable uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N uniformly L-Lipschitzian asymptotically quasi-nonexpansive in the intermediate sense random operators as in Theorem 3.1 which satisfy condition $(\overline{C})$. Let $F = \bigcap_{i=1}^N RF(T_i) \neq \emptyset$. Let $\{\xi_n(\omega)\}$ be the sequence defined by (2.7). Put*

$$A_n(\omega) \quad = \quad \max\Big\{ \sup_{\xi(\omega)\in F,\ n\geq 1} \Big(\|T_i^n(\omega, \xi_n(\omega)) - \xi(\omega)\| - \|\xi_n(\omega) - \xi(\omega)\| \Big) \vee 0 : i \in I \Big\}.$$

Assume that $\sum_{n=1}^\infty A_n(\omega) < \infty$, $\sum_{n=1}^\infty \gamma_n < \infty$ and $\{\alpha_n\}$ be a sequence as in Theorem 3.1. Then the sequence $\{\xi_n(\omega)\}$ converges to a common random fixed point of random operators $\{T_i : i = 1, 2, \ldots, N\}$.

Proof. By Theorem 3.1, we know that $\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\|$ and $\lim_{n\to\infty} d(\xi_n(\omega), F)$ exist. Let one of $T_i's$, say T_l, $l \in \{1, 2, \ldots, N\}$ satisfy condition (A), also by Theorem 3.1, we have $\lim_{n\to\infty} \|\xi_n(\omega) - T_l(\omega, \xi_n(\omega))\| = 0$, so we have $\lim_{n\to\infty} f(d(\xi_n(\omega), F)) = 0$, for each $\omega \in \Omega$. By the property of f and the fact that $\lim_{n\to\infty} d(\xi_n(\omega), F)$ exists, we have $\lim_{n\to\infty} d(\xi_n(\omega), F) = 0$ for each $\omega \in \Omega$. By Theorem 3.2, we obtain $\{\xi_n\}$ converges strongly to a common random fixed point in F. This completes the proof. □

Theorem 3.4. *Let E be a real separable uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N uniformly L-Lipschitzian asymptotically quasi-nonexpansive in the intermediate sense random operators as in Theorem 3.1 such that one of the mappings in $\{T_1, T_2, \ldots, T_N\}$ is semi-compact. Let $F = \bigcap_{i=1}^N RF(T_i) \neq \emptyset$. Let $\{\xi_n(\omega)\}$ be the sequence defined by (2.7). Put*

$$A_n(\omega) \quad = \quad \max\Big\{ \sup_{\xi(\omega)\in F,\ n\geq 1} \Big(\|T_i^n(\omega, \xi_n(\omega)) - \xi(\omega)\| - \|\xi_n(\omega) - \xi(\omega)\| \Big) \vee 0 : i \in I \Big\}.$$

Assume that $\sum_{n=1}^\infty A_n(\omega) < \infty$, $\sum_{n=1}^\infty \gamma_n < \infty$ and $\{\alpha_n\}$ be a sequence as in Theorem 3.1. Then the sequence $\{\xi_n(\omega)\}$ converges to a common random fixed point of random operators $\{T_i : i = 1, 2, \ldots, N\}$.

Proof. For any given $\xi(\omega) \in F$, we note that

$$\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\| = d, \tag{3.24}$$

where $d \geq 0$. By Theorem 3.1, we know that

$$\lim_{n\to\infty} \|\xi_n(\omega) - T_n(\omega, \xi_n(\omega))\| = 0 \tag{3.25}$$

for each $\omega \in \Omega$. Consequently, for any $j \in \{1, 2, \ldots, N\}$,

$$\begin{aligned}\|\xi_n(\omega) - T_{n+j}(\omega, \xi_n(\omega))\| &\leq \|\xi_n(\omega) - \xi_{n+j}(\omega)\| + \\ &\quad + \|\xi_{n+j}(\omega) - T_{n+j}(\omega, \xi_{n+j}(\omega))\| + \\ &\quad + \|T_{n+j}(\omega, \xi_{n+j}(\omega)) - T_{n+j}(\omega, \xi_n(\omega))\| \leq \\ &\leq (1 + L)\, \|\xi_n(\omega) - \xi_{n+j}(\omega)\| + \\ &\quad + \|\xi_{n+j}(\omega) - T_{n+j}(\omega, \xi_{n+j}(\omega))\| \to 0\end{aligned}$$

as $n \to \infty$ for each $\omega \in \Omega$ and $j \in \{1, 2, \ldots, N\}$.

Consequently, $\|\xi_n(\omega) - T_j(\omega, \xi_n(\omega))\| \to 0$ as $n \to \infty$ for each $\omega \in \Omega$ and $j \in \{1, 2, \ldots, N\}$. Assume that T_j is a semi-compact random operator. Therefore, there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ and a measurable mapping $\xi_0 \colon \Omega \to C$ such that ξ_{n_k} converges pointwise to ξ_0. Now

$$\lim_{n\to\infty} \|\xi_{n_k}(\omega) - T_j(\omega, \xi_{n_k}(\omega))\| = \|\xi_0(\omega) - T_j(\omega, \xi_0(\omega))\| = 0$$

for each $\omega \in \Omega$ and $j \in \{1, 2, \ldots, N\}$. It implies that $\xi_0 \in F$, and so $\liminf_{n\to\infty} d(\xi_n(\omega), F) = 0$. Hence, by Theorem 3.2, we obtain that $\{\xi_n\}$ converges strongly to a common random fixed point in F. This completes the proof. □

Remark 3.5. Our results extend and improve the corresponding results of Plubtieng *et al.* [20] to the case of a more general class of asymptotically quasi-nonexpansive random operators considered in this paper.

Remark 3.6. Our results also extend and improve the corresponding results of Saluja [21] to the case of a more general class of asymptotically quasi-nonexpansive random operator and implicit random iteration process with errors for a finite family of random operators considered in this paper.

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