Opuscula Mathematica • Vol. 32 • No. 2 • 2012

http://dx.doi.org/10.7494/OpMath.2012.32.2.317

AN APPLICATION OF THE CHOQUET THEOREM TO THE STUDY OF RANDOMLY-SUPERINVARIANT MEASURES

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Abstract. Given a real valued random variable Θ we consider Borel measures μ on $\mathcal{B}(\mathbb{R})$, which satisfy the inequality $\mu(B) \geq E\mu(B-\Theta)$ $(B \in \mathcal{B}(\mathbb{R}))$ (or the integral inequality $\mu(B) \geq \int_{-\infty}^{\infty} \mu(B-h)\gamma(dh)$). We apply the Choquet theorem to obtain an integral representation of measures μ satisfying this inequality. We give integral representations of these measures in the particular cases of the random variable Θ.

Keywords: backward translation operator, backward difference operator, integral inequality, extreme point.

Mathematics Subject Classification: 60E15, 26D10.

1. INTRODUCTION

In this paper we are going to study the solvability of an integral inequality of the type

$$\mu(B) \geq \int_{-\infty}^{\infty} \mu(B-h)\gamma(dh), \quad B \in \mathcal{B}(\mathbb{R}), \tag{1.1}$$

where γ is a probability distribution on the σ-field of Borel subsets of $\mathbb{R}$, $\mathcal{B}(\mathbb{R})$ (shortly on $\mathbb{R}$) and μ is a Borel measure on $\mathbb{R}$. Considering the distribution function, $F(x)$, corresponding to μ (under some additional assumptions), the inequality (1.1) implies

$$F(x) \geq \int_{-\infty}^{\infty} F(x-h)\gamma(dh), \quad x \in \mathbb{R},$$

where $F(x)$ is a non-decreasing function. Let us mention that the theory of integral equations and inequalities has many useful applications in describing numerous events and problems in the real word. Various types of integral operators were investigated in several papers (see [1–4]). We first recall some definitions and give some preliminary results.

We recall notion of the (*backward*) *translation* and (*backward*) *difference* operators τ_h and Δ_h. For a fixed number h these operators, acting on real functions $F\colon \mathbb{R} \to \mathbb{R}$ are defined by

$$\tau_h F(x) = F(x-h), \quad \Delta_h F(x) = F(x) - F(x-h), \quad x \in \mathbb{R}, \tag{1.2}$$

respectively. As is common and convenient, τ_h and Δ_h will also stand for operators acting on Borel measures μ on $\mathbb{R}$ as follows

$$\tau_h \mu(B) = \mu(B-h), \quad \Delta_h \mu(B) = \mu(B) - \mu(B-h), \quad B \in \mathcal{B}(\mathbb{R}). \tag{1.3}$$

Replacing in (1.2) and (1.3) the real number h by a random variable Θ and taking expectations, we obtain the randomized translation and the randomized difference operators $E\tau_\Theta$ and $E\Delta_\Theta$:

$$\begin{aligned} E\tau_\Theta F(x) &= EF(x-\Theta), & E\Delta_\Theta F(x) &= F(x) - EF(x-\Theta), \\ E\tau_\Theta \mu(B) &= E\mu(B-\Theta), & E\Delta_\Theta \mu(B) &= \mu(B) - E\mu(B-\Theta). \end{aligned}$$

Throughout this paper Θ will denote a real valued random variable with the distribution μ_Θ concentrated on $[0,\infty)$.

Let $M = M(\Theta)$ be the set of all Borel measures μ on $\mathbb{R}$ such that $\mu((-\infty,x)) < \infty$, $x \in \mathbb{R}$, and

$$E\Delta_\Theta \mu \geq 0. \tag{1.4}$$

We will call μ that satisfies (1.4) Θ-*superinvariant*. The probability measure concentrated at x ($x \in \mathbb{R}$) will be denoted by δ_x. Note that, if $\mu_\Theta = \delta_h$ ($h > 0$), then μ is Θ superinvariant if

$$\Delta_h \mu \geq 0. \tag{1.5}$$

Thus Θ-superinvariant measures can be regarded as a randomized version of measures satisfying (1.5) (or randomly-superinvariant with respect to Θ). In [9] we can find a characterization of measures which satisfy (1.5) for all $h \in H$, where $H \subset [0,\infty)$. The measures satisfying the inequality of type (1.5) appear in probability theory in the study of the classes L_c (see [6]). Then the Lévy spectral measures corresponding to infinitely divisible distributions from the class L_c satisfy a multiplicative version of the inequality (1.5).

Let μ be a Borel measure on $\mathbb{R}$. Note that if the condition (1.5) holds for $h_0 > 0$, then (1.5) also holds for all $h \in \{nh_0\}_{n=0}^\infty$. It is not difficult to check that for the measure $\mu = \sum_{n=0}^\infty \delta_{nh_0}$ we have that (1.5) holds if and only if $h \in \{nh_0\}_{n=0}^\infty$.

Assume now that (1.5) holds for all $h > 0$. Let $F(x) = F_\mu(x) = \mu((-\infty,x))$ be a distribution function corresponding to μ. Let B be a Borel set of the form

$$B = [x-h_1, x), \tag{1.6}$$

where $h_1 > 0$, $x \in \mathbb{R}$. Then $\mu(B) = F(x) - F(x-h_1) = \Delta_{h_1}F(x)$. By (1.5), this gives

$$\Delta_h \Delta_{h_1} F(x) \geq 0, \quad h_1, h_2 > 0, x \in \mathbb{R}. \tag{1.7}$$

Taking into account that $F(x)$ is a non-decreasing function, we have that (1.7) holds if and only if $F(x)$ is convex (see [5]). Moreover, it is not difficult to check that $\Delta_h\mu(B) \geq 0$ holds for all $B \in \mathcal{B}(\mathbb{R})$ if and only if this condition is satisfied for all B of the form (1.6). Consequently, we obtain that (1.5) holds for all $h > 0$ if and only if the distribution function corresponding to μ is convex. Obviously, if (1.5) holds for all $h > 0$, then for any random variable Θ the inequality (1.4) is satisfied.

In Section 4 we will prove that for the measure $\mu(du) = \delta_0(u)du + \chi_{(0,\infty)}(u)du$, there exists a random variable Θ, such that μ is Θ-superinvariant, however there exists no $h > 0$ for which (1.5) is satisfied (see Remark 4.5).

It is not difficult to prove that $\mu \in M(\Theta)$ if and only if $E(\Delta_\Theta F_\mu(x))$ is a non-decreasing function. In [8] a characterization of non-decreasing functions F such that $E\Delta_\Theta F(x)$ is a non-decreasing function, can be found. In this paper we study Θ-superinvariant measures using a different method. We apply the Choquet theorem to obtain an integral representation of a Θ-superinvariant measure in the general case without any additional assumptions on Θ. In addition to illustrating how our formula works in practice, it provides explicit formulas for the particular cases of Θ.

2. THE CLASS $M(\Theta)$

Let $\mu \in M(\Theta)$ and let $B \in \mathcal{B}(\mathbb{R})$. By the definition of the operators $E\tau_\Theta$ and $E\Delta_\Theta$ we have that

$$E\Delta_\Theta\mu(B) = \mu(B) - E\tau_\Theta\mu(B). \tag{2.1}$$

Therefore, the measure μ can be written in the form

$$\mu(B) = E\tau_\Theta\mu(B) + E\Delta_\Theta\mu(B). \tag{2.2}$$

From (2.1), (2.2) and the definition of $M(\Theta)$ we immediately obtain the following two lemmas.

Lemma 2.1. *$\mu \in M(\Theta)$ if and only if $E\Delta_\Theta\mu$ is a Borel measure.*

Lemma 2.2. *$\mu \in M(\Theta)$ if and only if μ can be written in the form*

$$\mu(B) = E\tau_\Theta\mu(B) + \nu(B), \quad B \in \mathcal{B}(\mathbb{R}),$$

where ν is a Borel measure on $\mathbb{R}$. Moreover, we have that

$$\nu(B) = E\Delta_\Theta\nu(B).$$

Theorem 2.3. *Let $\Theta_1, \Theta_2, \ldots$ be independent copies of a random variable Θ. Then $\mu \in M(\Theta)$ if and only if μ is of the form*

$$\mu(B) = \sum_{j=2}^{\infty} E\tau_{\Theta_j} \ldots E\tau_{\Theta_2}\nu(B) + \nu(B), \quad B \in \mathcal{B}(\mathbb{R}), \tag{2.3}$$

where ν is a Borel measure on $\mathbb{R}$. Moreover, we have

$$\nu(B) = E\Delta_{\Theta_1}\mu(B). \tag{2.4}$$

Proof. ($\Leftarrow$) Let μ be a measure of the form (2.3). Then

$$\begin{aligned}E\tau_{\Theta_1}\mu(B) &= \sum_{j=2}^{\infty} E\tau_{\Theta_j}\dots E\tau_{\Theta_2}E\tau_{\Theta_1}\nu(B) + E\tau_{\Theta_1}\nu(B) = \\ &= \sum_{j=1}^{\infty} E\tau_{\Theta_j}\dots E\tau_{\Theta_1\nu(B)} = \\ &= \sum_{j=2}^{\infty} E\tau_{\Theta_j}\dots E\tau_{\Theta_2}\nu(B).\end{aligned}$$

Consequently μ can be written in the form $\mu(B) = E\tau_{\Theta_1}\mu(B)+\nu(B)$. From Lemma 2.2 we conclude that $\mu \in M(\Theta)$ and the formula (2.4) holds.
($\Rightarrow$) Let $\mu \in M(\Theta)$. Let $\Theta_1, \Theta_2, \dots$ be independent copies of Θ. From Lemma 2.2 and taking expectations $E\tau_{\Theta_2}(\cdot), E\tau_{\Theta_2}(\cdot), \dots, E\tau_{\Theta_n}(\cdot)$, we obtain

$$\begin{aligned}\mu(B) &= E\tau_{\Theta_1}\mu(B) + E\Delta_{\Theta_1}\mu(B),\\ E\tau_{\Theta_2}\mu(B) &= E\tau_{\Theta_2}E\tau_{\Theta_1}\mu(B) + E\tau_{\Theta_2}E\Delta_{\Theta_1}\mu(B),\\ E\tau_{\Theta_3}E\tau_{\Theta_2}\mu(B) &= E\tau_{\Theta_3}E\tau_{\Theta_2}E\tau_{\Theta_1}\mu(B) + E\tau_{\Theta_3}E\tau_{\Theta_2}E\Delta_{\Theta_1}\mu(B),\\ E\tau_{\Theta_n}\dots E\tau_{\Theta_2}\mu(B) &= E\tau_{\Theta_n}\dots E\tau_{\Theta_2}E\tau_{\Theta_1}\mu(B) + E\tau_{\Theta_n}\dots E\tau_{\Theta_2}E\Delta_{\Theta_1}\mu(B).\end{aligned}$$

Taking into account that $E\tau_{\Theta_1}\mu(B) = E\tau_{\Theta_2}\mu(B)$, $E\tau_{\Theta_2}E\tau_{\Theta_1}\mu(B) = E\tau_{\Theta_3}E\tau_{\Theta_2}\mu(B)$, ..., $E\tau_{\Theta_{n-1}}\dots E\tau_{\Theta_1}\mu(B) = E\tau_{\Theta_n}\dots E\tau_{\Theta_2}\mu(B)$, we obtain

$$\begin{aligned}\mu(B) = E\Delta_{\Theta_1}\mu(B) + E\tau_{\Theta_2}E\Delta_{\Theta_1}\mu(B) + \dots\\ + E\tau_{\Theta_n}\dots E\tau_{\Theta_2}E\Delta_{\Theta_1}\mu(B) + E\tau_{\Theta_n}\dots E\tau_{\Theta_1}\mu(B).\end{aligned} \tag{2.5}$$

It is not difficult to prove that $E\tau_{\Theta_n}\dots E\tau_{\Theta_1}\mu(B) \xrightarrow[n\to\infty]{} 0$ (when $\mu(B) < \infty$). Letting $n \to \infty$, (2.5) gives (2.3) with ν given by (2.4). This completes the proof of the theorem. □

3. THE CLASS $M(\Theta, I_0)$

Let $I_a = (-\infty, a)$ and let $\tilde{I}_a = [-\infty, a]$, $a \in \mathbb{R}$. Let $M(\Theta, \tilde{I}_a)$ be the set of all measures on $\tilde{I}_a$ such that $E\Delta_\Theta\mu(B) \geq 0$, for all $B \in \mathcal{B}(\tilde{I}_a)$ (assuming $\mu([-\infty, x]) < \infty$, $-\infty \leqslant x \leqslant a$). Let $M(\Theta, I_a)$ be the set of all measures $\mu \in M(\Theta, \tilde{I}_a)$ for which $\mu(\{-\infty, a\}) = 0$. Similarly, we define the set $M(\Theta, (-\infty, a])$.

Taking into account the definitions of $M(\Theta)$ and $M(\Theta, (-\infty, a])$, and Theorem 2.3 we immediately obtain the following two lemmas.

Lemma 3.1. *Let $\Theta_1, \Theta_2, \dots$ be independent copies of a random variable Θ. Then $\mu \in M(\Theta, (-\infty, a))$ if and only if*

$$\mu(B) = \sum_{j=2}^{\infty} E\tau_{\Theta_j}\dots E\tau_{\Theta_2}\nu(B) + \nu(B), \tag{3.1}$$

for all $B \in \mathcal{B}((-\infty, a))$, where ν is a Borel measure on $(-\infty, a)$. Moreover, we have that

$$\nu(B) = E\Delta_{\Theta_1}\mu(B).$$

Lemma 3.2. *If $\mu \in M(\Theta)$, then $\mu\big|_{(-\infty,a]} \in M(\Theta, (-\infty, a])$ for all $a \in \mathbb{R}$.*

Lemma 3.3. *If $\mu \in M(\Theta, (-\infty, a))$, then there exists $\lambda \in M(\Theta)$ such that $\lambda\big|_{(-\infty,a)} = \mu$.*

Proof. Let $\mu \in M(\Theta, (-\infty, a))$. From Lemma 3.1, μ is of the form (3.1), where ν is a Borel measure on $(-\infty, a)$. Let λ be a Borel measure on $\mathbb{R}$ given by the formula

$$\lambda(B) = \sum_{j=2}^{\infty} E\tau_{\Theta_j} \ldots E\tau_{\Theta_2}\nu(B) + \nu(B), \tag{3.2}$$

for all $B \in \mathcal{B}(\mathbb{R})$. By Theorem 2.3, $\lambda \in M(\Theta)$. Taking into account (3.1) and (3.2), we have that $\mu(B) = \lambda(B)$ for $B \in \mathcal{B}((-\infty, a))$, hence $\lambda\big|_{(-\infty,a)} = \mu$. This completes the proof. □

Consider now $a = 0$. Let $K(\Theta, \tilde{I}_0)$ be the subset of $M(\Theta, \tilde{I}_0)$ consisting of all probability measures. By $e(K(\Theta, \tilde{I}_0))$ we denote the set of extreme points of $K(\Theta, \tilde{I}_0)$. Let $\nu_{(x)}$ $(-\infty < x < 0)$ be the measure given by the formula

$$\nu_{(x)} = A_x\delta_x, \tag{3.3}$$

where

$$A_x = \left[\sum_{j=2}^{\infty} E\tau_{\Theta_j} \ldots E\tau_{\Theta_2}\delta_x((-\infty, 0)) + \delta_x((-\infty, 0))\right]^{-1}. \tag{3.4}$$

Let $\mu_{(x)}(-\infty < x < 0)$ be the measure given by the formula

$$\mu_{(x)}(B) = \sum_{j=2}^{\infty} E\tau_{\Theta_j} \ldots E\tau_{\Theta_2}\nu_{(x)}(B) + \nu_{(x)}(B), \tag{3.5}$$

for $B \in \mathcal{B}(I_0)$. By (3.3), (3.4), (3.5) and Lemma 3.1, $\mu_{(x)} \in K(\Theta, I_0)$ $(-\infty < x < 0)$. Let $\mu_{(-\infty)} = \delta_{-\infty}$ and $\mu_{(0)} = \delta_0$.

By the definition of $K(\Theta, \tilde{I}_0)$, we can see that $\mu_{(-\infty)}, \mu_{(0)} \in K(\Theta, \tilde{I}_0)$. Indeed, clearly $\tau_h\mu_{(-\infty)} = \mu_{(-\infty)}$ and $\tau_h\mu_{(0)} = 0$, for all $h > 0$, hence $E\tau_\Theta\mu_{(-\infty)} = \mu_{(-\infty)}$ and $E\tau_\Theta\mu_{(0)} = 0$. Consequently, $\mu_{(-\infty)} = E\tau_\Theta\mu_{(-\infty)} + \nu_{(-\infty)}$ and $\mu_{(0)} = E\tau_\Theta\mu_{(0)} + \nu_{(0)}$, where $\nu_{(-\infty)} = 0$ and $\nu_{(0)} = \delta_0$. This proves that $\mu_{(-\infty)}, \mu_{(0)} \in K(\Theta, \tilde{I}_0)$.

It is not difficult to prove the following lemmas.

Lemma 3.4. *The extreme points of $K(\Theta, \tilde{I}_0)$ are measures concentrated on one of the following sets: $\{-\infty\}$, $(-\infty, 0)$ and $\{0\}$.*

We define the sets B_1, B_2 and B_3 by setting $B_1 = \{-\infty\}$, $B_2 = (-\infty, 0)$ and $B_3 = \{0\}$.

Lemma 3.5. *Let $\mu \in K(\Theta, \tilde{I}_0)$. Then:*

(i) *for $i = 2, 3$, μ is concentrated on B_i if and only if $E\Delta_\Theta \mu$ is concentrated on B_i,*
(ii) *μ is concentrated on B_1 if and only if $E\tau_\Theta \mu$ is concentrated on B_1.*

Theorem 3.6.

$$e(K(\Theta, \tilde{I}_0)) = \{\mu_{(x)} \colon x \in [-\infty, 0]\}.$$

Proof. Let $\mu \in e(K(\Theta, \tilde{I}_0))$. From Lemma 3.4 μ is concentrated on one of the following sets: $\{-\infty\}$, $\{0\}$ and $(-\infty, 0)$. If μ is concentrated on $\{-\infty\}$ or $\{0\}$, then μ equals $\mu_{(-\infty)}$ or $\mu_{(0)}$, respectively. Assume that μ is concentrated on $(-\infty, 0)$. From Lemmas 3.1 and 3.5, μ is of the form (3.1) with the measure $\nu = E\Delta_{\Theta_1}\mu$ concentrated on $(-\infty, 0)$.

Suppose that there exists $a < 0$ such that $\nu((-\infty, a]) > 0$ and $\nu((a, 0)) > 0$. Let $\nu_1 = \nu\left|_{(-\infty,a])}\right.$ and $\nu_2 = \nu\left|_{(a,0))}\right.$. Let $\mu_i(B) = C_i(\sum_{j=2}^{\infty} E\tau_{\Theta_j} \dots E\tau_{\Theta_2}\nu_i(B) + \nu_i(B))$, where $C_i^{-1} = \sum_{j=2}^{\infty} E\tau_{\Theta_j} \dots E\tau_{\Theta_2}\nu_i((-\infty, 0)) + \nu_i((-\infty, 0))$, $i = 1, 2$. Then $\mu(B) = 1/C_1\mu_1(B) + 1/C_2\mu_2(B)$, where $\mu_1, \mu_2 \in K(\Theta, \tilde{I}_0)$, $\mu_1 \neq \mu_2$ and $1/C_1, 1/C_2 \in (0, 1)$. This contradicts the assumption that $\mu \in e(K)$.

Thus for every $a < 0$, either $\nu((-\infty, a]) = 0$ or $\nu((a, 0)) = 0$. This yields that there exists $x < 0$ such that ν is concentrated at x, hence $\nu = \nu_{(x)} = A_x\delta_x$. Consequently, $\mu = \mu_{(x)}$.

To check the necessity let $\mu = \mu_{(x)}$, where $x \in [-\infty, 0]$. If $x = -\infty$ or $x = 0$, then by Lemma 3.4, $\mu_{(x)} \in e(K(\Theta, \tilde{I}_0))$. Assume that $x \in (-\infty, 0)$. Then $\mu_{(x)}$ is of the form (3.1) with $\nu_{(x)} = A_x\delta_x$ in place of ν. Suppose that $\nu_{(x)} = \alpha\nu_1 + (1-\alpha)\nu_2$, where $\mu_1, \mu_2 \in K(\Theta, \tilde{I}_0)$ and $0 < \alpha < 1$. Then μ_i is of the form (3.1) with the measures μ_i and ν_i in place of μ and ν, respectively ($i = 1, 2$). From this it follows that $\nu_{(x)} = \alpha\nu_1 + (1-\alpha)\nu_2$. Since $\nu_{(x)}$ is concentrated at x, so are ν_1 and ν_2. Consequently, $\nu_1 = \nu_2 = \nu_{(x)}$ and $\mu_1 = \mu_2 = \mu_{(x)}$. This implies that $\mu_{(x)}$ is an extreme point. The theorem is proved. □

4. REPRESENTATION THEOREM

The space of probability measures on $[-\infty, 0]$ with weak convergence is a metrizable compact space. We consider the induced topology on $K(\Theta, \tilde{I}_0)$. Observe that $e(K(\Theta, \tilde{I}_0))$ is closed, hence compact, and consequently, $K(\Theta, \tilde{I}_0)$ is compact.

Lemma 4.1. *$K(\Theta, \tilde{I}_0)$ is compact.*

Now we are ready to prove the theorem on a representation of a measure from $M(\Theta, I_0)$.

Theorem 4.2. *$\mu \in M(\Theta, I_0)$ if and only if μ takes one of the following forms:*

$$\mu(du) = \int_{-\infty}^{0} A_x \left[\sum_{j=2}^{\infty} E\tau_{\Theta_j} \ldots E\tau_{\Theta_2} \delta_x(u)du + \delta_x(u)du \right] \kappa d(x), \tag{4.1}$$

where κ is a finite measure on $(-\infty, 0)$ and the $A_x(-\infty < x < 0)$ is given by (3.4), or equivalently

$$\mu(du) = \int_{-\infty}^{0} \left[\sum_{j=2}^{\infty} E\tau_{\Theta_j} \ldots E\tau_{\Theta_2} \delta_x(u)du + \delta_x(u)du \right] \lambda(dx), \tag{4.2}$$

where λ is a Borel measure on $(-\infty, 0)$, such that $\int_{-\infty}^{0} A_x^{-1}\lambda(dx) < \infty$. Moreover, λ is unique, $\lambda = E\Delta_\Theta \mu$.

Proof. We will apply the Choquet theorem on a representation of the points of a compact set as barycenters of the extreme points [7, p. 17]. Then taking into account Theorem 3.6, we infer that μ is in $K(\Theta, \tilde{I}_0)$ if and only if

$$\mu = \int_{\{\mu_{(x)} \colon x \in [-\infty, 0]\}} \beta \kappa(d\beta), \tag{4.3}$$

where κ is a probability measure on $\{\mu_{(x)} \colon x \in [-\infty, 0]\}$.

It is not difficult to prove that $\{\mu_{(x)} \colon x \in [-\infty, 0]\}$ is homeomorphic to the set $[-\infty, 0]$. Then considering the measure κ as the measure on $[-\infty, 0]$, μ is of the form

$$\mu = \int_{[-\infty, 0]} \mu_{(x)} \kappa(dx). \tag{4.4}$$

Moreover, $\mu \in K(\Theta, I_0)$ if and only if the measure κ assigns the zero mass to the set $\{-\infty, 0\}$. Then μ in $K(\Theta, I_0)$ is given by (4.4) with $[-\infty, 0]$ replaced by $(-\infty, 0)$. Consequently, we can write μ from $K(\Theta, I_0)$ in the form

$$\mu = \int_{-\infty}^{0} \mu_{(x)} \kappa(dx), \tag{4.5}$$

where κ is a probability measure on $(-\infty, 0)$. Obviously, the measure $\mu \in M(\Theta, I_0)$ is given by (4.5), where κ is a finite measure on $(-\infty, 0)$.

By (3.3), (3.4), (3.5) and (4.5), this yields (4.1). Defining $\lambda(dx) = A_x \kappa(dx)$ we obtain (4.2).

To prove the necessity let μ be a measure of the form (4.2). Since $E\Delta_\Theta(\sum_{j=2}^{\infty} E\tau_{\Theta_j} \ldots E\tau_{\Theta_2}\ \delta_x(B) + \delta_x(B)) = \delta_x(B)$ $(x \in (-\infty, 0),\ B \in \mathcal{B}((-\infty, 0)))$, where $\Theta_1, \Theta_2, \ldots$ are independent copies of Θ, then

$$E\Delta_\Theta \mu(B) = \int_{-\infty}^{0} \delta_x(B)\lambda(dx) = \lambda(B). \tag{4.6}$$

From Lemma 2.1 this implies that $\mu \in M(\Theta, I_0)$. Moreover, by (4.6) we obtain that λ is determined uniquely. The theorem is proved. $\square$

Theorem 4.3. *$\mu \in M(\Theta)$ if and only if μ is of the form*

$$\mu(du) = \int_{-\infty}^{\infty} \left[\sum_{j=2}^{\infty} E\tau_{\Theta_j} \dots E\tau_{\Theta_2} \delta_x(u)du + \delta_x(u)du \right] \gamma(dx), \tag{4.7}$$

where γ is a Borel measure on $\mathbb{R}$. Moreover, γ is uniquely determined,

$$\gamma = E\Delta_\Theta \mu. \tag{4.8}$$

Proof. We divide the proof into several steps.
Step 1. First consider the case when $\mu \in M(\Theta, I_N)$, where $I_N = (-\infty, N)$ $(N = 1, 2, \dots)$. Set $\mu_N(B) = \mu(B + N)$, $B \in \mathcal{B}((-\infty, 0))$. Then $\mu_N \in M(\Theta, I_0)$. By (4.2), we conclude that

$$\mu_N(B) = \int_{-\infty}^{0} \left[\sum_{j=2}^{\infty} E\tau_{\Theta_j} \dots E\tau_{\Theta_2} \delta_x(B) + \delta_x(B) \right] \lambda_N(dx),$$

where λ_N is a measure on $(-\infty, 0)$. This gives for all $B \in \mathcal{B}((-\infty, N))$

$$\begin{aligned} \mu(B) &= \mu_N(B - N) = \\ &= \int_{-\infty}^{0} \left[\sum_{j=2}^{\infty} E\tau_{\Theta_j} \dots E\tau_{\Theta_2} \delta_x(B - N) + \delta_x(B - N) \right] \lambda_N(dx) = \\ &= \int_{-\infty}^{N} \left[\sum_{j=2}^{\infty} E\tau_{\Theta_j} \dots E\tau_{\Theta_2} \delta_y(B) + \delta_y(B) \right] \gamma_N(dy), \end{aligned}$$

where γ_N is the measure on $(-\infty, N)$, such that $\lambda_N(dx) = \gamma_N(d(x+N))$. Clearly, γ_N is uniquely determined.
Step 2. Let $\mu \in M(\Theta) = M(\Theta, \mathbb{R})$. Then $\mu|_{I_N} \in M(\Theta, I_N)$, $N = 1, 2, \dots$, and from Step 1 we conclude that

$$\mu(B) = \int_{-\infty}^{N} \left[\sum_{j=2}^{\infty} E\tau_{\Theta_j} \dots E\tau_{\Theta_2} \delta_y(B) + \delta_y(B) \right] \gamma_N(dy),$$

for $B \in \mathcal{B}(I_N)$, where γ_N is a measure on I_N. Since the measure γ_N is uniquely determined, then for any $N_1 < N_2$, $\gamma_{N_2}\big|_{I_{N_1}} = \gamma_{N_1}$. Letting $N \to \infty$, this implies that there exists a measure γ on $\mathbb{R}$ such that $\gamma|_{I_N} = \gamma_N$ and (4.7) is satisfied.

The proof that if μ is of the form (4.7) then $\mu \in M(\Theta)$, and that (4.8) is satisfied is similar to that of Theorem 4.2 and hence is omitted here. The theorem is proved. $\square$

As an application, consider Theorem 4.3 for the particular cases of the random variable Θ. Then, after some computations, we obtain the following representations of measures $\mu \in M(\Theta)$.

Theorem 4.4. *A measure $\mu \in M(\Theta)$ admits the following representation:*

(i) $\mu(du) = \int_{-\infty}^{\infty} \left[\chi_{(x,\infty)}(u)du + \delta_x(u)du\right] \gamma(dx)$, *when Θ has the exponential distribution, $\Theta \sim Exp(1)$ and $\mu_\Theta(dh) = e^{-h}\chi_{(0,\infty)}(h)$,*

(ii) $\mu(du) = \int_{-\infty}^{\infty} \left[\sum_{k=0}^{\infty} \frac{1}{p}\delta_{x+k}(u)du\right] \gamma(dx)$, *when $P(\Theta = 0) = 1 - p$ and $P(\Theta = 1) = p$ $(0 < p < 1)$,*

(iii) $\mu(du) = \int_{-\infty}^{\infty} \left[\delta_x(u)du + \sum_{n=1}^{\infty} \sum_{j=0}^{n} \binom{n}{j} q^j p^{n-j} \delta_{x+n+j}(u)du\right] \gamma(dx)$, *when $P(\Theta = 1) = q$ and $P(\Theta = 2) = p$ $(0 < p < 1,\ q = 1 - p)$,*

(iv) $\mu(du) = \int_{-\infty}^{\infty} \left[\sum_{j,k=0}^{\infty} q^j p^k \delta_{x+j+k\sqrt{2}}(u)du\right] \gamma(dx)$, *when $P(\Theta = 1) = q$ and $P(\Theta = \sqrt{2}) = p$ $(0 < p < 1,\ q = 1 - p)$, where γ is a Borel measure on $\mathbb{R}$.*

Remark 4.5. Let Θ be a random variable with the exponential distribution, $\Theta \sim \mathrm{Exp}(1)$ and $\mu_\Theta(dh) = e^{-h}\chi_{(0,\infty)}(h)$. Setting $\gamma = \delta_0$ in Theorem 4.4 (i), we obtain the measure $\mu(du) = \chi_{(0,\infty)}(u)du + \delta_0(u)du \in M(\Theta)$. By (1.4), this gives

$$E\Delta_\Theta(\chi_{(0,\infty)}(u)du + \delta_0(u)du) \geq 0. \tag{4.9}$$

On the other hand, it is not difficult to check that there is no $h > 0$, for which

$$\Delta_h \left(\chi_{(0,\infty)}(u)du + \delta_0(u)du\right) \geq 0. \tag{4.10}$$

In other words, there exists a random variable Θ for which (4.9) holds, but, for all $h > 0$, the condition (4.10) does not hold.

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Received: December 3, 2010.
Revised: June 3, 2011.
Accepted: June 3, 2011.