Opuscula Mathematica • Vol. 32 • No. 2 • 2012

http://dx.doi.org/10.7494/OpMath.2012.32.2.297

ON SELF-ADJOINT OPERATORS IN KREIN SPACES CONSTRUCTED BY CLIFFORD ALGEBRA $\mathcal{C}l_2$

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Abstract. Let J and R be anti-commuting fundamental symmetries in a Hilbert space $\mathfrak{H}$. The operators J and R can be interpreted as basis (generating) elements of the complex Clifford algebra $\mathcal{C}l_2(J,R) := \text{span}\{I, J, R, iJR\}$. An arbitrary non-trivial fundamental symmetry from $\mathcal{C}l_2(J,R)$ is determined by the formula $J_{\vec{\alpha}} = \alpha_1 J + \alpha_2 R + \alpha_3 iJR$, where $\vec{\alpha} \in \mathbb{S}^2$. Let S be a symmetric operator that commutes with $\mathcal{C}l_2(J,R)$. The purpose of this paper is to study the sets $\Sigma_{J_{\vec{\alpha}}}$ ($\forall \vec{\alpha} \in \mathbb{S}^2$) of self-adjoint extensions of S in Krein spaces generated by fundamental symmetries $J_{\vec{\alpha}}$ ($J_{\vec{\alpha}}$-self-adjoint extensions). We show that the sets $\Sigma_{J_{\vec{\alpha}}}$ and $\Sigma_{J_{\vec{\beta}}}$ are unitarily equivalent for different $\vec{\alpha}, \vec{\beta} \in \mathbb{S}^2$ and describe in detail the structure of operators $A \in \Sigma_{J_{\vec{\alpha}}}$ with empty resolvent set.

Keywords: Krein spaces, extension theory of symmetric operators, operators with empty resolvent set, J-self-adjoint operators, Clifford algebra $\mathcal{C}l_2$.

Mathematics Subject Classification: 47A55, 47B25, 47A57, 81Q15.

1. INTRODUCTION

Let $\mathfrak{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and with non-trivial fundamental symmetry J (i.e., $J = J^*$, $J^2 = I$, and $J \neq \pm I$).

The space $\mathfrak{H}$ endowed with the indefinite inner product (indefinite metric) $[\cdot,\cdot]_J := (J\cdot,\cdot)$ is called a Krein space $(\mathfrak{H}, [\cdot,\cdot]_J)$.

An operator A acting in $\mathfrak{H}$ is called J-self-adjoint if A is self-adjoint with respect to the indefinite metric $[\cdot,\cdot]_J$, i.e., if $A^*J = JA$.

In contrast to self-adjoint operators in Hilbert spaces (which necessarily have a purely real spectrum), a J-self-adjoint operator A, in general, has spectrum which is only symmetric with respect to the real axis. In particular, the situation where $\sigma(A) = \mathbb{C}$ (i.e., A has empty resolvent set $\rho(A) = \emptyset$) is also possible and it may indicate on a special structure of A. To illustrate this point we consider a simple symmetric[1)] operator S with deficiency indices $\langle 2,2\rangle$ which commutes with J:

$$SJ = JS.$$

It was recently shown [15, Theorem 4.3] that the existence at least one J-self-adjoint extension A of S with empty resolvent set *is equivalent* to the existence of an additional fundamental symmetry R in $\mathfrak{H}$ such that

$$SR = RS, \qquad JR = -RJ. \tag{1.1}$$

The fundamental symmetries J and R can be interpreted as basis (generating) elements of the complex Clifford algebra $\mathcal{C}l_2(J,R) := \text{span}\{I, J, R, iJR\}$ [11]. Hence, the existence of J-self-adjoint extensions of S with empty resolvent set *is equivalent* to the commutation of S with an arbitrary element of the Clifford algebra $\mathcal{C}l_2(J,R)$.

In the present paper we investigate nonself-adjoint extensions of a densely defined symmetric operator S assuming that S commutes with elements of $\mathcal{C}l_2(J,R)$. Precisely, we show that an arbitrary non-trivial fundamental symmetry $J_{\vec{\alpha}}$ constructed in terms of $\mathcal{C}l_2(J,R)$ is uniquely determined by the choice of vector $\vec{\alpha}$ from the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ (Lemma 2.1) and we study various collections $\Sigma_{J_{\vec{\alpha}}}$ of $J_{\vec{\alpha}}$-self-adjoint extensions of S. Such a 'flexibility' of fundamental symmetries is inspirited by the application to $\mathcal{PT}$-symmetric quantum mechanics [5], where $\mathcal{PT}$-symmetric Hamiltonians are not necessarily can be realized as $\mathcal{P}$-self-adjoint operators [1, 17]. Moreover, for certain models [11], the corresponding $\mathcal{PT}$-symmetric operator realizations can be interpreted as $J_{\vec{\alpha}}$-self-adjoint operators when $\vec{\alpha}$ runs $\mathbb{S}^2$.

We show that the sets $\Sigma_{J_{\vec{\alpha}}}$ and $\Sigma_{J_{\vec{\beta}}}$ are unitarily equivalent for different $\vec{\alpha}, \vec{\beta} \in \mathbb{S}^2$ (Theorem 2.9) and describe properties of $A \in \Sigma_{J_{\vec{\alpha}}}$ in terms of boundary triplets (subsections 2.4, 2.5).

Denote by $\Xi_{\vec{\alpha}}$ the collection of all operators $A \in \Sigma_{J_{\vec{\alpha}}}$ with empty resolvent set. It follows from our results that, as a rule, an operator $A \in \Xi_{\vec{\alpha}}$ is $J_{\vec{\beta}}$-self-adjoint (i.e., $A \in \Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}}$) for a special choice of $\vec{\beta} \in \mathbb{S}^2$ which depends on A. In this way, for the case of symmetric operators S with deficiency indices $\langle 2,2\rangle$, the complete description of $\Xi_{\vec{\alpha}}$ is obtained as the union of operators $A \in \Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}}$, $\rho(A) = \emptyset$, $\forall \vec{\beta} \in \mathbb{S}^2$ (Theorem 3.3). In the exceptional case when the Weyl function of S is a constant, the set $\Xi_{\vec{\alpha}}$ increases considerably (Corollary 3.5).

The one-dimensional Schrödinger differential expression with non-integrable singularity at zero (the limit-circle case at $x = 0$) is considered as an example of application (Proposition 3.6).

Throughout the paper, $\mathcal{D}(A)$ denotes the domain of a linear operator A. $A \upharpoonright_{\mathcal{D}}$ means the restriction of A onto a set $\mathcal{D}$. The notation $\sigma(A)$ and $\rho(A)$ are used for the spectrum and the resolvent set of A.

[1)] with respect to the initial inner product $(\cdot,\cdot)$

2. SETS $\Sigma_{J_{\vec{\alpha}}}$ AND THEIR PROPERTIES

2.1. PRELIMINARIES

Let $\mathfrak{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and let J and R be fundamental symmetries in $\mathfrak{H}$ satisfying (1.1).

Denote by $\mathcal{C}l_2(J,R) := \operatorname{span}\{I, J, R, iJR\}$ a complex Clifford algebra with generating elements J and R. Since the operators I, J, R, and iJR are linearly independent (due to (1.1)), an arbitrary operator $K \in \mathcal{C}l_2(J,R)$ can be presented as:

$$K = \alpha_0 I + \alpha_1 J + \alpha_2 R + \alpha_3 iJR, \qquad \alpha_j \in \mathbb{C}. \tag{2.1}$$

Lemma 2.1 ([15])**.** *An operator K defined by* (2.1) *is a non-trivial fundamental symmetry in $\mathfrak{H}$ (i.e., $K^2 = I$, $K = K^*$, and $K \neq I$) if and only if*

$$K = \alpha_1 J + \alpha_2 R + \alpha_3 iJR, \tag{2.2}$$

where $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ and $\alpha_j \in \mathbb{R}$.

Proof. The reality of α_j in (2.2) follows from the self-adjointness of I, J, R, and iJR. The condition $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ is equivalent to the relation $K^2 = I$. □

Remark 2.2. The formula (2.2) establishes a one-to-one correspondence between the set of non-trivial fundamental symmetries K in $\mathcal{C}l_2(J,R)$ and vectors $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ of the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. To underline this relationship we will use the notation $J_{\vec{\alpha}}$ for the fundamental symmetry K determined by (2.2), i.e.,

$$J_{\vec{\alpha}} = \alpha_1 J + \alpha_2 R + \alpha_3 iJR. \tag{2.3}$$

In particular, this means that $J_{\vec{\alpha}} = J$ with $\vec{\alpha} = (1,0,0)$ and $J_{\vec{\alpha}} = R$ when $\vec{\alpha} = (0,1,0)$.

Lemma 2.3. *Let $\vec{\alpha}, \vec{\beta} \in \mathbb{S}^2$. Then*

$$J_{\vec{\alpha}} J_{\vec{\beta}} = -J_{\vec{\beta}} J_{\vec{\alpha}} \qquad \textit{if and only if} \qquad \vec{\alpha} \cdot \vec{\beta} = 0. \tag{2.4}$$

and

$$J_{\vec{\alpha}} + J_{\vec{\beta}} = |\vec{\alpha} + \vec{\beta}| J_{\frac{\vec{\alpha}+\vec{\beta}}{|\vec{\alpha}+\vec{\beta}|}} \qquad \textit{if} \qquad \vec{\alpha} \neq -\vec{\beta}. \tag{2.5}$$

Proof. It immediately follows from Lemma 2.1 and identities (1.1), (2.3). □

2.2. DEFINITION AND PROPERTIES OF $\Sigma_{J_{\vec{\alpha}}}$

1. Let S be a closed densely defined symmetric operator *with equal deficiency indices* in the Hilbert space $\mathfrak{H}$. In what follows we suppose that S *commutes with all elements of* $\mathcal{C}l_2(J,R)$ or, that is equivalent, S *commutes with* J and R:

$$SJ = JS, \qquad SR = RS \tag{2.6}$$

Denote by Υ *the set of all self-adjoint extensions* A *of* S *which commute with* J *and* R:

$$\Upsilon = \{ \ A \supset S \ : \ A^* = A, \quad AJ = JA, \quad AR = RA \ \}. \tag{2.7}$$

It follows from (2.3) and (2.7) that Υ contains self-adjoint extensions of S *which commute with all fundamental symmetries* $J_{\vec{\alpha}} \in \mathcal{C}l_2(J, R)$.

Let us fix one of them $J_{\vec{\alpha}}$ and denote by $(\mathfrak{H}, [\cdot,\cdot]_{J_{\vec{\alpha}}})$ the corresponding Krein space[2)] with the indefinite inner product $[\cdot,\cdot]_{J_{\vec{\alpha}}} := (J_{\vec{\alpha}}\cdot,\cdot)$.

Denote by $\Sigma_{J_{\vec{\alpha}}}$ *the collection of all* $J_{\vec{\alpha}}$*-self-adjoint extensions of* S:

$$\Sigma_{J_{\vec{\alpha}}} = \{ \ A \supset S \ : \ J_{\vec{\alpha}} A^* = A J_{\vec{\alpha}} \ \}. \tag{2.8}$$

An operator $A \in \Sigma_{J_{\vec{\alpha}}}$ is a self-adjoint extension of S with respect to the indefinite metric $[\cdot,\cdot]_{J_{\vec{\alpha}}}$.

Proposition 2.4. *The following relation holds*

$$\bigcap_{\forall \vec{\alpha} \in \mathbb{S}^2} \Sigma_{J_{\vec{\alpha}}} = \Upsilon.$$

Proof. It follows from the definitions above that $\Sigma_{J_{\vec{\alpha}}} \supset \Upsilon$. Therefore,

$$\bigcap_{\forall \vec{\alpha} \in \mathbb{S}^2} \Sigma_{J_{\vec{\alpha}}} \supset \Upsilon.$$

Let $A \in \bigcap_{\forall \vec{\alpha} \in \mathbb{S}^2} \Sigma_{J_{\vec{\alpha}}}$. In particular, this means that $A \in \Sigma_J$, $A \in \Sigma_R$, and $A \in \Sigma_{iJR}$. It follows from the first two relations that $JA^* = AJ$ and $RA^* = AR$. Therefore, $iJRA^* = iJAR = A^* iJR$. Simultaneously, $iJRA^* = AiJR$ since $A \in \Sigma_{iJR}$. Comparing the obtained relations we deduce that $A^* iJR = AiJR$ and hence, $A^* = A$. Thus A is a self-adjoint operator and it commutes with an arbitrary fundamental symmetry $J_{\vec{\alpha}} \in \mathcal{C}l_2(J, R)$. Therefore, $A \in \Upsilon$. Proposition 2.4 is proved. □

Simple analysis of the proof of Proposition 2.4 leads to the conclusion that

$$\Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}} \cap \Sigma_{J_{\vec{\gamma}}} = \Upsilon$$

for any three linearly independent vectors $\vec{\alpha}, \vec{\beta}, \vec{\gamma} \in \mathbb{S}^2$. However,

$$\Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}} \supset \Upsilon, \qquad \forall \vec{\alpha}, \vec{\beta} \in \mathbb{S}^2 \tag{2.9}$$

and the intersection $\Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}}$ contains operators A *with empty resolvent set* (i.e., $\rho(A) = \emptyset$ or, that is equivalent, $\sigma(A) = \mathbb{C}$). Let us discuss this phenomena in detail.

Consider two linearly independent vectors $\vec{\alpha}, \vec{\beta} \in \mathbb{S}^2$. If $\vec{\alpha} \cdot \vec{\beta} \neq 0$, we define new vector $\vec{\beta'}$ in $\mathbb{S}^2$:

$$\vec{\beta'} = \frac{\vec{\alpha} + c\vec{\beta}}{|\vec{\alpha} + c\vec{\beta}|}, \qquad c = -\frac{1}{\vec{\alpha} \cdot \vec{\beta}}$$

2) We refer to [4, 10] for the terminology of the Krein spaces theory.

such that $\vec{\alpha} \cdot \vec{\beta}' = 0$. Then the fundamental symmetry

$$J_{\vec{\beta}'} = \frac{1}{|\vec{\alpha} + c\vec{\beta}|} J_{\vec{\alpha}} + \frac{c}{|\vec{\alpha} + c\vec{\beta}|} J_{\vec{\beta}} \tag{2.10}$$

anti-commutes with $J_{\vec{\alpha}}$ (due to Lemma 2.3).

The operator

$$J_{\vec{\gamma}} = iJ_{\vec{\alpha}} J_{\vec{\beta}'} = \begin{vmatrix} J & R & iJR \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta'_1 & \beta'_2 & \beta'_3 \end{vmatrix} \tag{2.11}$$

is a fundamental symmetry in $\mathfrak{H}$ which commutes with S. Therefore, the orthogonal decomposition of $\mathfrak{H}$ constructed by $J_{\vec{\gamma}}$:

$$\mathfrak{H} = \mathfrak{H}_+^\gamma \oplus \mathfrak{H}_-^\gamma, \qquad \mathfrak{H}_+^\gamma = \frac{1}{2}(I + J_{\vec{\gamma}})\mathfrak{H}, \quad \mathfrak{H}_-^\gamma = \frac{1}{2}(I - J_{\vec{\gamma}})\mathfrak{H} \tag{2.12}$$

reduces S:

$$S = \begin{pmatrix} S_{\gamma+} & 0 \\ 0 & S_{\gamma-} \end{pmatrix}, \qquad S_{\gamma+} = S\upharpoonright_{\mathfrak{H}_+^\gamma}, \quad S_{\gamma-} = S\upharpoonright_{\mathfrak{H}_-^\gamma}. \tag{2.13}$$

Since $J_{\vec{\gamma}}$ anti-commutes with $J_{\vec{\alpha}}$ (see (2.11)), the operator $J_{\vec{\alpha}}$ maps $\mathfrak{H}_\pm^\gamma$ onto $\mathfrak{H}_\mp^\gamma$ and operators $S_{\gamma+}$ and $S_{\gamma-}$ are unitarily equivalent. Precisely, $S_{\gamma-}x = J_{\vec{\alpha}} S_{\gamma+} J_{\vec{\alpha}} x$ for all elements $x \in \mathcal{D}(S_{\gamma-})$. This means that $S_{\gamma+}$ and $S_{\gamma-}$ have equal deficiency indices.[3)]

Denote

$$A_\gamma = \begin{pmatrix} S_{\gamma+} & 0 \\ 0 & S_{\gamma-}^* \end{pmatrix}, \qquad A_\gamma^* = \begin{pmatrix} S_{\gamma+}^* & 0 \\ 0 & S_{\gamma-} \end{pmatrix}. \tag{2.14}$$

The operators A_γ and A_γ^* are extensions of S and $\sigma(A_\gamma) = \sigma(A_\gamma^*) = \mathbb{C}$ (since $S_{\gamma\pm}$ are symmetric operators), i.e., these operators *have empty resolvent set.*

Theorem 2.5. *Let* $\vec{\alpha}, \vec{\beta} \in \mathbb{S}^2$ *be linearly independent vectors. Then the operators* A_γ *and* A_γ^* *defined by* (2.14) *belong to* $\Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}}$.

Proof. Assume that $A \in \Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}}$, where $\vec{\alpha}, \vec{\beta} \in \mathbb{S}^2$ are linearly independent vectors. Then

$$J_{\vec{\alpha}} A = A^* J_{\vec{\alpha}}, \qquad J_{\vec{\beta}} A = A^* J_{\vec{\beta}} \tag{2.15}$$

and hence, $J_{\vec{\beta}'} A = A^* J_{\vec{\beta}'}$ due to (2.10). In that case

$$J_{\vec{\gamma}} A = iJ_{\vec{\alpha}} J_{\vec{\beta}'} A = iJ_{\vec{\alpha}} A^* J_{\vec{\beta}'} = iAJ_{\vec{\alpha}} J_{\vec{\beta}'} = AJ_{\vec{\gamma}},$$

where $J_{\vec{\gamma}} = iJ_{\vec{\alpha}} J_{\vec{\beta}'}$ (see (2.11)) is the fundamental symmetry in $\mathfrak{H}$.

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3) This also implies that the symmetric operator S commuting with $\mathcal{C}l_2(J, R)$ may have only *even* deficiency indices.

Since A commutes with $J_{\vec{\gamma}}$, the decomposition (2.12) reduces A and

$$A = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}, \qquad A_+ = A \upharpoonright_{\mathfrak{H}_+^\gamma}, \quad A_- = A \upharpoonright_{\mathfrak{H}_-^\gamma}, \tag{2.16}$$

where $S_{\gamma+} \subseteq A_+ \subseteq S_{\gamma+}^*$ and $S_{\gamma-} \subseteq A_- \subseteq S_{\gamma-}^*$. This means that

$$A^* = \begin{pmatrix} A_+^* & 0 \\ 0 & A_-^* \end{pmatrix}, \qquad A_+^* = A^* \upharpoonright_{\mathfrak{H}_+^\gamma}, \quad A_-^* = A^* \upharpoonright_{\mathfrak{H}_-^\gamma}. \tag{2.17}$$

Since $J_{\vec{\alpha}}$ anti-commutes with $J_{\vec{\gamma}}$, the operator $J_{\vec{\alpha}}$ maps $\mathfrak{H}_\pm^\gamma$ onto $\mathfrak{H}_\mp^\gamma$. Therefore, the first relation in (2.15) can be rewritten with the use of formulas (2.16) and (2.17) as follows:

$$J_{\vec{\alpha}}Ax = J_{\vec{\alpha}}(A_+x_+ + A_-x_-) = A_-^*J_{\vec{\alpha}}x_+ + A_+^*J_{\vec{\alpha}}x_- = A^*J_{\vec{\alpha}}x, \tag{2.18}$$

where $x = x_+ + x_- \in \mathcal{D}(A),\ x_\pm \in \mathcal{D}(A_\pm)$.

The identity (2.18) holds for all $x_\pm \in \mathcal{D}(A_\pm)$. This means that

$$J_{\vec{\alpha}}A_+ = A_-^*J_{\vec{\alpha}}, \qquad J_{\vec{\alpha}}A_- = A_+^*J_{\vec{\alpha}}. \tag{2.19}$$

It follows from (2.10) and (2.11) that the fundamental symmetry $J_{\vec{\beta}}$ anti-commutes with $J_{\vec{\gamma}}$. Repeating the arguments above for the second relation in (2.15) we obtain

$$J_{\vec{\beta}}A_+ = A_-^*J_{\vec{\beta}}, \qquad J_{\vec{\beta}}A_- = A_+^*J_{\vec{\beta}}. \tag{2.20}$$

Thus an operator A belongs to $\Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}}$ *if and only if its counterparts* A_+ *and* A_- *in* (2.16) *satisfy relations* (2.19) *and* (2.20). In particular, these relations are satisfied for the cases when $A_+ = S_{\gamma+},\ A_- = S_{\gamma-}^*$ and $A_+ = S_{\gamma+}^*,\ A_- = S_{\gamma-}$. Hence, the operators A_γ, A_γ^* defined by (2.14) belong to $\Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}}$. Theorem 2.5 is proved. □

Remark 2.6. The operators A_γ and A_γ^* constructed above depend on the choice of $\vec{\beta} \in \mathbb{S}^2$. Considering various vectors $\vec{\beta} \in \mathbb{S}^2$ in (2.10), (2.11), we obtain a collection of fundamental symmetries $J_{\vec{\gamma}(\vec{\beta})}$. This gives rise to a one-parameter set of different operators $A_{\gamma(\vec{\beta})}$ and $A_{\gamma(\vec{\beta})}^*$ with empty resolvent set which belong to $\Sigma_{J_{\vec{\alpha}}}$.

Corollary 2.7. *Let* $\vec{\alpha}, \vec{\beta} \in \mathbb{S}^2$ *be linearly independent vectors and let* (2.12) *be the decomposition of* $\mathfrak{H}$ *constructed by these vectors. Then, with respect to* (2.12)*, all operators* $A \in \Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}}$ *are described by the formula*

$$A = \begin{pmatrix} A_+ & 0 \\ 0 & J_{\vec{\alpha}}A_+^*J_{\vec{\alpha}} \end{pmatrix}, \tag{2.21}$$

where A_+ *is an arbitrary intermediate extension of* $S_{\gamma+} = S \upharpoonright_{\mathfrak{H}_+^\gamma}$ *(i.e.,* $S_{\gamma+} \subseteq A_+ \subseteq S_{\gamma+}^*$*).*

Proof. If $A \in \Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}}$, then the presentation (2.21) follows from (2.16) and the second identity in (2.19).

Conversely, assume that an operator A is defined by (2.21). Since $J_{\vec{\alpha}}$ and $J_{\vec{\beta}}$ anti-commute with $J_{\vec{\gamma}}$, they admit the presentations $J_{\vec{\alpha}} = \begin{pmatrix} 0 & J_{\vec{\alpha}} \\ J_{\vec{\alpha}} & 0 \end{pmatrix}$ and $J_{\vec{\beta}} = \begin{pmatrix} 0 & J_{\vec{\beta}} \\ J_{\vec{\beta}} & 0 \end{pmatrix}$ with respect to (2.12). Then, the operator equality $J_{\vec{\alpha}}A = A^*J_{\vec{\alpha}}$ is established by the direct multiplication of the corresponding operator entries. The same procedure for $J_{\vec{\beta}}A = A^*J_{\vec{\beta}}$ leads to the verification of relations

$$J_{\vec{\alpha}}J_{\vec{\beta}}A_+ = A_+J_{\vec{\alpha}}J_{\vec{\beta}}, \qquad J_{\vec{\beta}}J_{\vec{\alpha}}A_+^* = A_+^*J_{\vec{\beta}}J_{\vec{\alpha}}. \tag{2.22}$$

To this end we recall that $J_{\vec{\gamma}}$ commutes with A_+ and

$$J_{\vec{\alpha}}J_{\vec{\beta}} = -\frac{1}{c}I - i\frac{|\vec{\alpha} + c\vec{\beta}|}{c}J_{\vec{\gamma}}$$

due to (2.10) and (2.11). Therefore, A_+ commutes with $J_{\vec{\alpha}}J_{\vec{\beta}}$ and the first relation in (2.22) holds. The second relation is established in the same manner, if we take into account that $J_{\vec{\gamma}}$ commutes with A_+^* and $J_{\vec{\beta}}J_{\vec{\alpha}} = -\frac{1}{c}I + i\frac{|\vec{\alpha}+c\vec{\beta}|}{c}J_{\vec{\gamma}}$. Corollary 2.7 is proved. □

Remark 2.8. It follows from the proof that the choice of $J_{\vec{\alpha}}$ in (2.21) is not essential and the similar description of $\Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}}$ can be obtained with the help of $J_{\vec{\beta}}$.

2. Denote

$$W_{\vec{\alpha},\vec{\beta}} = \begin{cases} J_{\frac{\vec{\alpha}+\vec{\beta}}{|\vec{\alpha}+\vec{\beta}|}} & \text{if} \quad \vec{\alpha} \neq -\vec{\beta}, \\ I & \text{if} \quad \vec{\alpha} = -\vec{\beta}. \end{cases} \tag{2.23}$$

It is clear that $W_{\vec{\alpha},\vec{\beta}}$ is a fundamental symmetry in $\mathfrak{H}$ and $W_{\vec{\alpha},\vec{\beta}} = W_{\vec{\beta},\vec{\alpha}}$ for any $\vec{\alpha}, \vec{\beta} \in \mathbb{S}^2$.

Theorem 2.9. *For any $\vec{\alpha}, \vec{\beta} \in \mathbb{S}^2$ the sets $\Sigma_{J_{\vec{\alpha}}}$ and $\Sigma_{J_{\vec{\beta}}}$ are unitarily equivalent and $A \in \Sigma_{J_{\vec{\alpha}}}$ if and only if $W_{\vec{\alpha},\vec{\beta}}AW_{\vec{\alpha},\vec{\beta}} \in \Sigma_{J_{\vec{\beta}}}$.*

Proof. Since $J_{-\vec{\alpha}} = -J_{\vec{\alpha}}$ (see (2.3)), the sets $\Sigma_{J_{\vec{\alpha}}}$ and $\Sigma_{J_{-\vec{\alpha}}}$ coincide and therefore, the case $\vec{\alpha} = -\vec{\beta}$ is trivial.

Assume that $A \in \Sigma_{J_{\vec{\alpha}}}$, $\vec{\alpha} \neq -\vec{\beta}$ and consider the operator $W_{\vec{\alpha},\vec{\beta}}AW_{\vec{\alpha},\vec{\beta}}$, which we denote B for brevity. Taking into account that S commutes with $J_{\vec{\alpha}}$ for any choice of $\vec{\alpha} \in \mathbb{S}^2$, we deduce from (2.23) that $W_{\vec{\alpha},\vec{\beta}}S = SW_{\vec{\alpha},\vec{\beta}}$ and $W_{\vec{\alpha},\vec{\beta}}S^* = S^*W_{\vec{\alpha},\vec{\beta}}$. This means that $Bx = W_{\vec{\alpha},\vec{\beta}}AW_{\vec{\alpha},\vec{\beta}}x = W_{\vec{\alpha},\vec{\beta}}SW_{\vec{\alpha},\vec{\beta}}x = Sx$ for all $x \in \mathcal{D}(S)$ and $By = W_{\vec{\alpha},\vec{\beta}}AW_{\vec{\alpha},\vec{\beta}}y = W_{\vec{\alpha},\vec{\beta}}S^*W_{\vec{\alpha},\vec{\beta}}y = S^*y$ for all $y \in \mathcal{D}(B) = W_{\vec{\alpha},\vec{\beta}}\mathcal{D}(A)$. Therefore, B is an intermediate extension of S (i.e., $S \subseteq B \subseteq S^*$).

It follows from (2.5) and (2.23) that

$$J_{\vec{\beta}}W_{\vec{\alpha},\vec{\beta}} = J_{\vec{\beta}}\frac{J_{\vec{\alpha}} + J_{\vec{\beta}}}{|\vec{\alpha} + \vec{\beta}|} = \frac{J_{\vec{\beta}}J_{\vec{\alpha}} + I}{|\vec{\alpha} + \vec{\beta}|} = \frac{J_{\vec{\beta}} + J_{\vec{\alpha}}}{|\vec{\alpha} + \vec{\beta}|}J_{\vec{\alpha}} = W_{\vec{\alpha},\vec{\beta}}J_{\vec{\alpha}}. \tag{2.24}$$

Using (2.15) and (2.24), we arrive at the conclusion that

$$J_{\vec{\beta}}B^* = J_{\vec{\beta}}W_{\vec{\alpha},\vec{\beta}}A^*W_{\vec{\alpha},\vec{\beta}} = W_{\vec{\alpha},\vec{\beta}}AJ_{\vec{\alpha}}W_{\vec{\alpha},\vec{\beta}} = W_{\vec{\alpha},\vec{\beta}}AW_{\vec{\alpha},\vec{\beta}}J_{\vec{\beta}} = BJ_{\vec{\beta}}.$$

Therefore, condition $A \in \Sigma_{J_{\vec{\alpha}}}$ implies that $B = W_{\vec{\alpha},\vec{\beta}}AW_{\vec{\alpha},\vec{\beta}} \in \Sigma_{J_{\vec{\beta}}}$. The inverse implication $B \in \Sigma_{J_{\vec{\beta}}} \Rightarrow A = W_{\vec{\alpha},\vec{\beta}}BW_{\vec{\alpha},\vec{\beta}} \in \Sigma_{J_{\vec{\alpha}}}$ is established in the same manner. Theorem 2.9 is proved. □

Remark 2.10. Due to Theorem 2.9, for any $J_{\vec{\alpha}}$-self-adjoint extension $A \in \Sigma_{J_{\vec{\alpha}}}$ there exists a unitarily equivalent $J_{\vec{\beta}}$-self-adjoint extension $B \in \Sigma_{J_{\vec{\beta}}}$. This means that, the spectral analysis of operators from $\bigcup_{\alpha\in\mathbb{S}^2}\Sigma_{J_{\vec{\alpha}}}$ can be reduced to the spectral analysis of $J_{\vec{\alpha}}$-self-adjoint extensions from $\Sigma_{J_{\vec{\alpha}}}$, where $\vec{\alpha}$ is a fixed vector from $\mathbb{S}^2$.

2.3. BOUNDARY TRIPLETS AND WEYL FUNCTION

1. Let S be a closed symmetric operator with equal deficiency indices in the Hilbert space $\mathfrak{H}$. A triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$, where $\mathcal{H}$ is an auxiliary Hilbert space and Γ_0, Γ_1 are linear mappings of $\mathcal{D}(S^*)$ into $\mathcal{H}$, is called a *boundary triplet of* S^* if the abstract Green identity

$$(S^*x, y) - (x, S^*y) = (\Gamma_1 x, \Gamma_0 y)_{\mathcal{H}} - (\Gamma_0 x, \Gamma_1 y)_{\mathcal{H}}, \quad x, y \in \mathcal{D}(S^*) \tag{2.25}$$

is satisfied and the map $(\Gamma_0, \Gamma_1) : \mathcal{D}(S^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective [7, 9].

Lemma 2.11. *Assume that S satisfies the commutation relations* (2.6) *and $J_{\vec{\tau}}, J_{\vec{\gamma}} \in \mathcal{C}l_2(J, R)$ are fixed anti-commuting fundamental symmetries. Then there exists a boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ of S^* such that the formulas*

$$\mathcal{J}_{\vec{\tau}}\Gamma_j := \Gamma_j J_{\vec{\tau}}, \qquad \mathcal{J}_{\vec{\gamma}}\Gamma_j := \Gamma_j J_{\vec{\gamma}}, \qquad j = 0, 1 \tag{2.26}$$

correctly define anti-commuting fundamental symmetries $\mathcal{J}_{\vec{\tau}}$ and $\mathcal{J}_{\vec{\gamma}}$ in the Hilbert space $\mathcal{H}$.

Proof. If S satisfies (2.6), then S commutes with an arbitrary fundamental symmetry $J_{\vec{\gamma}} \in \mathcal{C}l_2(J, R)$ and hence, S admits the representation (2.13) for any vector $\vec{\gamma} \in \mathbb{S}^2$.

Let $S_{\gamma+}$ be a symmetric operator in $\mathfrak{H}_+^{\gamma}$ from (2.13) and let $(N, \Gamma_0^+, \Gamma_1^+)$ be an arbitrary boundary triplet of $S_{\gamma+}^*$.

Since $J_{\vec{\tau}}$ anti-commutes with $J_{\vec{\gamma}}$, the symmetric operator $S_{\gamma-}$ in (2.13) can be described as $S_{\gamma-} = J_{\vec{\tau}}S_{\gamma+}J_{\vec{\tau}}$. This means that $(N, \Gamma_0^+J_{\vec{\tau}}, \Gamma_1^+J_{\vec{\tau}})$ is a boundary triplet of $S_{\gamma-}$.

It is easy to see that the operators

$$\Gamma_j f = \Gamma_j(f_+ + f_-) = \begin{pmatrix} \Gamma_j^+ f_+ \\ \Gamma_j^+ J_{\vec{\tau}} f_- \end{pmatrix} \tag{2.27}$$

$(f = f_+ + f_- \in \mathcal{D}(S^*),\ f_\pm \in \mathcal{D}(S_{\gamma\pm}^*))$ map $\mathcal{D}(S^*)$ onto the Hilbert space

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \qquad \mathcal{H}_+ = \begin{pmatrix} N \\ 0 \end{pmatrix}, \quad \mathcal{H}_- = \begin{pmatrix} 0 \\ N \end{pmatrix}$$

and they form a boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ of S^* which satisfies (2.26) with

$$\mathcal{J}_{\vec{\tau}} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \mathcal{J}_{\vec{\gamma}} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{2.28}$$

It is clear that $\mathcal{J}_{\vec{\tau}}$ and $\mathcal{J}_{\vec{\gamma}}$ are anti-commuting fundamental symmetries in the Hilbert space $\mathcal{H}$. □

Remark 2.12. The fundamental symmetries $\mathcal{J}_{\vec{\tau}}$ and $\mathcal{J}_{\vec{\gamma}}$ are defined by (2.26) in a similar way that in [13], where symmetric operators commuting with involution has been studied.

Remark 2.13. Since J and R can be expressed as linear combinations of $J_{\vec{\tau}}$, $J_{\vec{\gamma}}$, and $iJ_{\vec{\tau}}J_{\vec{\gamma}}$, formulas (2.26) imply that

$$\mathcal{J}\Gamma_j := \Gamma_j J, \qquad \mathcal{R}\Gamma_j := \Gamma_j R, \qquad j = 0, 1,$$

where $\mathcal{J}$ and $\mathcal{R}$ are anti-commuting fundamental symmetries in $\mathcal{H}$. Therefore, an arbitrary boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ of S^* with property (2.26) allows one *to establish a bijective correspondence between elements of the initial Clifford algebra $\mathcal{C}l_2(J,R)$ and its "image" $\mathcal{C}l_2(\mathcal{J},\mathcal{R})$ in the auxiliary space $\mathcal{H}$*. In particular, *for every $J_{\vec{\alpha}} \in \mathcal{C}l_2(J,R)$* defined by (2.3),

$$\mathcal{J}_{\vec{\alpha}}\Gamma_j = \Gamma_j J_{\vec{\alpha}}, \qquad j = 0, 1, \tag{2.29}$$

where $\mathcal{J}_{\vec{\alpha}} = \alpha_1\mathcal{J} + \alpha_2\mathcal{R} + \alpha_3 i\mathcal{J}\mathcal{R}$ belongs to $\mathcal{C}l_2(\mathcal{J},\mathcal{R})$.

2. Let $(\mathcal{H}, \Gamma_0, \Gamma_1)$ be a boundary triplet of S^*. The Weyl function of S associated with $(\mathcal{H}, \Gamma_0, \Gamma_1)$ is defined as follows:

$$M(\mu)\Gamma_0 f_\mu = \Gamma_1 f_\mu, \quad \forall f_\mu \in \ker(S^* - \mu I), \quad \forall \mu \in \mathbb{C} \setminus \mathbb{R}. \tag{2.30}$$

Lemma 2.14. *Let $(\mathcal{H}, \Gamma_0, \Gamma_1)$ be a boundary triplet of S^* with properties* (2.26)*. Then the corresponding Weyl function $M(\cdot)$ commutes with every fundamental symmetry $\mathcal{J}_{\vec{\alpha}} \in \mathcal{C}l_2(\mathcal{J},\mathcal{R})$:*

$$M(\mu)\mathcal{J}_{\vec{\alpha}} = \mathcal{J}_{\vec{\alpha}}M(\mu), \qquad \forall \mu \in \mathbb{C} \setminus \mathbb{R}.$$

Proof. It follows from (2.3) and (2.6) that $S^*J_{\vec{\alpha}} = J_{\vec{\alpha}}S^*$ for all $\vec{\alpha} \in \mathbb{S}^2$. Therefore, $J_{\vec{\alpha}} : \ker(S^* - \mu I) \to \ker(S^* - \mu I)$. In that case, relations (2.29) and (2.30) lead to $M(\mu)\mathcal{J}_{\vec{\alpha}}\Gamma_0 f_\mu = \mathcal{J}_{\vec{\alpha}}\Gamma_1 f_\mu$. Thus, $\mathcal{J}_{\vec{\alpha}}M(\mu)\mathcal{J}_{\vec{\alpha}} = M(\mu)$ or $M(\mu)\mathcal{J}_{\vec{\alpha}} = \mathcal{J}_{\vec{\alpha}}M(\mu)$. □

2.4. DESCRIPTION OF $\Sigma_{J_{\vec{\alpha}}}$ IN TERMS OF BOUNDARY TRIPLETS

Theorem 2.15. *Let $(\mathcal{H}, \Gamma_0, \Gamma_1)$ be a boundary triplet of S^* with properties* (2.26) *for a fixed anti-commuting fundamental symmetries $J_{\vec{\tau}}, J_{\vec{\gamma}} \in \mathcal{C}l_2(J,R)$ and let $J_{\vec{\alpha}}$ be an arbitrary fundamental symmetry from $\mathcal{C}l_2(J,R)$. Then operators $A \in \Sigma_{J_{\vec{\alpha}}}$ coincide with the restriction of S^* onto the domains*

$$\mathcal{D}(A) = \{f \in \mathcal{D}(S^*) \ : \ U(\mathcal{J}_{\vec{\alpha}}\Gamma_1 + i\Gamma_0)f = (\mathcal{J}_{\vec{\alpha}}\Gamma_1 - i\Gamma_0)f\}, \tag{2.31}$$

where U runs the set of unitary operators in $\mathcal{H}$. The correspondence $A \leftrightarrow U$ determined by (2.31) *is a bijection between the set $\Sigma_{J_{\vec{\alpha}}}$ of all $J_{\vec{\alpha}}$-self-adjoint extensions of S and the set of unitary operators in $\mathcal{H}$.*

Proof. An operator A is a $J_{\vec{\alpha}}$-self-adjoint extension of S if and only if $J_{\vec{\alpha}}A$ is a self-adjoint extension of the symmetric operator $J_{\vec{\alpha}}S$. Since

$$(J_{\vec{\alpha}}S)^* = S^* J_{\vec{\alpha}} = J_{\vec{\alpha}} S^*, \tag{2.32}$$

the Green identity (2.25) can be rewritten with the use of (2.29) as follows:

$$(S^* J_{\vec{\alpha}} x, y) - (x, S^* J_{\vec{\alpha}} y) = (\mathcal{J}_{\vec{\alpha}}\Gamma_1 x, \Gamma_0 y)_{\mathcal{H}} - (\Gamma_0 x, \mathcal{J}_{\vec{\alpha}}\Gamma_1 y)_{\mathcal{H}}.$$

Recalling the definition of boundary triplet we conclude that $(\mathcal{H}, \Gamma_0, \mathcal{J}_{\vec{\alpha}}\Gamma_1)$ is a boundary triplet of $J_{\vec{\alpha}}S$. Therefore [9, Chapter 3, Theorem 1.6], self-adjoint extensions $J_{\vec{\alpha}}A$ of $J_{\vec{\alpha}}S$ coincide with the restriction of $(J_{\vec{\alpha}}S)^*$ onto

$$\mathcal{D}(J_{\vec{\alpha}}A) = \{f \in \mathcal{D}((J_{\vec{\alpha}}S)^*) \ : \ U(\mathcal{J}_{\vec{\alpha}}\Gamma_1 + i\Gamma_0)f = (\mathcal{J}_{\vec{\alpha}}\Gamma_1 - i\Gamma_0)f\}$$

where U runs the set of unitary operators in $\mathcal{H}$ and the correspondence $J_{\vec{\alpha}}A \leftrightarrow U$ is bijective. By virtue of (2.32), $\mathcal{D}((J_{\vec{\alpha}}S)^*) = \mathcal{D}(S^*)$. Hence, $J_{\vec{\alpha}}$-self-adjoint extensions A of S coincide with the restriction of S^* onto $\mathcal{D}(J_{\vec{\alpha}}A)$ that implies (2.31). □

Corollary 2.16. *If $A \in \Sigma_{J_{\vec{\alpha}}}$ and $A \leftrightarrow U$ in* (2.31)*, then the $J_{\vec{\beta}}$-self-adjoint operator $B = W_{\vec{\alpha},\vec{\beta}} A W_{\vec{\alpha},\vec{\beta}} \in \Sigma_{J_{\vec{\beta}}}$ $(\vec{\alpha} \neq -\vec{\beta})$ is determined by the formula*

$$B = S^* \restriction \{g \in \mathcal{D}(S^*) \ : \ \mathcal{W}_{\vec{\alpha},\vec{\beta}} U \mathcal{W}_{\vec{\alpha},\vec{\beta}} (\mathcal{J}_{\vec{\beta}}\Gamma_1 + i\Gamma_0)g = (\mathcal{J}_{\vec{\beta}}\Gamma_1 - i\Gamma_0)g\},$$

where $\mathcal{W}_{\vec{\alpha},\vec{\beta}} = \mathcal{J}_{\frac{\vec{\alpha}+\vec{\beta}}{|\vec{\alpha}+\vec{\beta}|}}$ is a fundamental symmetry in $\mathcal{H}$.

Proof. Let $A \in \Sigma_{J_{\vec{\alpha}}}$. Then $B = W_{\vec{\alpha},\vec{\beta}} A W_{\vec{\alpha},\vec{\beta}} \in \Sigma_{J_{\vec{\beta}}}$ by Theorem 2.9 and, in view of Theorem 2.15,

$$B = S^* \restriction \{g \in \mathcal{D}(S^*) \ : \ U'(\mathcal{J}_{\vec{\beta}}\Gamma_1 + i\Gamma_0)g = (\mathcal{J}_{\vec{\beta}}\Gamma_1 - i\Gamma_0)g\}, \tag{2.33}$$

where U' is a unitary operator in $\mathcal{H}$.

It follows from the definition of B that $f \in \mathcal{D}(A)$ if and only if $g = W_{\vec{\alpha},\vec{\beta}} f \in \mathcal{D}(B)$. Hence, we can rewrite (2.33) with the use of (2.24):

$$\begin{aligned} U'(\mathcal{J}_{\vec{\beta}}\Gamma_1 + i\Gamma_0)g &= U'\mathcal{W}_{\vec{\alpha},\vec{\beta}}(\mathcal{J}_{\vec{\alpha}}\Gamma_1 + i\Gamma_0)f = \\ &= (\mathcal{J}_{\vec{\beta}}\Gamma_1 - i\Gamma_0)g = \\ &= \mathcal{W}_{\vec{\alpha},\vec{\beta}}(\mathcal{J}_{\vec{\alpha}}\Gamma_1 - i\Gamma_0)f, \end{aligned} \tag{2.34}$$

where $\mathcal{W}_{\vec{\alpha},\vec{\beta}}\Gamma_j = \Gamma_j W_{\vec{\alpha},\vec{\beta}},\ j = 0, 1$ (cf. (2.29)).

It follows from (2.23) that $\mathcal{W}_{\vec{\alpha},\vec{\beta}} = \mathcal{J}_{\frac{\vec{\alpha}+\vec{\beta}}{|\vec{\alpha}+\vec{\beta}|}}$ and hence, $\mathcal{W}_{\vec{\alpha},\vec{\beta}}$ is a fundamental symmetry in $\mathcal{H}$. Comparing (2.34) with (2.31), we arrive at the conclusion that $U' = \mathcal{W}_{\vec{\alpha},\vec{\beta}} U \mathcal{W}_{\vec{\alpha},\vec{\beta}}$. Corollary 2.16 is proved. □

Corollary 2.17. *A $J_{\vec{\alpha}}$-self-adjoint operator $A \in \Sigma_{J_{\vec{\alpha}}}$ commutes with $J_{\vec{\beta}}$, where $\vec{\alpha} \cdot \vec{\beta} = 0$ if and only if the corresponding unitary operator U in* (2.31) *satisfies the relation*

$$\mathcal{J}_{\vec{\beta}} U = U^{-1} \mathcal{J}_{\vec{\beta}}. \tag{2.35}$$

Proof. Assume that $\vec{\beta} \in \mathbb{S}^2$ and $\vec{\alpha} \cdot \vec{\beta} = 0$. Then $J_{\vec{\beta}} J_{\vec{\alpha}} = -J_{\vec{\alpha}} J_{\vec{\beta}}$ due to Lemma 2.3. Since $J_{\vec{\beta}} S^* = S^* J_{\vec{\beta}}$, the commutation relation $AJ_{\vec{\beta}} = J_{\vec{\beta}} A$ is equivalent to the condition

$$\forall f \in \mathcal{D}(A) \Rightarrow J_{\vec{\beta}} f \in \mathcal{D}(A). \tag{2.36}$$

Let $f \in \mathcal{D}(S^*)$. Recalling that $\mathcal{J}_{\vec{\beta}} \Gamma_j = \Gamma_j J_{\vec{\beta}}$, we obtain

$$(\mathcal{J}_{\vec{\alpha}} \Gamma_1 + i\Gamma_0) J_{\vec{\beta}} f = -\mathcal{J}_{\vec{\beta}} (\mathcal{J}_{\vec{\alpha}} \Gamma_1 - i\Gamma_0) f,$$
$$(\mathcal{J}_{\vec{\alpha}} \Gamma_1 - i\Gamma_0) J_{\vec{\beta}} f = -\mathcal{J}_{\vec{\beta}} (\mathcal{J}_{\vec{\alpha}} \Gamma_1 + i\Gamma_0) f.$$

Combining the last two relations with (2.31), we conclude that (2.36) is equivalent to the identity $\mathcal{J}_{\vec{\beta}} U^{-1} \mathcal{J}_{\vec{\beta}} = U$. Corollary 2.17 is proved. □

Corollary 2.18. *A $J_{\vec{\alpha}}$-self-adjoint operator $A \in \Sigma_{J_{\vec{\alpha}}}$ belongs to the subset Υ (see* (2.7)*) if and only if the corresponding unitary operator U in* (2.31) *satisfies the equality* (2.35) for all $\vec{\beta} \in \mathbb{S}^2$ *such that $\vec{\alpha} \cdot \vec{\beta} = 0$.*

Proof. Since U satisfies (2.35) for all $\vec{\beta} \in \mathbb{S}^2$ such that $\vec{\alpha} \cdot \vec{\beta} = 0$, the operator $A \in \Sigma_{J_{\vec{\alpha}}}$ commutes with an arbitrary $J_{\vec{\beta}}$ such that $J_{\vec{\alpha}} J_{\vec{\beta}} = -J_{\vec{\beta}} J_{\vec{\alpha}}$ (due to Lemma 2.3 and Corollary 2.17). In particular, the fundamental symmetry $J_{\vec{\gamma}} = iJ_{\vec{\alpha}} J_{\vec{\beta}}$ anti-commutes with $J_{\vec{\alpha}}$ and hence, $J_{\vec{\gamma}} A = AJ_{\vec{\gamma}}$. On the other hand, since $A \in \Sigma_{J_{\vec{\alpha}}}$, we have $J_{\vec{\alpha}} A = A^* J_{\vec{\alpha}}$ and

$$J_{\vec{\gamma}} A = iJ_{\vec{\alpha}} J_{\vec{\beta}} A = iJ_{\vec{\alpha}} AJ_{\vec{\beta}} = A^* iJ_{\vec{\alpha}} J_{\vec{\beta}} = A^* J_{\vec{\gamma}}.$$

Thus $AJ_{\vec{\gamma}} = A^* J_{\vec{\gamma}}$ and hence, $A = A^*$. This means that the self-adjoint extension $A \supset S$ commutes with all fundamental symmetries from the Clifford algebra $\mathcal{C}l_2(J, R)$. Therefore, $A \in \Upsilon$. □

Corollary 2.19. *Let $A \in \Sigma_{J_{\vec{\alpha}}}$ be defined by* (2.31) *with $U = \mathcal{J}_{\vec{\gamma}}$, where $\vec{\gamma} \in \mathbb{S}^2$ is an arbitrary vector such that $\vec{\alpha} \cdot \vec{\gamma} = 0$. Then $\sigma(A) = \mathbb{C}$, i.e., A has empty resolvent set.*

Proof. Taking into account that $\mathcal{J}_{\vec{\alpha}} \mathcal{J}_{\vec{\gamma}} = -\mathcal{J}_{\vec{\gamma}} \mathcal{J}_{\vec{\alpha}}$ (since $\vec{\alpha} \cdot \vec{\gamma} = 0$), we rewrite the definition (2.31) of A:

$$A = S^* \restriction \{f \in \mathcal{D}(S^*) \ : \ \mathcal{J}_{\vec{\alpha}} (\mathcal{J}_{\vec{\gamma}} + I)\Gamma_1 f = i(\mathcal{J}_{\vec{\gamma}} + I)\Gamma_0 f\}. \tag{2.37}$$

Since relation (2.35) holds when $U = \mathcal{J}_{\vec{\gamma}}$ and $\vec{\beta} = \vec{\gamma}$, the operator $A \in \Sigma_{J_{\vec{\alpha}}}$ commutes with $J_{\vec{\gamma}}$ (Corollary 2.17). Therefore (cf. (2.16)),

$$A = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}, \qquad A_+ = A \restriction_{\mathfrak{H}_+^\gamma}, \quad A_- = A \restriction_{\mathfrak{H}_-^\gamma} \tag{2.38}$$

with respect to the decomposition (2.12). Here $S_{\gamma+} \subseteq A_+ \subseteq S_{\gamma+}^*$ and $S_{\gamma-} \subseteq A_- \subseteq S_{\gamma-}^*$, where $S_{\gamma\pm} = S \restriction_{\mathfrak{H}_\pm^\gamma}$.

Denote $\mathcal{H}_+^\gamma = \frac{1}{2}(I + \mathcal{J}_{\vec{\gamma}})\mathcal{H}$ and $\mathcal{H}_-^\gamma = \frac{1}{2}(I - \mathcal{J}_{\vec{\gamma}})\mathcal{H}$. Then

$$\mathcal{H} = \mathcal{H}_+^\gamma \oplus \mathcal{H}_-^\gamma \tag{2.39}$$

and $(\mathcal{H}^{\gamma}_{\pm}, \Gamma_0, \Gamma_1)$ are boundary triplets of operators $S^*_{\gamma\pm}$ (due to (2.29) and Lemma 2.11).

Let $f \in \mathcal{D}(S^*_{\gamma+})$. Then $\Gamma_j f \in \mathcal{H}^{\gamma}_{+}$, $j = 0, 1$ and the identity in (2.37) takes the form

$$\mathcal{J}_{\vec{\alpha}}\Gamma_1 f = i\Gamma_0 f. \tag{2.40}$$

Since $\mathcal{J}_{\vec{\alpha}}\mathcal{J}_{\vec{\gamma}} = -\mathcal{J}_{\vec{\gamma}}\mathcal{J}_{\vec{\alpha}}$, the operator $\mathcal{J}_{\vec{\alpha}}$ maps $\mathcal{H}^{\gamma}_{+}$ onto $\mathcal{H}^{\gamma}_{-}$. Thus, (2.40) may only hold in the case where $\Gamma_0 f = \Gamma_1 f = 0$. Therefore, the operator A_+ in (2.38) coincides with S^{γ}_{+}.

Assume now $f \in \mathcal{D}(S^*_{\gamma-})$. Then $\Gamma_j f \in \mathcal{H}^{\gamma}_{-}$, $j = 0, 1$ and the identity in (2.37) vanishes (i.e., $0 = 0$). This means that $A_- = S^*_{\gamma-}$. Therefore, $A = A_\gamma$, where A_γ is defined by (2.14) and $\sigma(A_\gamma) = \mathbb{C}$. □

2.5. THE RESOLVENT FORMULA

Let $\gamma(\mu) = (\Gamma_0 \upharpoonright_{\ker(S^*-\mu I)})^{-1}$ be the γ-field corresponding to the boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ of S^* with properties (2.26). Since $J_{\vec{\alpha}}$ maps $\ker(S^*-\mu I)$ onto $\ker(S^*-\mu I)$, formula (2.29) implies

$$\gamma(\mu)\mathcal{J}_{\vec{\alpha}} = J_{\vec{\alpha}}\gamma(\mu), \quad \forall \mu \in \mathbb{C} \setminus \mathbb{R}$$

for an arbitrary fundamental symmetry $J_{\vec{\alpha}} \in \mathcal{C}l_2(J, R)$.

Let $A_0 = S^* \upharpoonright \ker \Gamma_0$. Then A_0 is a self-adjoint extension of S (due to the general properties of boundary triplets [9]). Moreover, it follows from (2.7) and Remark 2.13 that $A_0 \in \Upsilon$.

Proposition 2.20. *Let* $(\mathcal{H}, \Gamma_0, \Gamma_1)$ *be a boundary triplet of* S^* *with properties* (2.26) *and let* $A \in \Sigma_{J_{\vec{\alpha}}}$ *be defined by* (2.31). *Assume that* A *is disjoint with* A_0 *(i.e.,* $\mathcal{D}(A) \cap \mathcal{D}(A_0) = \mathcal{D}(S)$*) and* $\mu \in \rho(A) \cap \rho(A_0)$*, then*

$$(A - \mu I)^{-1} = (A_0 - \mu I)^{-1} - \gamma(\mu)[M(\mu) - T]^{-1}\gamma^*(\overline{\mu}), \tag{2.41}$$

where $T = i\mathcal{J}_{\vec{\alpha}}(I + U)(I - U)^{-1}$ *is a* $\mathcal{J}_{\vec{\alpha}}$*-self-adjoint operator in the Krein space* $(\mathcal{H}, [\cdot, \cdot]_{\mathcal{J}_{\vec{\alpha}}})$.

Proof. Since A and A_0 are disjoint, the unitary operator U which corresponds to the operator $A \in \Sigma_{J_{\vec{\alpha}}}$ in (2.31) satisfies the relation $\ker(I - U) = \{0\}$. This relation and (2.29) allow one to rewrite (2.31) as follows:

$$A = S^* \upharpoonright \{f \in \mathcal{D}(S^*) \mid T\Gamma_0 f = \Gamma_1 f\}, \tag{2.42}$$

where $T = i\mathcal{J}_{\vec{\alpha}}(I + U)(I - U)^{-1}$ is a $\mathcal{J}_{\vec{\alpha}}$-self-adjoint operator in the Krein space $(\mathcal{H}, [\cdot, \cdot]_{\mathcal{J}_{\vec{\alpha}}})$ (due to self-adjointness of $i(I + U)(I - U)^{-1}$). Repeating the standard arguments (see, e.g., [8, p.14]), we deduce (2.41) from (2.42). □

Remark 2.21. The condition of disjointness of A and A_0 in Proposition 2.20 is not essential and it is assumed for simplifying the exposition. In particular, this allows one to avoid operators A with empty resolvent set (see Corollary 2.19 and relation

(2.37)) for which the formula (2.41) has no sense. In the case of an arbitrary $A \in \Sigma_{J_{\vec{\alpha}}}$ with non-empty resolvent set, the formula (2.41) also remains true if we interpret T as a $\mathcal{J}_{\vec{\alpha}}$-self-adjoint relation in $\mathcal{H}$ (see [12, Theorem 3.22] for a similar result and [6] for the basic definitions of linear relations theory).

3. THE CASE OF DEFICIENCY INDICES $\langle 2, 2\rangle$

In what follows, the symmetric operator S has deficiency indices $\langle 2, 2\rangle$.

1. Let $(\mathcal{H}, \Gamma_0, \Gamma_1)$ be a boundary triplet of S^* with properties (2.26) or, that is equivalent, with properties (2.29). Let us fix an arbitrary fundamental symmetry $\mathcal{J}_{\vec{\gamma}} \in \mathcal{C}l_2(\mathcal{J}, \mathcal{R})$ and consider the decomposition $\mathcal{H} = \mathcal{H}_+^{\gamma} \oplus \mathcal{H}_-^{\gamma}$ constructed by $\mathcal{J}_{\vec{\gamma}}$ (see (2.39)). Then the Weyl function $M(\cdot)$ associated with $(\mathcal{H}, \Gamma_0, \Gamma_1)$ can be rewritten as

$$M(\cdot) = \begin{pmatrix} m_{++}(\cdot) & m_{+-}(\cdot) \\ m_{-+}(\cdot) & m_{--}(\cdot) \end{pmatrix}, \qquad m_{xy}(\cdot) : \mathcal{H}_y^{\gamma} \to \mathcal{H}_x^{\gamma}, \quad x, y \in \{+, -\},$$

where $m_{xy}(\cdot)$ are scalar functions (since $\dim \mathcal{H} = 2$ and $\dim \mathcal{H}_\pm^{\gamma} = 1$).

According to Lemma 2.14, $M(\cdot)$ commutes with every fundamental symmetry from $\mathcal{C}l_2(\mathcal{J}, \mathcal{R})$. In particular, $\sigma_j M(\cdot) = M(\cdot)\sigma_j$ $(j = 1, 3)$, where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are Pauli matrices. This is possible only in the case

$$m_{+-}(\cdot) = m_{-+}(\cdot) = 0, \qquad m_{++}(\cdot) = m_{--}(\cdot),$$

i.e.,

$$M(\cdot) = m(\cdot)E, \tag{3.1}$$

where $m(\cdot) = m_{++}(\cdot) = m_{--}(\cdot)$ is a scalar function defined on $\mathbb{C} \setminus \mathbb{R}$ and E is the identity 2×2-matrix.

Recalling that $(\mathcal{H}_+^{\gamma}, \Gamma_0, \Gamma_1)$ is a boundary triplet of $S_{\gamma+}^*$ (see the proof of Corollary 2.19) and taking into account the definition (2.30) of Weyl functions, we arrive at the conclusion that $m(\cdot)$ is the Weyl function of $S_{\gamma+} = S \restriction_{\mathfrak{H}_+^{\gamma}}$ associated with boundary triplet $(\mathcal{H}_+^{\gamma}, \Gamma_0^+, \Gamma_1^+)$.

The following statement is proved.

Proposition 3.1. *Let $(\mathcal{H}, \Gamma_0, \Gamma_1)$ be a boundary triplet of S defined above. Then the Weyl function $M(\cdot)$ is defined by* (3.1), *where $m(\cdot)$ is the Weyl function of $S_{\gamma+}$ associated with boundary triplet $(\mathcal{H}_+^{\gamma}, \Gamma_0, \Gamma_1)$. The function $m(\cdot)$ does not depend on the choice of $\vec{\gamma} \in \mathbb{S}^2$.*

2. Let $\vec{\alpha}, \vec{\beta} \in \mathbb{S}^2$ be linearly independent vectors. According to Corollary 2.7 all operators $A \in \Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}}$ are described by the formula (2.21). This means that spectra of these operators are completely characterized by the spectra of their counterparts A_+ in (2.21).

The operator A_+ is supposed to be an intermediate extension of $S_{\gamma+}$. Two different situations may occur: 1. $A_+ = S_{\gamma+}$ or $A_+ = S^*_{\gamma+}$; 2. A_+ is a quasi-self-adjoint extension[4)] of S, i.e., $S_{\gamma+}\subset A_+\subset S^*_{\gamma+}$. In the first case, the operators $A \in \Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}}$ have empty resolvent set (Theorem 2.5); in the second case, the spectral properties of A_+ (and hence, A) are well known (see, e.g., [3, Theorem 1, Appendix I]). Summing up, we arrive at the following conclusion.

Proposition 3.2. *Let S be a simple symmetric operator with deficiency indices $\langle 2, 2\rangle$ and $A \in \Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}}$. Then or $\sigma(A) = \mathbb{C}$ or the spectrum of A consists of the spectral kernel of S and the set of eigenvalues which can have only real accommodation points.*

3. Denote by $\Xi_{\vec{\alpha}}$ the collection of all operators $A \in \Sigma_{J_{\vec{\alpha}}}$ with empty resolvent set:

$$\Xi_{\vec{\alpha}} = \{A \in \Sigma_{J_{\vec{\alpha}}} \ : \ \rho(A) = \emptyset \ \}$$

and by $\Xi_{\vec{\alpha},\vec{\beta}}$ the pair of two operators $A_{\gamma(\vec{\beta})}$ and $A^*_{\gamma(\vec{\beta})}$ with empty resolvent set which are defined by (2.14) for a fixed $\vec{\alpha}$ and $\vec{\beta}$.

Theorem 3.3. *Assume that S is a symmetric operator with deficiency indices $\langle 2, 2\rangle$ and its Weyl function (associated with an arbitrary boundary triplet) differs from constant on $\mathbb{C} \setminus \mathbb{R}$. Then*

$$\Xi_{\vec{\alpha}} = \bigcup_{\forall\vec{\beta}\in\mathbb{S}^2,\ \vec{\alpha}\cdot\vec{\beta}=0} \Xi_{\vec{\alpha},\vec{\beta}}. \tag{3.2}$$

Proof. By Theorem 2.5, $\Xi_{\vec{\alpha}} \supset \Xi_{\vec{\alpha},\vec{\beta}}$ for all $\vec{\beta} \in \mathbb{S}^2$ such that $\vec{\alpha} \cdot \vec{\beta} = 0$. Therefore, $\Xi_{\vec{\alpha}} \supset \bigcup \Xi_{\vec{\alpha},\vec{\beta}}$.

In the case of deficiency indices $\langle 2, 2\rangle$ of S, the set $\Xi_{\vec{\alpha}}$ of all $J_{\vec{\alpha}}$-self-adjoint extensions with empty resolvent set is described in [15]. We briefly outline the principal results.

Denote by $\mathfrak{N}_\mu = \ker(S^* - \mu I)$, $\mu \in \mathbb{C} \setminus \mathbb{R}$, the defect subspaces of S and consider the Hilbert space $\mathfrak{M} = \mathfrak{N}_i\dot{+}\mathfrak{N}_{-i}$ with the inner product

$$(x, y)_{\mathfrak{M}} = 2[(x_i, y_i) + (x_{-i}, y_{-i})],$$

where $x = x_i + x_{-i}$ and $y = y_i + y_{-i}$ with $x_i, y_i \in \mathfrak{N}_i$, $x_{-i}, y_{-i} \in \mathfrak{N}_{-i}$.

The operator Z that acts as identity operator I on $\mathfrak{N}_i$ and minus identity operator $-I$ on $\mathfrak{N}_{-i}$ is an example of fundamental symmetry in $\mathfrak{M}$. Other examples can be constructed due to the fact that S commutes with $J_{\vec{\beta}}$ for all $\vec{\beta} \in \mathbb{S}^2$. This means that the subspaces $\mathfrak{N}_{\pm i}$ reduce $J_{\vec{\beta}}$ and the restriction $J_{\vec{\beta}} \restriction \mathfrak{M}$ gives rise to a fundamental symmetry in the Hilbert space $\mathfrak{M}$. Moreover, according to the properties of Z mentioned above, $J_{\vec{\beta}}Z = ZJ_{\vec{\beta}}$ and $J_{\vec{\beta}}Z$ is a fundamental symmetry in $\mathfrak{M}$. Therefore, the sesquilinear form

$$[x, y]_{J_{\vec{\beta}}Z} = (J_{\vec{\beta}}Zx, y)_{\mathfrak{M}} = 2[(J_{\vec{\beta}}x_i, y_i) - (J_{\vec{\beta}}x_{-i}, y_{-i})]$$

defines an indefinite metric on $\mathfrak{M}$.

[4)] This class includes self-adjoint extensions also.

According to the von-Neumann formulas, any closed intermediate extension A of S (i.e., $S \subseteq A \subseteq S^*$) is uniquely determined by the choice of a subspace $M \subset \mathfrak{M}$:

$$A = S^* \restriction_{\mathcal{D}(A)}, \qquad \mathcal{D}(A) = \mathcal{D}(S) \dot{+} M. \tag{3.3}$$

In particular, $J_{\vec{\beta}}$-self-adjoint extensions A of S correspond to *hypermaximal neutral subspaces M with respect to* $[\cdot,\cdot]_{J_{\vec{\beta}}Z}$. This means that $A \in \Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}}$ if and only if the corresponding subspace M in (3.3) is *simultaneously hypermaximal neutral with respect to two different indefinite metrics* $[\cdot,\cdot]_{J_{\vec{\alpha}}Z}$ *and* $[\cdot,\cdot]_{J_{\vec{\beta}}Z}$.

Without loss of generality we assume that $J_{\vec{\alpha}}$ coincides with J in (2.3), i.e., $\vec{\alpha} = (1,0,0)$. Then fundamental symmetries $J_{\vec{\beta}}$ which anti-commute with J have the form

$$J_{\vec{\beta}} = \beta_2 R + \beta_3 iJR, \qquad \beta_2^2 + \beta_3^2 = 1. \tag{3.4}$$

To specify M we consider an orthonormal basis $\{e_{++}, e_{+-}, e_{-+}, e_{--}\}$ of $\mathfrak{M}$ which satisfies the relations

$$\begin{gathered} Ze_{++} = e_{++}, \quad Ze_{+-} = e_{+-}, \quad Ze_{-+} = -e_{-+}, \quad Ze_{--} = -e_{--}, \\ J_{\vec{\alpha}}e_{++} = e_{++}, \quad J_{\vec{\alpha}}e_{+-} = -e_{+-}, \quad J_{\vec{\alpha}}e_{-+} = e_{-+}, \quad J_{\vec{\alpha}}e_{--} = -e_{--}, \\ Re_{++} = e_{+-}, \quad Re_{+-} = e_{++}, \quad Re_{--} = e_{-+}, \quad Re_{-+} = e_{--}. \end{gathered} \tag{3.5}$$

The existence of this basis was established in [2] and it was used in [15, Corollary 3.2] to describe the collection of all M in (3.3) which correspond to J-self-adjoint extensions of S with empty resolvent set. Such a description depends on properties of Weyl function of S. In particular, if the Weyl function differs from the constant for a fixed boundary triplet, then this property remains true for Weyl functions associated with an arbitrary boundary triplet of S. Then, using relations (2.7)–(2.9) in [15], we deduce that the Straus characteristic function of S (see [18]) differs from the zero-function on $\mathbb{C} \setminus \mathbb{R}$. In this case, Corollary 3.2 in [15] says that a J-self-adjoint extension A has empty resolvent set if and only if the corresponding subspace M coincides with linear span $M = \mathsf{span}\{d_1, d_2\}$, where $d_1 = e_{++} + e^{i\gamma}e_{+-}$, $d_2 = e_{--} + e^{-i\gamma}e_{-+}$, and $\gamma \in [0, 2\pi)$ is an arbitrary parameter.

The operator A will belong to $\Sigma_{J_{\vec{\beta}}}$ if and only if the subspace $M = \mathsf{span}\{d_1, d_2\}$ turns out to be hypermaximal neutral with respect to $[\cdot,\cdot]_{J_{\vec{\beta}}Z}$. Since $\dim M = 2$ and $\dim \mathfrak{M} = 4$, it suffices to check the neutrality of M. The last condition is equivalent to the relations

$$[d_1, d_2]_{J_{\vec{\beta}}Z} = 0, \qquad [d_1, d_1]_{J_{\vec{\beta}}Z} = 0, \qquad [d_2, d_2]_{J_{\vec{\beta}}Z} = 0.$$

Using (3.4), (3.5), and remembering the orthogonality of $e_{\pm,\pm}$ in $\mathfrak{M}$, we establish that $[d_1, d_2]_{J_{\vec{\beta}}Z} = 0$ for all $\gamma \in [0, 2\pi)$. The next two conditions are transformed to the linear equation

$$(\cos\gamma)\beta_2 - (\sin\gamma)\beta_3 = 0, \tag{3.6}$$

which has the nontrivial solution $\beta_2 = \sin\gamma$, $\beta_3 = \cos\gamma$ for any $\gamma \in [0, 2\pi)$. This means that an arbitrary J-self-adjoint extension A with empty resolvent set is also a $J_{\vec{\beta}}$-self-adjoint operator under choosing β_2 and β_3 in (3.4) as solutions of (3.6). Theorem 3.3 is proved. □

Corollary 3.4. *Let S be a symmetric operator with deficiency indices $\langle 2,2\rangle$ and let $(\mathcal{H},\Gamma_0,\Gamma_1)$ be a boundary triplet of S^* with properties* (2.26). *If the Weyl function of S differs from constant on $\mathbb{C}\setminus\mathbb{R}$, then the set $\Xi_{\vec{\alpha}}$ is described by* (2.31) *where U runs the set of all fundamental symmetries $\mathcal{J}_{\vec{\beta}}\in\mathcal{C}l_2(\mathcal{J},\mathcal{R})$ such that $\vec{\alpha}\cdot\vec{\beta}=0$.*

Proof. It follows from Corollary 2.19 and Theorem 3.3. □

Theorem 3.3 and Corollary 3.4 are not true when the Weyl function of S is a constant. In that case, the set $\Xi_{\vec{\alpha}}$ of $J_{\vec{\alpha}}$-self-adjoint extensions increases considerably and $\Xi_{\vec{\alpha}}\supset\bigcup\Xi_{\vec{\alpha},\vec{\beta}}$.

Corollary 3.5. *Let S be a simple symmetric operator with deficiency indices $\langle 2,2\rangle$. Then the following statements are equivalent:*

(i) *the strict inclusion*

$$\Xi_{\vec{\alpha}}\supset\bigcup_{\forall\vec{\beta}\in\mathbb{S}^2,\ \vec{\alpha}\cdot\vec{\beta}=0}\Xi_{\vec{\alpha},\vec{\beta}}$$

holds,

(ii) *the Weyl function $M(\cdot)$ of S is a constant on $\mathbb{C}\setminus\mathbb{R}$,*

(iii) *S is unitarily equivalent to the symmetric operator in $L_2(\mathbb{R},\mathbb{C}^2)$:*

$$S'=i\frac{d}{dx},\qquad \mathcal{D}(S')=\{u\in W_2^1(\mathbb{R},\mathbb{C}^2)\ :\ u(0)=0\}. \tag{3.7}$$

Proof. Assume that the Weyl function $M(\cdot)$ of S is a constant. By (3.1), the Weyl function $m(\cdot)$ of $S_{\gamma+}=S\upharpoonright_{\mathfrak{H}_+^\gamma}$ is also constant. This means that the Straus characteristic function of the simple symmetric operator $S_{\gamma+}$ with deficiency indices $\langle 1,1\rangle$ is zero on $\mathbb{C}\setminus\mathbb{R}$ (see the proof of Theorem 3.3). Therefore, $S_{\gamma+}$ is unitarily equivalent to the symmetric operator $S'_+=i\frac{d}{dx}$, $\mathcal{D}(S'_+)=\{u\in W_2^1(\mathbb{R})\ :\ u(0)=0\}$ in $L_2(\mathbb{R})$ [16, Subsection 3.4].

Recalling the decomposition (2.13) of S, where the simple symmetric operator $S_{\gamma-}=S\upharpoonright_{\mathfrak{H}_-^\gamma}$ also has deficiency indices $\langle 1,1\rangle$ and zero characteristic function, we conclude that S is unitarily equivalent to the symmetric operator S' defined by (3.7). This establishes the equivalence of (ii) and (iii).

Assume again that the Weyl function of S is a constant. Then the Straus characteristic function of S is zero. In that case, Corollary 3.2 in [15] yields that $A\in\Xi_{\vec{\alpha}}$ if and only if the corresponding subspace M in (3.3) coincides with linear span

$$M=\mathsf{span}\{d_1,d_2\},\quad d_1=e_{++}+e^{i(\phi+\gamma)}e_{+-},\quad d_2=e_{--}+e^{i(\phi-\gamma)}e_{-+}, \tag{3.8}$$

where $\phi,\gamma\in[0,2\pi)$ are *two arbitrary parameters.* Thus the set $\Xi_{\vec{\alpha}}$ is described by two independent parameters ϕ and γ.

Due to the proof of Theorem 3.3, the operator $A\in\Xi_{\vec{\alpha}}$ belongs to the subset $\Xi_{\vec{\alpha},\vec{\beta}}$ if and only if the subspace M in (3.8) is neutral with respect to $[\cdot,\cdot]_{J_{\vec{\beta}}Z}$. Repeating the argumentation above, we conclude that the neutrality of M is equivalent to the existence of nontrivial solution β_2,β_3 of the system (cf. (3.6))

$$\begin{cases}\cos(\phi+\gamma)\beta_2-\sin(\phi+\gamma)\beta_3=0,\\ \cos(\phi-\gamma)\beta_2+\sin(\phi-\gamma)\beta_3=0.\end{cases} \tag{3.9}$$

The determinant of (3.9) is $\sin 2\phi$. Therefore, there are no nontrivial solutions for $\phi \notin \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$. This means the existence of operators $A \in \Xi_{\vec{\alpha}}$ which, simultaneously, do not belong to $\bigcup \Xi_{\vec{\alpha},\vec{\beta}}$. Thus, we establish the equivalence of (ii) and (i). Corollary 3.5 is proved. □

4. Consider the one-dimensional Schrödinger differential expression

$$l(\phi)(x) = -\phi''(x) + q(x)\phi(x), \qquad x \in \mathbb{R}, \tag{3.10}$$

where q is an *even* real-valued measurable function that has a non-integrable singularity at zero and is integrable on every finite subinterval of $\mathbb{R} \setminus \{0\}$.

Assume in what follows that the potential $q(x)$ is in the limit point case at $x \to \pm\infty$ and is in the limit-circle case at $x = 0$. Denote by $\mathcal{D}$ the set of all functions $\phi(x) \in L_2(\mathbb{R})$ such that ϕ and ϕ' are absolutely continuous on every finite subinterval of $\mathbb{R} \setminus \{0\}$ and $l(\phi) \in L_2(\mathbb{R})$. On $\mathcal{D}$ we define the operator L as follows:

$$L\phi = l(\phi), \qquad \forall \phi \in \mathcal{D}.$$

The operator L commutes with the space parity operator $\mathcal{P}\phi(x) = \phi(-x)$ and with the operator of multiplication by $(\mathsf{sgn}\ x)I$. These operators are anti-commuting fundamental symmetries in $L_2(\mathbb{R})$. Therefore, L commutes with elements of the Clifford algebra $\mathcal{C}l_2(\mathcal{P}, (\mathsf{sgn}\ x)I)$. However, L is not a symmetric operator.

Denote for brevity $J_{\vec{\gamma}} = (\mathsf{sgn}\ x)I$. Then, the decomposition (2.12) takes the form $L_2(\mathbb{R}) = L_2(\mathbb{R}_+) \oplus L_2(\mathbb{R}_-)$ and with respect to it

$$L = \begin{pmatrix} L_+ & 0 \\ 0 & \mathcal{P}L_+\mathcal{P} \end{pmatrix}, \qquad L_+ = L \upharpoonright_{L_2(\mathbb{R}_+)} . \tag{3.11}$$

The operator L_+ is the maximal operator for differential expression $l(\phi)$ considered on semi-axes $\mathbb{R}_+ = (0, \infty)$. Denote by S_+ the minimal operator generated $l(\phi)$ in $L_2(\mathbb{R}_+)$. The symmetric operator S_+ has deficiency indices $\langle 1, 1\rangle$.

Let $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$ be an arbitrary boundary triplet of $L_+ = S_+^*$ in $L_2(\mathbb{R}_+)$. Then, the boundary triplet $(\mathbb{C}^2, \Gamma_0, \Gamma_1)$ determined by (2.27) with $J_{\vec{\tau}} = \mathcal{P}$ and $N = \mathbb{C}$ is a boundary triplet of $L = S^*$ in the space $L_2(\mathbb{R})$. Here $S = \begin{pmatrix} S_+ & 0 \\ 0 & \mathcal{P}S_+\mathcal{P} \end{pmatrix}$ is the symmetric operator in $L_2(\mathbb{R})$ (cf. (3.11)) with deficiency indices $\langle 2, 2\rangle$.

Let $J_{\vec{\alpha}}$ be an arbitrary fundamental symmetry from $\mathcal{C}l_2(\mathcal{P}, (\mathsf{sgn}\ x)I)$. By Theorem 2.15, $J_{\vec{\alpha}}$-self-adjoint extensions $A \in \Sigma_{J_{\vec{\alpha}}}$ of S are defined as the restrictions of L:

$$A = L \upharpoonright \{f \in \mathcal{D} \ : \ U(\mathcal{J}_{\vec{\alpha}}\Gamma_1 + i\Gamma_0)f = (\mathcal{J}_{\vec{\alpha}}\Gamma_1 - i\Gamma_0)f\}, \tag{3.12}$$

where U runs the set of 2×2-unitary matrices. The operators A can be interpreted as $J_{\vec{\alpha}}$-self-adjoint operator realizations of differential expression (3.10) in $L_2(\mathbb{R})$.

Since the sets $\Sigma_{J_{\vec{\alpha}}}$ are unitarily equivalent for different $\vec{\alpha} \in \mathbb{S}^2$ (Theorem 2.9) one can set $J_{\vec{\alpha}} = \mathcal{P}$ for definiteness.

Proposition 3.6. *The collection of all $\mathcal{P}$-self-adjoint extensions $A \in \Sigma_{\mathcal{P}}$ with empty resolvent set coincides with the restrictions of L onto the sets of functions $f \in \mathcal{D}$ satisfying the condition*

$$\begin{pmatrix} i\sin\theta & 1-\cos\theta \\ 1+\cos\theta & -i\sin\theta \end{pmatrix}\Gamma_1 f = i\begin{pmatrix} 1+\cos\theta & -i\sin\theta \\ i\sin\theta & 1-\cos\theta \end{pmatrix}\Gamma_0 f, \quad \forall\theta\in[0,2\pi).$$

Proof. Since $J_{\vec{\tau}} = \mathcal{P}$ and $J_{\vec{\gamma}} = (\mathsf{sgn}\ x)I$, relations (2.26), (2.28) mean that $\sigma_1\Gamma_j = \Gamma_j\mathcal{P}$ and $\sigma_3\Gamma_j = \Gamma_j(\mathsf{sgn}\ x)I$ $(j=0,1)$, where σ_1 and σ_3 are Pauli matrices. Therefore, the 'image' of the Clifford algebra $\mathcal{C}l_2(\mathcal{P}, (\mathsf{sgn}\ x)I)$ coincides with $\mathcal{C}l_2(\sigma_1,\sigma_3)$ in $\mathbb{C}^2$ (see Remark 2.13).

The fundamental symmetries $\mathcal{J}_{\vec{\beta}} \in \mathcal{C}l_2(\sigma_1,\sigma_3)$ anti-commuting with σ_1 have the form $\mathcal{J}_{\vec{\beta}} = \beta_2\sigma_2 + \beta_3\sigma_3$, $\beta_2^2+\beta_3^2=1$, where $\sigma_2 = i\sigma_1\sigma_3$. Hence,

$$\mathcal{J}_{\vec{\beta}} = \begin{pmatrix} \beta_3 & -i\beta_2 \\ i\beta_2 & -\beta_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix}, \quad \theta\in[0,2\pi). \tag{3.13}$$

Here, we set $\beta_3 = \cos\theta$ and $\beta_2 = \sin\theta$ (since $\beta_2^2+\beta_3^2=1$). Applying Corollary 3.4 and rewriting (2.31) in the form (2.37) with $\mathcal{J}_{\vec{\alpha}} = \sigma_1$, $\mathcal{J}_{\vec{\gamma}} = \mathcal{J}_{\vec{\beta}}$ (here $\mathcal{J}_{\vec{\beta}}$ is determined by (3.13)), we complete the proof of Proposition 3.6. □

Remark 3.7. To apply Proposition 3.6 for concrete potentials $q(x)$ in (3.13) one needs only to construct a boundary triplet $(\mathbb{C}^2, \Gamma_0, \Gamma_1)$ of L with the help of a boundary triplet $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$ of the differential expression (3.10) on semi-axis $\mathbb{R}_+$ (see (2.27)). To do that one can use [14], where simple explicit formulas for operators Γ_j^+ constructed in terms of asymptotic behavior of $q(x)$ as $x\to 0$ were obtained for great number of singular potentials.

Acknowledgements
The first author (S.K.) expresses his gratitude to the DFG (project AL 214/33-1) and JRP IZ73Z0 (28135) of SCOPES 2009–2012 for the support.

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Received: March 17, 2011.
Revised: September 15, 2011.
Accepted: September 19, 2011.