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A NOTE ON A RELATION BETWEEN THE WEAK AND STRONG DOMINATION NUMBERS OF A GRAPH

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Abstract. In a graph $G = (V, E)$ a vertex is said to dominate itself and all its neighbors. A set $D \subseteq V$ is a weak (strong, respectively) dominating set of G if every vertex $v \in V - S$ is adjacent to a vertex $u \in D$ such that $d_G(v) \geq d_G(u)$ ($d_G(v) \leq d_G(u)$, respectively). The weak (strong, respectively) domination number of G, denoted by $\gamma_w(G)$ ($\gamma_s(G)$, respectively), is the minimum cardinality of a weak (strong, respectively) dominating set of G. In this note we show that if G is a connected graph of order $n \geq 3$, then $\gamma_w(G) + t\gamma_s(G) \leq n$, where $t = 3/(\Delta + 1)$ if G is an arbitrary graph, $t = 3/5$ if G is a block graph, and $t = 2/3$ if G is a claw free graph.

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1. INTRODUCTION

We consider finite, undirected, simple graphs. Let G be a graph, with vertex set V and edge set E. The *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. For a subset $S \subseteq V$, the *open neighborhood* is $N(S) = \cup_{v \in S} N(v)$ and the *closed neighborhood* is $N[S] = N(S) \cup S$. By $G[S]$ we denote the *subgraph* induced by the vertices of S. If v is a vertex of V, then the *degree* of v denoted by $d_G(v)$, is the size of its open neighborhood. A tree is a connected graph that contains no cycle. A *star* $K_{1,q}$ is a tree of order $q + 1$ with at least q vertices of degree 1. A *subdivided star* SS_q is obtained from a star $K_{1,q}$ by replacing each edge uv of the star by a vertex w and edges uw and vw. The *claw* is the star $K_{1,3}$. Given any graph H, a graph G is *H-free* if it does not have any induced subgraph isomorphic to H. A *block graph* is a graph in which every block (maximal 2-connected graph) is a clique. It is well-known that block graphs are exactly chordal graphs that do not contain $K_4 - \{e\}$ as induced subgraph.

In [5], Sampathkumar and Pushpa Latha have introduced the concept of weak and strong domination in graphs. A subset $D \subseteq V$ is a *weak dominating set* (wd-set) if every vertex $v \in V - S$ is adjacent to a vertex $u \in D$, where $d_G(v) \geq d_G(u)$. The subset D is a *strong dominating set* (sd-set) if every vertex $v \in V - S$ is adjacent to a vertex $u \in D$, where $d_G(u) \geq d_G(v)$. The *weak* (*strong*, respectively) *domination number* $\gamma_w(G)$ ($\gamma_s(G)$, respectively) is the minimum cardinality of a wd-set (an sd-set, respectively) of G. If D is an sd-set of G of size $\gamma_s(G)$, then we call D a $\gamma_s(G)$-set. Strong and weak domination have been studied for example in [1–4].

In their paper introducing weak and strong domination in graphs, Sampathkumar and Pushpa Latha showed that a graph G of order n satisfies $\gamma_w(G) + \gamma_s(G) \leq n$ if G is a d-balanced graph (G has an sd-set D_1 and a wd-set D_2 such that $D_1 \cap D_2 = \emptyset$). However there exist graphs G for which $\gamma_w(G) + \gamma_s(G) > n$. For example if G is a subdivided star SS_q with $q \geq 3$, then $\gamma_w(SS_q) = \gamma_s(SS_q) = q + 1 = (n+1)/2$.

2. RESULTS

We begin by giving an observation and two useful lemmas.

Observation 2.1. 1) For a cycle C_n we have $\gamma_w(C_n) = \gamma_s(C_n) = \lceil n/3 \rceil$.
2) For a nontrivial path P_n we have

$$\gamma_s(P_n) = \lceil n/3 \rceil \quad \text{and} \quad \gamma_w(P_n) = \begin{cases} \lceil n/3 \rceil, & \text{if } n \equiv 1 (\operatorname{mod} 3), \\ \lceil n/3 \rceil + 1, & \text{otherwise.} \end{cases}$$

Lemma 2.2. *Let $G = (V, E)$ be a nontrivial connected graph. Then G has a $\gamma_s(G)$-set D such that for every vertex $x \in D$ having at least one neighbor in $V - D$, there is a vertex $y \in V - D$ adjacent to x such that $d_G(y) \leq d_G(x)$.*

Proof. Among all $\gamma_s(G)$-sets let D be a one such vertex such that $\sum_{u \in D} d_G(u)$ is maximum. Obviously the result is valid if $|V| = 2$. Hence let $|V| \geq 3$ and assume that D contains a vertex x such that $N(x) \cap (V - D) \neq \emptyset$ and $d_G(y) > d_G(x)$ for every $y \in N(x) \cap (V - D)$. Then $\{y\} \cup D - \{x\} = D'$ is a $\gamma_s(G)$-set such that $\sum_{u \in D'} d_G(u) > \sum_{u \in D} d_G(u)$, contradicting our choice of D. □

Lemma 2.3. *Let X be an independent set of a connected graph G such that every vertex of X has degree at least three. Then:*

(i) *if G is a claw free graph, then $3|X| \leq 2|N(X)|$,*
(ii) *if G is a block graph, then $2|X| + 1 \leq |N(X)|$.*

Proof. (i) Let E' be the set of edges between X and $N(X)$. Then $3|X| \leq |E'|$. Also since G is claw free and X is independent, every vertex of G has at most two neighbors in X, implying that $|E'| \leq 2|N(X)|$. Therefore, $3|X| \leq |E'| \leq 2|N(X)|$.

(ii) Assume now that G is a block graph and let $A = N(X)$. Consider the graph $G[(X, A)]$ induced by the vertices of X and A. We can suppose that $G[(X, A)]$ is connected, for otherwise we can repeat the procedure below for each component. Let

$v_1, v_2, \ldots, v_t$ be the vertices of X and $A_1, A_2, \ldots, A_t$ the subsets of A ordering as follow: $A_1 = N(v_1) \cap A$ and for $2 \leqslant k \leqslant t$, x_k is a vertex of X adjacent to a vertex of $\cup_{j=1}^{k-1} A_j$ with $A_k = N(v_k) \cap (A - \cup_{j=1}^{k-1} A_j)$. Since every vertex of X has degree at least three, we have $|A_1| \geqslant 3$. Also, since $G[(X, A)]$ is a connected block graph, each vertex x_k for $k \geq 2$ has exactly one neighbor in $\cup_{j=1}^{k-1} A_j$. Using this fact and the fact that every vertex of X has degree at least three, it follows that $|A_k| \geqslant 2$ for $2 \leq k \leq t$. Therefore, $|N(X)| = |A| = |A_1| + |A_2| + \ldots + |A_t| \geq 3 + 2(t-1) = 2|X| + 1$. □

Now we are ready to state our main result.

Theorem 2.4. *Let G be a connected graph of order $n \geq 3$ and maximum degree Δ. Then $\gamma_w(G) + 3\gamma_s(G)/(\Delta + 1) \leq n$. Moreover,*

(i) *if G is a claw free graph, then $\gamma_w(G) + 3\gamma_s(G)/5 \leq n$, and*
(ii) *if G is a block graph, then $\gamma_w(G) + 2\gamma_s(G)/3 \leq (3n-1)/3$.*

Proof. Clearly since $n \geq 3$, we have $\Delta \geq 2$. If $\Delta = 2$, then G is either a cycle C_n or a path P_n, and by Observation 2.1 the result holds. Thus we may assume that $\Delta \geq 3$. Let D be a $\gamma_s(G)$-set satisfying the conditions of Lemma 2.2. Let $A = \{x \in D : N(x) \cap (V - D) \neq \emptyset\}$ and $X = D - A$. Observe that by our choice of D, the set $V - D$ weakly dominates A. If $X = \emptyset$, then $A = D$, and consequently, $\gamma_w(G) \leq |V - D| = n - \gamma_s(G)$. Hence the result is valid even for (i) and (ii) when G is claw free or a block graph, respectively. From now on we will assume that $X \neq \emptyset$. If X contains two adjacent vertices u and v, then one of $D - \{u\}$ or $D - \{v\}$ is a strong dominating set of G, a contradiction. Hence X is an independent set. Note that every vertex of D has degree at least two, otherwise $n = 2$ or G is not connected. Also since $N(X) \subseteq A$ we have $d_G(u) \geq 3$ for every $u \in X$; otherwise $D - \{u\}$ is an sd-set of G, a contradiction. Now since $V - D$ weakly dominates A, the set $(V - D) \cup X$ weakly dominates G, and therefore

$$\gamma_w(G) \leq |(V - D) \cup X| = n - |D| + |X|.$$

Now let us show how to bound $|X|$ by $|D|$ when G is an arbitrary graph, claw free, or a block graph. Note that $|D| = |X| + |A| \geq |X| + |N(X)|$. Let $E(X, N(X))$ be the set of edges between X and $N(X)$. Since $d_G(u) \geq 3$ for every $u \in X$ and $N(X) \subset D$ we have $3|X| \leq |E(X, N(X))|$. Also each vertex y of $N(X)$ has degree at most $\Delta - 1$, otherwise $D - N(y) \cap X$ would be an sd-set of G, a contradiction. It follows that every vertex of $N(X)$ has at most $\Delta - 2$ neighbors in X, thus $|E(X, N(X))| \leq (\Delta - 2)|N(X)|$. This implies that $3|X| \leq |E(X, N(X))| \leq (\Delta - 2)|N(X)|$, and consequently, $|N(X)| \geq 3|X|/(\Delta - 2)$. Since $|D| \geq |X| + |N(X)|$, we obtain $|X| \leq (\Delta - 2)|D|/(\Delta + 1)$. Now we get $\gamma_w(G) \leq n - |D| + |X| = n - 3|D|/(\Delta + 1)$.

Using Lemma 2.3, one can improve the previous result when G is a claw free graph or a block graph. Hence we obtain (i) and (ii), respectively. We omit the details. □

Since the class of trees is contained in the class of block graphs we obtain the following corollary.

Corollary 2.5. *If T is a tree of order $n \geq 3$, then $\gamma_w(T) + 2\gamma_s(T)/3 \leq (3n-1)/3$.*

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