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ON THE EXTENDED AND ALLAN SPECTRA AND TOPOLOGICAL RADII

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Abstract. In this paper we prove that the extended spectrum $\Sigma(x)$, defined by W. Żelazko, of an element x of a pseudo-complete locally convex unital complex algebra A is a subset of the spectrum $\sigma_A(x)$, defined by G.R. Allan. Furthermore, we prove that they coincide when $\Sigma(x)$ is closed. We also establish some order relations between several topological radii of x, among which are the topological spectral radius $R_t(x)$ and the topological radius of boundedness $\beta_t(x)$.

Keywords: topological algebra, bounded element, spectrum, pseudocomplete algebra, topologically invertible element, extended spectral radius, topological spectral radius.

Mathematics Subject Classification: 46H05.

1. INTRODUCTION

A complex algebra A with a topology τ is a *locally convex* algebra if it is a Hausdorff locally convex space and its multiplication $(x, y) \to xy$ is jointly continuous. The topology of A can be given by the family of all continuous seminorms on A.

Throughout this paper $A = (A, \tau)$ will be a locally convex complex algebra with unit e, A' its topological dual and $\{\|\cdot\|_\alpha : \alpha \in \Lambda\}$ the family of all continuous seminorms on A.

An element $x \in A$ is called *bounded* if for some non-zero complex number λ, the set $\{(\lambda x)^n : n = 1, 2, \ldots\}$ is a bounded set of A. The set of all bounded elements of A is denoted by A_0.

For $x \in A$ define the *radius of boundedness* $\beta(x)$ of x by

$$\beta(x) = \inf\left\{\lambda > 0 : \left\{\left(\frac{x}{\lambda}\right)^n : n \geq 1\right\} \text{ is bounded}\right\}$$

adopting the usual convention that $\inf \varnothing = \infty$. Henceforth we shall use this convention without further mention.

Notice that $\lambda_0 > 0$ and $\left\{ \left(\frac{x}{\lambda_0} \right)^n : n \geqq 1 \right\}$ bounded imply $\left\| \left(\frac{x}{\lambda} \right)^n \right\|_\alpha \to 0$ for all $|\lambda| > \lambda_0$ and $\alpha \in \Lambda$. Using this fact it is easy to see that $\beta(x) = \beta_0(x)$, where

$$\beta_0(x) = \inf \left\{ \lambda > 0 : \lim_{n\to\infty} \left(\frac{x}{\lambda} \right)^n = 0 \right\}.$$

In [1], by $\mathcal{B}_1$ it is denoted the collection of all subsets B of A such that:

(i) B is absolutely convex and $B^2 \subset B$,
(ii) B is bounded and closed.

For any $B \in \mathcal{B}_1$, let $A(B)$ be the subalgebra of A generated by B. From (i) we get

$$A(B) = \{\lambda x : \lambda \in \mathbb{C}, x \in B\}.$$

The formula

$$\|x\|_B = \inf \{\lambda > 0 : x \in \lambda B\}$$

defines a norm in $A(B)$, which makes it a normed algebra. It will always be assumed that $A(B)$ carries the topology induced by this norm. Since B is bounded in (A, τ), the norm topology on $A(B)$ is finer than its topology as a subspace of (A, τ).

The algebra A is called *pseudo-complete* if each of the normed algebras $A(B)$, for $B \in \mathcal{B}_1$, is a Banach algebra. It is proved in [1, Proposition 2.6] that if A is sequentially complete, then A is pseudo-complete.

In [1], it is also introduced by G. R. Allan the *spectrum* $\sigma_A(x)$ of $x \in A$ as the subset of the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ defined as follows:

(a) for $\lambda \neq \infty$, $\lambda \in \sigma_A(x)$ if and only if $\lambda e - x$ has no inverse belonging to A_0,
(b) $\infty \in \sigma_A(x)$ if and only if $x \notin A_0$.

In [1, Corollary 3.9] it is proved that $\sigma_A(x) \neq \emptyset$ for all x. We shall call $\sigma_A(x)$ the *Allan spectrum.*

The *Allan spectral radius* $r_A(x)$ of x is defined by

$$r_A(x) = \sup \{|\lambda| : \lambda \in \sigma_A(x)\},$$

where $|\infty| = \infty$.

On the other hand, W. Żelazko defined in [4] the concept of *extended spectrum* of $x \in A$ in the way that we now recall.

As usual

$$\sigma(x) = \{\lambda \in \mathbb{C} : \lambda e - x \notin G(A)\},$$

where $G(A)$ is the set of all invertible elements of A. The resolvent

$$\lambda \to R(\lambda, x) = (\lambda e - x)^{-1}$$

is then defined on $\mathbb{C} \setminus \sigma(x)$, but it is not always a continuous map. Put

$$\sigma_d(x) = \{\lambda_0 \in \mathbb{C} \setminus \sigma(x) : R(\lambda, x) \text{ is discontinuous at } \lambda = \lambda_0\}$$

and

$$\sigma_\infty(x) = \begin{cases} \varnothing & \text{if } \lambda \to R(1, \lambda x) \text{ is continuous at } \lambda = 0, \\ \infty & \text{otherwise.} \end{cases}$$

Then the extended spectrum of x is the set

$$\Sigma(x) = \sigma(x) \cup \sigma_d(x) \cup \sigma_\infty(x).$$

It is proved in [4, Theorem 15.2] that if A is complete, then $\Sigma(x)$ is a non empty set of $\mathbb{C}_\infty$ for every x, and the *extended spectral radius* $R(x)$ is defined by

$$R(x) = \sup\{|\lambda| : \lambda \in \Sigma(x)\}.$$

We shall not assume that A is complete. Nevertheless, from now on we assume that $\Sigma(x)$ is a non empty set of $\mathbb{C}_\infty$ for every $x \in A$.

2. COMPARISON OF $\Sigma(x)$ AND $\sigma_A(x)$

Theorem 2.1. *If A is pseudo-complete, then $\Sigma(x) \subset \sigma_A(x)$ for any $x \in A$.*

Proof. Let $\lambda \notin \sigma_A(x)$ with $\lambda \neq \infty$, then $\lambda \notin \sigma(x)$ and $R(\lambda, x)$ is bounded. Hence $R(\lambda, x) \in A(B)$ for some $B \in \mathcal{B}_1$ ([1, Proposition 2.4]).

For any $\mu \in \mathbb{C}$, we have that $(\mu e - x) = (\lambda e - x) + (\mu - \lambda) e$. Let $0 < \gamma < \|R(\lambda, x)\|_B^{-1}$, then for $|\mu - \lambda| < \gamma$, the formula

$$S_n(\mu) = R(\lambda, x) - (\mu - \lambda) R(\lambda, x)^2 + (\mu - \lambda)^2 R(\lambda, x)^3 - \ldots + (-1)^n (\mu - \lambda)^n R(\lambda, x)^{n+1},$$

defines a Cauchy sequence in the Banach algebra $A(B)$. Therefore, it converges in $A(B)$ to $R(\mu, x)$.

Given $\varepsilon > 0$, there exists $0 < \delta < \gamma$ such that

$$\|S_n(\mu) - R(\lambda, x)\|_B \leq |\mu - \lambda| \, \|R(\lambda, x)\|_B^2 \left(\frac{1}{1 - \gamma \|R(\lambda, x)\|_B} \right) < \varepsilon$$

for all n if $|\lambda - \mu| < \delta$, which implies that $\|R(\mu, x) - R(\lambda, x)\| \leq \varepsilon$ if $|\lambda - \mu| < \delta$. Hence $R(\mu, x) \to R(\lambda, x)$ as $\mu \to \lambda$, in $A(B)$ and also in (A, τ), therefore $\lambda \notin \sigma_d(x)$. Thus, $\lambda \notin \Sigma(x)$.

If $\infty \notin \sigma_A(x)$, then x is bounded and there exists $r > 0$ such that the idempotent set $\left\{ \left(\frac{x}{r}\right)^n : n \geq 1 \right\}$ is bounded. The closed absolutely convex hull B of $\left\{ \left(\frac{x}{r}\right)^n : n \geq 1 \right\}$ belongs to B_1. Consider the Banach algebra $A(B)$. Since $\left\| \frac{x}{\beta} \right\|_B < 1$ for every $|\beta| > r$, we obtain

$$R\left(1, \frac{x}{\beta}\right) = e + \frac{x}{\beta} + \left(\frac{x}{\beta}\right)^2 + \ldots$$

in the Banach algebra $A(B)$.

Since
$$\left\| R\left(1, \frac{x}{\beta}\right) - e \right\|_B \to 0$$
as $|\beta| \to \infty$, we have that $R(1, tx) \to e$ as $t \to 0$, in $A(B)$ and hence in (A, τ) as well. Thus $R(1, tx)$ is continuous in $t = 0$ and $\infty \notin \Sigma(x)$. □

Lemma 2.2. *Suppose A is pseudo-complete and let $x \in A$ be such that the extended spectral radius $R(x) < \infty$. Then for each $f \in A'$ the function $F(\lambda) = f(R(1, \lambda x))$ is holomorphic in the open disc $D(0, \delta)$, with $\delta = \frac{1}{R(x)}$, where $D(0, \delta) = \mathbb{C}$ when $R(x) = 0$. Furthermore,*

$$F^{(n)}(\lambda) = n! f\left(R(1, \lambda x)^{n+1} x^n\right) \tag{2.1}$$

for every $\lambda \in D(0, \delta)$ and $n = 0, 1, 2, \ldots$. In particular,

$$F^{(n)}(0) = n! f(x^n)$$

for all $n \geq 0$.

Proof. We have that $\lambda \notin \Sigma(x)$ whenever $|\lambda| > R(x)$. This implies that the function

$$\lambda \to R(1, \lambda x)$$

is continuous in the open disc $D = D(0, \delta)$. By definition $F^{(0)}(0) = f(e)$ and $F(\lambda) = f(R(1, \lambda x))$ is holomorphic in D since

$$\begin{aligned} F'(\lambda_0) &= \lim_{\lambda \to \lambda_0} \frac{f(R(1, \lambda x)) - f(R(1, \lambda_0 x))}{\lambda - \lambda_0} = \\ &= \lim_{\lambda \to \lambda_0} f\left(\frac{R(1, \lambda x) R(1, \lambda_0 x)(\lambda - \lambda_0) x}{\lambda - \lambda_0}\right) = \\ &= f\left(R(1, \lambda_0 x)^2 x\right) \end{aligned}$$

for every $\lambda_0 \in D$.

It is easy to obtain (2.1) by induction. □

Theorem 2.3. *If A is pseudo-complete, then for any $x \in A$ we have that $\Sigma(x) = \sigma_A(x)$ if $\Sigma(x)$ is closed in $\mathbb{C}_\infty$.*

Proof. Let $x \in A$ and assume that $\Sigma(x)$ is closed, then by Theorem 2.1 we only have to prove that $\lambda_0 \notin \Sigma(x)$ implies $\lambda_0 \notin \sigma_A(x)$.

Let $\lambda_0 \notin \Sigma(x)$, with $\lambda_0 \neq \infty$, then $\lambda_0 e - x \in G(A)$. We shall show that $(\lambda_0 e - x)^{-1}$ is bounded. Since $\Sigma(x)$ is closed, then there exists an open disc $D(\lambda_0)$ around λ_0 such that $\lambda e - x \in G(A)$ if $\lambda \in D(\lambda_0)$ and $R(\lambda, x)$ is continuous at $\lambda = \lambda_0$. Using the identity

$$(\lambda e - x)^{-1} - (\lambda_0 e - x)^{-1} = (\lambda_0 - \lambda)(\lambda e - x)^{-1}(\lambda_0 e - x)^{-1},$$

we obtain

$$\lim_{\lambda \to \lambda_0} \frac{R(\lambda, x) - R(\lambda_0, x)}{\lambda - \lambda_0} = -R(\lambda_0, x)^2.$$

Then for any $f \in A'$ we get

$$\lim_{\lambda \to \lambda_0} \frac{f\left(R(\lambda, x)\right) - f\left(R(\lambda_0, x)\right)}{\lambda - \lambda_0} = -f\left(R(\lambda_0, x)^2\right),$$

which implies that $R(\lambda, x)$ is weakly holomorphic in $\lambda = \lambda_0$. By [1, Theorem 3.8 (i)] we obtain that $(\lambda_0 e - x)^{-1}$ is bounded in A. Therefore, $\lambda_0 \notin \sigma_A(x)$.

If $\infty \notin \Sigma(x)$, then some neighborhood of ∞ does not intersect $\Sigma(x)$ and we have that $R(x) < \infty$. Let $f \in A'$. By Lemma 2.2, the Taylor expansion of $F(\lambda) = f\left(R\left(1, \lambda x\right)\right)$ around 0 is

$$F(\lambda) = f\left(e\right) + \lambda f(x) + \frac{2\lambda^2}{2!} f\left(x^2\right) + \dots$$

for $|\lambda| < \frac{1}{R(x)}$. In particular, $\lim_{n \to \infty} f\left(\lambda_0^n x^n\right) = 0$ for some $\lambda_0 > 0$ and then $\{f\left(\lambda_0^n x^n\right) : n \geq 1\}$ is bounded; therefore $\{(\lambda_0 x)^n : n \geq 1\}$ is bounded. Thus $x \in A_0$ and $\infty \notin \sigma_A(x)$. □

3. COMPARISON BETWEEN TOPOLOGICAL RADII

Let $x \in A$, we say that x is *topologically invertible* if $\overline{xA} = \overline{Ax} = A$, i.e. for each neighborhood V of e there exist $a_V, a'_V \in A$ such that $xa_V \in V$ and $a'_V x \in V$.

The *topological spectrum* $\sigma_t(x)$ of x is the set

$$\sigma_t(x) = \{\lambda \in \mathbb{C} : \lambda e - x \text{ is not topological invertible}\}.$$

The *topological spectral radius* $R_t(x)$ is defined by

$$R_t(x) = \sup\{|\lambda| : \lambda \in \sigma_t(x)\}.$$

Having in mind the definition of $\beta_0(x)$ we define the *topological radius of boundedness* $\beta_t(x)$ of x by

$$\beta_t(x) = \inf\left\{\lambda > 0 : \liminf_n \left\|\left(\frac{x}{\lambda}\right)^n\right\|_\alpha = 0 \text{ for all } \alpha \in \Lambda\right\}.$$

In [2] the first author defined the *lower extended spectral radius* of x by

$$R_*(x) = \sup_{\alpha \in \Lambda} \liminf_n \sqrt[n]{\|x^n\|_\alpha}$$

and in [3] it is proved that if A is a complete locally convex unital algebra, then for any $x \in A$ we have $R_*(x) \leq r_0(x)$, and $R_*(x) = r_0(x)$ if A is a unital B_0-algebra (metrizable complete locally convex algebra), where

$$\begin{aligned} r_0(x) = \inf\{\, 0 < r \leqq \infty &: \text{there exists } (a_n)_0^\infty, a_n \in \mathbb{C}, \text{ such that} \\ &\textstyle\sum a_n \lambda^n \text{ has radius of convergence } r \text{ and} \\ &\textstyle\sum a_n x^n \text{ converges in } A\} \end{aligned}$$

(In [3] this radius is denoted by $r_6(x)$).

Here we have the following result.

Proposition 3.1. *Let $x \in A$. Then*

$$R_t(x) \leq \beta_t(x) = R_*(x) \leq \beta(x) \leq r_A(x).$$

Proof. The first inequality is obvious if $\beta_t(x) = \infty$, therefore let $\beta_t(x) < \infty$. Given $\lambda > \beta_t(x)$ and $\alpha \in \Lambda$, there exists a subsequence $(n_k)_k = (n_k(\alpha))_k$ of the natural sequence (n) such that $\lim\limits_{k\to\infty} \left\| \left(\frac{x}{\lambda}\right)^{n_k} \right\|_\alpha = 0$. Then

$$\lim_{k\to\infty} \left\| \left(\frac{e}{\lambda} + \frac{x}{\lambda^2} + \ldots + \frac{x^{n_k - 1}}{\lambda^{n_k}} \right) (\lambda e - x) - e \right\|_\alpha = 0.$$

Hence $\lambda e - x$ is topologically invertible for any such λ and it follows that $R_t(x) \leq \beta_t(x)$.

If $R_*(x) = \infty$, then $\beta_t(x) \leq R_*(x)$. Now suppose $R_*(x) < \mu < \lambda < \infty$. Then given $\alpha \in \Lambda$ there exists a subsequence $(n_k)_k = (n_k(\alpha))_k$ of (n) such that $\sqrt[n_k]{\|x^{n_k}\|_\alpha} < \mu < \lambda$, which implies that $\left\| \left(\frac{x}{\lambda}\right)^{n_k} \right\|_\alpha < \left(\frac{\mu}{\lambda}\right)^{n_k}$. Therefore, $\beta_t(x) \leq \lambda$ and we have $\beta_t(x) \leq R_*(x)$.

Assume that $\beta_t(x) < R_*(x)$, then there exist $\lambda > 0$ and $\alpha_0 \in \Lambda$ such that $\beta_t(x) < \lambda < R_*(x)$ and $\lambda < \liminf\limits_n \sqrt[n]{\|x^n\|_{\alpha_0}}$. Hence $\liminf\limits_n \sqrt[n]{\left\| \left(\frac{x}{\lambda}\right)^n \right\|_{\alpha_0}} > 1$. On the other hand, $\lambda > \beta_t(x)$ implies that $\liminf\limits_n \sqrt[n]{\left\| \left(\frac{x}{\lambda}\right)^n \right\|_{\alpha_0}} = 0$, which contradicts the previous statement. Thus, $\beta_t(x) = R_*(x)$.

Since $\beta(x) = \beta_0(x)$ it is clear that $\beta_t(x) \leq \beta(x)$. Finally, $\beta(x) \leq r_A(x)$ by [1, Theorem 3.12]. □

Corollary 3.2. *If A is pseudo-complete, then*

$$R_t(x) \leq R_*(x) = \beta_t(x) \leq \beta(x) = r_A(x) \leq R(x)$$

for every $x \in A$.

Proof. Let $x \in A$. We have by [1, Theorem 3.12] that $\beta(x) = r_A(x)$. Thus we only have to prove that $\beta(x) \leq R(x)$. This is obvious if $R(x) = \infty$, so assume that $R(x) < \infty$, therefore $\infty \notin \Sigma(x)$. Applying Lemma 2.2 we obtain that the Taylor expansion about 0 of $F(\lambda) = f(R(1, \lambda x))$ is

$$F(\lambda) = f(e) + \lambda f(x) + \frac{2\lambda^2}{2!} f\left(x^2\right) + \ldots$$

for $f \in A'$ and $|\lambda| < \frac{1}{R(x)}$.

Then $\lim\limits_{n\to\infty} f((\lambda x)^n) = 0$ for any $0 < \lambda < \frac{1}{R(x)}$ and $f \in A'$. In particular, for any such λ the set $\{(\lambda x)^n : n \geq 1\}$ is weakly bounded and therefore $\{(\lambda x)^n : n \geq 1\}$ is bounded in A. It follows that $\lambda \geq \beta(x)$ for every $\lambda > R(x)$ and then $\beta(x) \leq R(x)$. □

Proposition 3.3. *If A is complete, then $r_A(x) = \beta(x) = R(x)$ for all $x \in A$.*

Proof. It remains to prove that $R(x) \le \beta(x)$. We can assume that $\beta(x) < \infty$. Let $r > \beta(x)$, then we have that $f\left(\left(\frac{x}{r}\right)^n\right) \to 0$ for every $f \in A'$, therefore

$$\limsup_n \sqrt[n]{|f(x^n)|} \le r$$

for every $f \in A'$. We get from [4, Theorem 15.6] that

$$R(x) = R_2(x) = \sup_{f \in A'} \limsup_n \sqrt[n]{|f(x^n)|} \le r.$$

Therefore, $R(x) \le \beta(x)$. □

Remark 3.4. In [2] it is constructed a unital B_0-algebra A in which there is an element x such $R_*(x) = 1$ and $R(x) = \infty$. On the other hand, if we consider the non-complete algebra $A = (P(t), \|\cdot\|)$ of all complex polynomials with the norm $\|p(t)\| = \max_{0 \le t \le 1} |p(t)|$, then for every $\lambda \ne 0$ we have that $\left\|\left(\frac{t}{\lambda}\right)^n\right\| = \frac{1}{|\lambda|^n}$. Therefore $\beta(t) = 1$, nevertheless $R(t) = \infty$ since $\lambda - t$ does not have an inverse for all $\lambda \in \mathbb{C}$.

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