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NOTE ON THE STABILITY OF FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

Florin Bojor

Abstract. In this paper, we will prove the generalized Hyers-Ulam stability of the linear differential equation of the form $y'(x)+f(x)\,y(x)+g(x)=0$ under some additional conditions.

Keywords: fixed point method, differential equation, Hyers-Ulam stability.

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1. INTRODUCTION

The study of the stability functional equations is strongly related to Ulam's question concerning the stability of group homomorphisms. We mention that the concept of stability for a functional equation appears when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question for functional equations shows "how the solutions of the inequality differ from those of the given functional equation." D.H. Hyers [3] excellently answered the question of Ulam and proved the following result:

Theorem 1.1 (Hyers, [3])**.** *Let E and E' be two Banach spaces and $f : E \to E'$ a given function such that there exists $\delta \geq 0$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta, \quad \forall\, x, y \in X. \tag{1.1}$$

Then the limit $L(x) = \lim\limits_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$, L is an additive function and the inequality

$$\|f(x) - L(x)\| < \delta \tag{1.2}$$

is true for all $x \in E$. Moreover, $L(x)$ is the only additive function which satisfies the inequality (1.2).

Since Hyers' result, a great number of papers on the subject have been published, extending and generalizing the Ulam's problem and the Hyers' theorem in various directions (see [3, 9, 10]).

In [9] V. Radu proposed a new method for obtaining the existence of exact solutions and error estimations, based on the fixed point alternative and this theorem is:

Theorem 1.2 (The fixed point alternative)**.** *Suppose we are given a complete generalized metric space* (Ω, d) *and a strictly contractive mapping* $T : \Omega \to \Omega$ *with the Lipschitz constant* a*. Then, for each given element* $x \in \Omega$*, either*

$$d\left(T^n x,\, T^{n+1} x\right) = \infty,\ \forall n \geq 0,$$

or there exists a natural number n_0 *such that:*

(i) $d\left(T^n x,\, T^{n+1} x\right) < \infty$ *for all* $n \geq n_0$.
(ii) *The sequence* $\left(T^n x\right)_{n \geq 0}$ *is convergent to a fixed point* $y*$ *of* T.
(iii) $y*$ *is the unique fixed point of* T *in the set* $\Delta = \{y \in \Omega \,|\, d\left(T^{n_0} x, y\right) < \infty\}$.
(iv) $d\left(y,\, y*\right) \leq \frac{1}{1-a} d\left(y,\, Ty\right)$ *for all* $y \in \Delta$.

Let $a_0, a_1, \ldots, a_{n-1}$ be real numbers and let I be an interval. For $y \in C^n\left(I, \mathbb{R}\right)$, $\varepsilon > 0$ and $\varphi \in C\left(I, \mathbb{R}_+\right)$ we consider the following equation:

$$y^{(n)}(t) = \sum_{k=0}^{n-1} a_k y^{(k)}(t), \quad t \in I \tag{1.3}$$

and the following inequations

$$\left| y^{(n)}(t) - \sum_{k=0}^{n-1} a_k y^{(k)}(t) \right| \leq \varepsilon, \quad t \in I \tag{1.4}$$

and

$$\left| y^{(n)}(t) - \sum_{k=0}^{n-1} a_k y^{(k)}(t) \right| \leq \varphi(t), \quad t \in I. \tag{1.5}$$

Definition 1.3. The equation (1.3) is Hyers-Ulam stable if there exists a real number $c > 0$ such that for each $\varepsilon > 0$ and for each solution $s \in C^{(n)}\left(I, \mathbb{R}\right)$ of (1.4) there exists a solution $y \in C^{(n)}\left(I, \mathbb{R}\right)$ of (1.3) with

$$|s(t) - y(t)| \leq c \cdot \varepsilon, \quad \forall\, t \in I.$$

Definition 1.4. The equation (1.3) is Hyers-Ulam-Rassias stable, with respect to φ, if there exists a real number $c_\varphi > 0$ such that for each solution $s \in C^{(n)}\left(I, \mathbb{R}\right)$ of (1.5) there exists a solution $y \in C^{(n)}\left(I, \mathbb{R}\right)$ of (1.3) with

$$|s(t) - y(t)| \leq c_\varphi \cdot \varphi(t), \quad \forall\, t \in I.$$

Alsina and Ger were the first authors who investigated the Hyers-Ulam stability of differential equations. In 1998, they proved in [1] the stability of differential equation

$$y'(t) = y(t). \tag{1.6}$$

Following the same approach as in [1], Miura [8] proved the Hyers-Ulam stability of differential equation

$$y'(t) = \lambda y(t). \tag{1.7}$$

S.M. Jung [4–7] applied the fixed point method for proving the Hyers-Ulam-Rassias stability of a Volterra integral equation of the second kind and for differential equations of first order. Using the same technique we prove the Hyers-Ulam-Rassias stability and Hyers-Ulam stability of differential equation

$$y'(x) + f(x)y(x) + g\,(x) = 0 \tag{1.8}$$

under some conditions, others than the conditions from [4].

2. MAIN RESULTS

In this paper, by using the idea of Cădariu and Radu [2], we will prove the Hyers-Ulam-Rassias stability for the equation (1.8) on the intervals $I = [a, b)$, where $-\infty < a < b \leq \infty$.

Theorem 2.1. *Let $f, g : I \to \mathbb{R}$ be continuous functions and let for a positive constant M, $|f(x)| \geq M$ for all $x \in I$. Assume that $\psi : I \to [0, \infty)$ is an integrable function with the property that there exists $P \in (0, 1)$ such that*

$$\int_a^x |f(t)|\psi(t)dt \leq P\psi(x) \tag{2.1}$$

for all $x \in I$. If a continuously differentiable function $y : I \to \mathbb{R}$ verifies the relation:

$$|y'(x) + f(x)y(x) + g\,(x)| \leq \psi(x) \tag{2.2}$$

for all $x \in I$, then there exists a unique solution $S : I \to \mathbb{R}$ of the equation (1.8) which verifies the following relations:

$$|y(x) - S(x)| \leq \frac{P}{M - MP}\psi(x) \tag{2.3}$$

for all $x \in I$ and $S\,(a) = y\,(a)$.

Proof. Let us consider the set $\Omega = \{h : I \to \mathbb{R} \,|\, h \text{ is continuous and } h\,(a) = y\,(a)\}$ and the generalized metric $d\,(h_1, h_2)$ defined on Ω as

$$d\,(h_1, h_2) = d_\psi\,(h_1, h_2) = \inf\left\{k > 0 \,|\, |h_1(x) - h_2(x)| \leq k\psi(x),\ \forall x \in I\right\}.$$

Then (Ω, d) is a generalized complete metric space (see [4]). We define the operator $T : \Omega \to \Omega$,

$$Th(x) = y(a) - \int_a^x (f(t)h(t) + g(t))dt \quad x \in I,$$

for all $h \in \Omega$. Indeed Th is a continuously differentiable function on I, since f and g are continuous function and $Th(a) = y(a)$.

Now, let $h_1, h_2 \in \Omega$. Then we have

$$|Th_1(x) - Th_2(x)| = \left| \int_a^x f(t)\left(h_1(t) - h_2(t)\right) dt \right| \leq \int_a^x |f(t)| \left|h_1(t) - h_2(t)\right| dt \leq$$

$$\leq d(h_1, h_2) \int_a^x |f(t)|\psi(t)dt \leq P\psi(x)d(h_1, h_2)$$

for all $x \in I$. Therefore,

$$d(Th_1, Th_2) \leq Pd(h_1, h_2), \tag{2.4}$$

thus the operator T is a contraction with the constant P.

Now integrating the both sides of the relation (2.2) on $[a, x]$ we obtain

$$\left| y(x) - y(a) + \int_a^x (f(t)y(t) + g(t))\, dt \right| \leq \frac{P}{M}\psi(x) \tag{2.5}$$

for all $x \in I$, which means $d(y, Ty) \leq \frac{P}{M} < \infty$. By the fixed point alternative there exists an element $S = \lim_{n\to\infty} T^n y$ and S is unique fixed point of T in the set $\Delta = \{h \in \Omega \,|\, d(T^{n_0}y, h) < \infty\}$. It may be proved that

$$\Delta = \{h \in \Omega \,|\, d(y, h) < \infty\}.$$

Therefore the set Δ is independent of n_0. To prove that the function S is a solution to the equation (1.8), we derive with respect to x the both sides of the relation

$$S(x) = TS(x), \quad x \in I. \tag{2.6}$$

Thus

$$S'(x) = -f(x)S(x) - g(x) \tag{2.7}$$

for all $x \in I$ which implies that the function S is a solution to the equation (1.8) and verifies the relation $S(a) = y(a)$.

Applying again the fixed point alternative we obtain

$$d(h, S) \leq \frac{1}{1-P} d(h, Th) \quad \text{for all} \quad h \in \Delta.$$

Since $y \in \Delta$, we have

$$d(y, S) \leq \frac{1}{1-P} d(y, Ty) \leq \frac{P}{M(1-P)},$$

whence

$$|y(x) - S(x)| \leq \frac{P}{M - MP} \psi(x)$$

for all $x \in I$. This inequality proves the relation (2.3). □

In the same manner it can be proved the following theorem of the Hyers-Ulam-Rassias stability of the equation (1.8) on the interval $J = (b, a]$, where $-\infty \leq b < a < \infty$.

Theorem 2.2. *Let $f, g : J \to \mathbb{R}$ be continuous functions and let for some positive constant M, $|f(x)| \geq M$ for all $x \in J$. Assume that $\psi : J \to [0, \infty)$ is an integrable function with the property that there exists $P \in (0, 1)$ such that*

$$\int_x^a |f(t)| \psi(t) dt \leq P\psi(x) \tag{2.8}$$

for all $x \in J$. If a continuously differentiable function $y : J \to \mathbb{R}$ verifies the relation:

$$|y'(x) + f(x)y(x) + g(x)| \leq \psi(x) \tag{2.9}$$

for all $x \in J$, then there exists a unique solution $S : J \to \mathbb{R}$ of the equation (1.8) *which verifies the following relations:*

$$|y(x) - S(x)| \leq \frac{P}{M - MP} \psi(x) \tag{2.10}$$

for all $x \in J$ and $S(a) = y(a)$.

The Hyers-Ulam-Rassias stability equation (1.8) on $\mathbb{R}$ will be proved by Theorem 2.1 and Theorem 2.2.

Corollary 2.3. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions and let for some positive constant M, $|f(x)| \geq M$ for all $x \in \mathbb{R}$. Assume that $\psi : \mathbb{R} \to [0, \infty)$ is an integrable function with the property that there exists $P \in (0, 1)$ such that*

$$\left| \int_0^x |f(t)| \psi(t) dt \right| \leq P\psi(x) \tag{2.11}$$

for all $x \in \mathbb{R}$. If a continuously differentiable function $y : \mathbb{R} \to \mathbb{R}$ verifies the relation:

$$|y'(x) + f(x)y(x) + g(x)| \leq \psi(x) \tag{2.12}$$

for all $x \in \mathbb{R}$*, then there exists a unique solution* $S : \mathbb{R} \to \mathbb{R}$ *of equation* (1.8) *which verifies the following relations:*

$$|y(x) - S(x)| \leq \frac{P}{M - MP}\psi(x) \tag{2.13}$$

for all $x \in \mathbb{R}$ *and* $S(0) = y(0)$.

Proof. By the relation (2.11) we have

$$\int_0^x |f(t)|\psi(t)dt \leq P\psi(x) \tag{2.14}$$

for all $x \geq 0$. Applying Theorem 2.1, there exists a solution of equation (1.8), $S_1 : [0, \infty) \to \mathbb{R}$ which verifies the relations (2.3) and $S_1(0) = y(0)$.

From (2.11) we also obtain

$$\int_x^0 |f(t)|\psi(t)dt \leq P\psi(x) \tag{2.15}$$

for all $x \leq 0$. Applying Theorem 2.2, there exists a solution of equation (1.8), $S_2 : (-\infty, 0] \to \mathbb{R}$ which verifies (2.10) and $S_2(0) = y(0)$. It is easy to check that the function

$$S(x) = \begin{cases} S_1(x), & x \geq 0, \\ S_2(x), & x < 0, \end{cases} \tag{2.16}$$

is a continuously differentiable function on $\mathbb{R}$, a solution of equation (1.8) on $\mathbb{R}$ and it verifies relation (2.13). □

Using Theorem 2.1 it can be shown the Hyers-Ulam stability for the equation (1.8) on $I = [a, b)$, where $-\infty < a < b \leq \infty$.

Corollary 2.4. *Let* ε, $M > 0$ *and let* $f : I \to [M, \infty)$ *and* $g : I \to \mathbb{R}$ *be continuous. If a continuously differentiable function* $y : I \to \mathbb{R}$ *verifies the relation*

$$|y'(x) + f(x)y(x) + g(x)| \leq \varepsilon \tag{2.17}$$

for all $x \in I$*, then there exists a unique solution* $S : I \to \mathbb{R}$ *of equation* (1.8) *which verifies the relations:*

$$|y(x) - S(x)| \leq \frac{\varepsilon}{M(2q - 1)} \tag{2.18}$$

for all $x \in I$*, where* $q \in \left(\frac{1}{2}, 1\right)$ *and* $S(a) = y(a)$.

Proof. Let $q \in \left(\frac{1}{2}, 1\right)$. Multiplying relation (2.17) by $e^{q\int_a^x f(t)dt}$, and denoting

$$z(x) := y(x)e^{q\int_a^x f(t)dt}, \quad x \in I \tag{2.19}$$

we have
$$\left|z'(x) + (1-q)\, f(x)z(x) + g(x)e^{q\int_a^x f(t)dt}\right| \le \varepsilon \cdot e^{q\int_a^x f(t)dt} \tag{2.20}$$

for all $x \in I$. Then the function $F(x) = (1-q)\, f(x)$, where $x \in I$, is continuous on I and satisfies the relation $|F(x)| > (1-q)\, M$ for all $x \in I$.

Let $\psi(x) = \varepsilon \cdot e^{q\int_a^x f(t)dt}$, when $x \in I$. We see that

$$\int_a^x |F(t)|\, \psi(t)dt = (1-q)\, \varepsilon \int_a^x f(t)e^{q\int_a^t f(u)du}dt \le \frac{1-q}{q}\psi(x) \tag{2.21}$$

for all $x \in I$, thus the function $\psi : I \to [0, \infty)$ verifies relation (2.1) with $P = \frac{1-q}{q} \in (0, 1)$.

By Theorem 2.1, there exists $s \in C^1(I, \mathbb{R})$, which is a unique solution for the equation
$$z'(x) + (1-q)\, f(x)z(x) + g(x)e^{q\int_a^x f(t)dt} = 0 \tag{2.22}$$

and verifies the relations

$$|z(x) - s(x)| \le \frac{1}{M(2q-1)} \cdot \varepsilon \cdot e^{q\int_a^x f(t)dt} \tag{2.23}$$

for all $x \in I$ and $s(a) = z(a)$.

Then the function $S(x) = s(x)e^{-q\int_a^x f(t)dt}$ is a solution of equation (1.8) and verifies relation (2.18). □

Equation (1.8) is not Hyers-Ulam stable on the intervals $J = (-\infty, a]$ in general, as we can see in the following example.

Example 2.5. Let us consider equation (1.8) where $f(x) = x^2$ and $g(x) = 0$. The solution of this equation $S : J \to \mathbb{R}$ which verifies the condition $S(a) = p$ is

$$S(x) = p \cdot e^{\frac{a^3 - x^3}{3}}. \tag{2.24}$$

A continuously differentiable function $y : J \to \mathbb{R}$ which verifies inequality (2.17) is

$$y(x) = p \cdot e^{\frac{a^3 - x^3}{3}} + \varepsilon \cdot e^{-\frac{x^3}{3}} \int_a^x e^{\frac{t^3}{3}} dt. \tag{2.25}$$

Considering equation (1.8) being Hyers-Ulam stable, there exists $k > 0$ such that

$$|y(x) - S(x)| \le \varepsilon \cdot k \tag{2.26}$$

for all $x \in J$. By substitution, we have

$$\left|\int_a^x e^{\frac{t^3}{3}} dt\right| \le k e^{\frac{x^3}{3}} \tag{2.27}$$

for all $x \in J$. Now letting $x \to -\infty$ it generates a contradiction. So equation (1.8) is not Hyers-Ulam stable.

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Florin Bojor
florin.bojor@yahoo.com

North University of Baia Mare
Faculty of Sciences
Department of Mathematics and Computer Sciences
Str. Victoriei, nr. 76
430122 Baia Mare, Romania

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